ABSTRACT—This paper attempts to bridge the gap between rational expectations equilibrium analysis and the standard CAPM. It finds that although the market portfolio is not mean-variance efficient under asymmetric information, the security market capital line still exists. Moreover, this security market line is determined by the Information-adjusted Average Market Portfolio (IAMP) such that the expected rate of return to an asset equals the product of IAPM’s return and the asset’s beta with respect to the IAMP. The paper also investigates Bayesian decision-making based on lognormal prior and lognormal signals, which overcomes a major technical difficulty in asset pricing under asymmetric information.

KEYWORDS: Asymmetric Information, Bayesian Inference, CAPM, Mean-Variance Efficiency, Optimal Portfolio, Rational Expectations Equilibrium

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1. INTRODUCTION

Capital asset pricing theory is one of the corner stones of modern finance theory. The classic CAPM developed by Sharpe (1964), Lintner (1965a, b), and Mossin (1966) contributes to modern finance theory in two folds. First, it demonstrates that the market portfolio is mean-variance efficient and investors should allocate their investment between the risk-free asset and the market portfolio. Second, the CAPM formula states that the expected risk premium of a risky asset in equilibrium equals the risk premium of the market portfolio times the risk of the asset relative to the market as measured by the asset’s beta. So, the market portfolio plays a central role in the standard CAPM: on the one hand, it is the reference for investors to determine their optimal investment decisions; on the other hand, it acts as an index for investors and econometricians to empirically test risky securities traded in the market and separate outperformed securities from underperformed securities. An important assumption implicitly made by the standard CAPM is information symmetry among all traders in the market and “asset pricing models were based on the notion that prices could be viewed as arising from a symmetric information world” (O’Hara (2003), p. 1337).

Following the pioneering works of Grossman (1976), and Grossman and Stiglitz (1980), rational expectations models developed by, for example, Hellwig (1980), Admati (1985), Easley and O’Hara (2004), focus on the effect of asymmetric
information on capital asset pricing. The consensus shared by the rational expectations analysis with multiple risky assets is that the market portfolio is no longer mean-variance efficient under asymmetric information since investors have diverse beliefs on the values of risky assets. This means that there does not exist a market capital line specifying all mean-variance efficient portfolios. Then, how is the other pillar of the CAPM—the security market line under asymmetric information? Is there a security market line for each financial market when traders have different information? If it exists, how should the standard CAPM formula be amended to accommodate asymmetric information? To our knowledge, existing rational expectations models or other models of capital asset pricing are silent on these issues. The primary aim of this paper is to bridge this gap between rational expectations equilibrium analysis and capital asset pricing, and develop a modified CAPM formula to estimate the expected risk premium under asymmetric information.

To incorporate asymmetric information into asset pricing, the challenge is how to develop a model that is analytically tractable. Most rational expectations models of perfectly competitive securities markets with asymmetric information assume that the prices of risky assets and signals conveying price information are normally distributed random variables. The normal distributions, accompanied with

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1 Merton (1987) studies asset returns with incomplete information where investors may be not aware of some securities in a market but for investors knowing a security they have symmetric information. Earlier, Lintner (1969) considers investors with diverse judgments about returns and risk of risky assets and Gonedes (1976) assumes expectation heterogeneity arising from the disagreement in assigning weights to a set of basic economic activities. In both models, investors do not abstract information from price signals. The literature on the rational expectations equilibrium of financial markets is rich. Easley and O’Hara (2004) provide an excellent literature review on the relevant research, see references therein.
constant absolute risk aversion utility functions, can generate elegant linear demand functions and a closed-form solution facilitating analysis. The tradeoff is that they bound the analysis to prices and returns per share. In other words, a model with normally distributed prices can hardly analyze the expected rates of return because the rate of return involves the ratio of a stock’s prices at two different time points, which makes the analysis extremely difficult if not impossible.\(^2\) So, if we want to generalize the CAPM to study the effects of asymmetric information on the market and establish a link between the risk premium of a particular asset or portfolio and a market index, it seems unavoidable to abandon the assumption of normal stock prices.

Actually, the assumption of normally distributed prices encounters some criticisms because prices are unbounded from below, although the probability of negative prices is infinitesimal when the mean is sufficiently large and the variance is sufficiently small. Another assumption of random asset prices widely accepted in financial economics, options theory in particular, is lognormal distribution. Undoubtedly, lognormal price distributions are more plausible and reasonable than normal price distributions (Benninga (2000), ch. 15). More importantly, the logarithm of the ratio of two lognormal variables is the subtraction of two normal variables. Thus, the rates of return on risky assets (defined as continuously compounded rates) are normally distributed, which enable us to reach a closed-form solution.

In this paper, we follow the approach of the rational expectations analysis developed by Grossman and Stiglitz (1980) and extend the Easley-O’Hara (2004)

\(^2\) For instance, even for two independent standard normal variables, the ratio of the two variables follows a Cauchy distribution, of which both mean and variance do not exist (Hogg and Craig (1995), p174).
model by adopting lognormal distributions of asset values and informative signals. Based on partial revealing rational expectations equilibrium, we derive a closed-form of the expected rates of return on risky assets. More importantly, we demonstrate that there exists a portfolio—the Information-adjusted Average Market Portfolio (IAMP), which plays a similar role as the market portfolio in the standard CAPM formula. The IAMP is the average market portfolio adjusted for asymmetric information and observable to all traders in the market. Therefore, utilizing the expected rate of return to each individual security, we can compute the expected rate of return on the IAMP and define a security market line for a financial market under asymmetric information. This implies that under asymmetric information, the portfolio determining the risk premium of each individual risky asset or a risky portfolio is the IAMP rather than the market portfolio or average market portfolio. Like the standard CAPM formula, this new formula also proposes a hypothesis for testing the relationship between the expected rate of return on an asset and its risk when the information of asset values is asymmetrically distributed among investors. However, limited by the scope of the paper, we leave such an empirical study for the further research.

Consistent with previous models of rational expectations equilibrium, we find that each investor holds an efficient portfolio from his/her perception but the market portfolio is not mean-variance efficient to any investor. A novel finding from our model is that the IAMP is mean-variance efficient with respect to the average market belief.

In the comparative statics analysis, we find that on average investors demand a lower rate of return if there are more private signals (observable only to some investors) and public signals (observable to all investors) about the future value of a
stock. The risk premium of a stock also becomes lower if more investors can observe private signals and/or more initial wealth is held by investors who observe private signals. But when some public signals are converted to private, the market requires a higher expected rate of return in equilibrium. All these predictions are based on the same simple mechanism: information changes investors’ posterior beliefs and when it reduces the posterior variance of the return rate investors perceive the stock is less risky and demand a lower expected rate of return. We also find that preference-adjust initial wealth (defined as the aggregate initial wealth divided by the preference parameter) has a negative impact on risk premia of all risky assets. Such a wealth effect does not exist in existing rational expectation models since the demand for risky assets is independent of initial wealth within a normal-CARA framework. This wealth effect also results in scale economies of information in our model—large investors have more incentives to collect information.

This paper also has a methodology contribution. Departing from the approach of existing rational expectations models,3 our model assumes lognormal prices (or future values of risky assets) and lognormal signals partially revealing price information as pointed earlier. So, we have developed a framework for rational expectations equilibrium analysis based on lognormal prior and lognormal signals of asset prices. This overcomes the technical difficulty in the analysis of expected rates of return with asymmetric information.

3 In an early paper, Ohlson and Ziemba (1976) assume lognormally distributed prices for competitive securities markets. However, they adopt an approximation that the sum of lognormal variables is still lognormal without an estimation of approximation accuracy. More importantly, asymmetric information is not considered in their model.
The rest of the paper is organized as follows. Section 2 specifies the Bayesian belief updating process based on lognormally distributed variables and signals. Then it completes an asymmetric information model by establishing rational expectations equilibrium. Section 3 calculates the expected rates of return in equilibrium, derives a modified CAPM formula under asymmetric information, and discusses the implications of this security market line. The first part of Section 4 is a comparative statics analysis, providing important properties of the expected rate of return. It is followed by an analysis of investment strategy and the value of information. Concluding remarks are given in the final section. All proofs are given in Appendix B.

2. AN ASYMMETRIC INFORMATION MODEL WITH LOGNORMAL PRICES AND CRRA UTILITY

2.1 Bayesian Inference with Lognormal Prices and Signals

Consider an economy, similar to Easley and O’Hara (2004), that there are \( K \) risky assets and one risk-free asset. Agents in this economy optimize their asset portfolios through trading in an open securities market at date 0 to maximize their expected utility at date 1. Although nobody knows the future values of risky assets, \( v_k \) (\( k = 1, \ldots, K \)), before trading, their prior distributions are common knowledge to all agents in the market. Different from Easley and O’Hara (2004), it is assumed in the model that \( v_k \) are independently, lognormally distributed with mean \( \bar{v}_k \) and variance

\[
\bar{v}_k^2 \left[ \exp(\rho_k^{-1}) - 1 \right], \quad \text{i.e.,}^{4}
\]

\[
v_k \sim LN(\ln \bar{v}_k - (2 \rho_k^{-1}), \rho_k^{-1}), \quad k = 1, \ldots, K. \quad (1)
\]

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\( ^4 \) For a brief review of lognormal distributions, see Appendix A.
In addition to the knowledge of prior distributions, there are $I_k$ signals revealing the future value of risky asset $k$ before trading. These signals, $s_{k1}, s_{k2}, \ldots, s_{kI_k}$, are drawn independently from a lognormal distribution, conditional on a realization of $v_k$, with mean $v_k$ and variance $v_k^2[\exp(\gamma_k^{-1}) - 1]$, i.e.,

$$s_{kj} \mid v_k \sim LN(\ln v_k - (2\gamma_k)^{-1}, \gamma_k^{-1}), \quad j = 1, \ldots, I_k. \tag{2}$$

Some of these signals are public information observed by all investors in the market but others are private information observed only by a portion of investors. The fraction of private signals is denoted as $\alpha_k$ so that the fraction of signals publicly observable is $1 - \alpha_k$. Investors who can observe both private and public signals of asset $k$ are called informed investors, denoted by set $\Omega_k^I$, while investors observe only public signals are called uninformed investors, denoted by set $\Omega_k^U$. To facilitate analysis and simplify notations, we define two statistics of these signals:

$$\ln g_k = \sum_{i=1}^{\alpha_k I_k} \ln s_{ki} / \alpha_k I_k + \{1 - (\alpha_k I_k)^{-1}\}(2\gamma_k)^{-1},$$

$$\ln h_k = \sum_{i=\alpha_k I_k+1}^{I_k} \ln s_{ki} / (1 - \alpha_k) I_k + \{1 - [(1 - \alpha_k) I_k]^{-1}\}(2\gamma_k)^{-1}.$$  

It can be easily verified that $g_k$ and $h_k$ are lognormal, conditional on $v_k$, that

$$g_k \mid v_k \sim LN(\ln v_k - (2\gamma_k \alpha_k I_k)^{-1}, (\gamma_k \alpha_k I_k)^{-1}), \tag{3}$$

$$h_k \mid v_k \sim LN(\ln v_k - [2\gamma_k (1 - \alpha_k) I_k]^{-1}, [\gamma_k (1 - \alpha_k) I_k]^{-1}). \tag{4}$$

Hence both $g_k$ and $h_k$ have a conditional mean of $v_k$. Moreover, it can be shown that $g_k$ and $h_k$ are sufficient statistics for the collections of private signals and public

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5 An investor is informed of asset $k$ can be uninformed of asset $j$ if he/she does not receive the private signals of asset $j$.  

CAPM-Asymmetry4.doc 7
signals, respectively. Thus, instead of considering signals \( s_{k1}, s_{k2}, \ldots, s_{klk} \) individually, we only need to investigate these two sufficient statistics.

In the securities market, investors update their knowledge about the future values of risky assets with the information they have obtained. More specifically, since informed investors receive both private and public information, they can use \( g_k \) and \( h_k \) given by (3) and (4) to update their beliefs on the future values of risky assets. For uninformed investors they only receive public information, \( h_k \). However, the equilibrium prices of risky assets partially reveal the information of private signals. So they use both \( h_k \) and price signals to update their beliefs. The following lemma illustrates how Bayesian investors use the observed signals to update their beliefs of asset values.

**Lemma 1:** If an investor has a prior belief \( \nu \sim \text{LN}(a_0, b_0^{-1}) \) and receives a signal, \( s \mid \nu \sim \text{LN}(\ln \nu + a_1, b_1^{-1}) \), then the investor’s posterior distribution of \( \nu \) is the lognormal distribution given by

\[
\nu \mid s \sim \text{LN}\left(\frac{\nu_0 + b_0 (\ln s - a_1)}{(b_0 + b_1)/(b_0 + b_1)^{-1}}\right).
\]

Thus, applying Lemma 1 twice to informed investors to incorporate information \( g_k \) and \( h_k \), we obtain their posterior beliefs on the future value of asset \( k \):

\[
\nu'_i \mid (g_k, h_k) \sim \text{LN}(\ln \bar{\nu}_k', (\lambda'_k)^{-1}), \quad i \in \Omega_k',
\]

where

\[\footnote{See Lemma A3 in Appendix A for details.}\]

\[\footnote{In the paper, we use the convention that two bars over a variable indicate a conditional expectation while one bar over a variable indicates unconditional expectation.}\]
\[
\ln \bar{v}_k^i = [\rho_k \ln \bar{v}_k + \gamma_k \alpha_k I_k \ln g_k + \gamma_k (1 - \alpha_k) I_k \ln h_k + 1/2] / \lambda_k^i, \quad (7)
\]
\[
\lambda_k^i \equiv \rho_k + \gamma_k I_k. \quad (8)
\]

For uninformed investors, the Bayesian learning process is more complicated. Let \( p_k \) be the equilibrium price of risky asset \( k \) when it is traded in the market at date 0. Assume that the supply of the \( k \)th asset in terms of its total value, \( y_k \), is a normal random variable, independent of other random variables in the model, with mean \( \bar{y}_k > 0 \) and variance \( \eta^{-1}_k \). Suppose that uninformed investors conjecture the following equilibrium price (it will be verified later that this conjecture is rational and self-fulfilling):
\[
\ln p_k = a \ln \bar{v}_k + b \ln g_k + c \ln h_k - d y_k + e \bar{y}_k + f, \quad (9)
\]

where \( a, b, c, d, e \) and \( f \) are constants and will be given in Proposition 1 below. According to (9), \( \ln p_k \) is a linear combination of normal variables, hence it is also a normal random variable. To facilitate analysis, we define a random variable that
\[
\ln \theta_k = [(\ln p_k - a \ln \bar{v}_k - c \ln h_k - \bar{y}_k (d - e) - f) / b = \ln g_k - (y_k - \bar{y}_k)(d / b)]. \quad (10)
\]

Since uninformed investors can compute \( \theta_k \) by price and public information, observing signal \( \theta_k \) is equivalent to observing price signal \( p_k \). It is clear that \( \ln \theta_k \) is normally distributed. In addition, we have

\[\text{It is well-known that there must some noise exist in the market to avoid a perfectly revealing equilibrium (see Grossman (1976) or Grossman and Stiglitz (1980)). Asymmetric information models (e.g., Admati (1985), Easley and O’Hara (2004)) typically assume that the number of per-capita supply of a risky asset is a normal random variable. Since the supply randomness is usually considered as the result of trading by liquidity traders who buy or sell an asset for their liquidity purposes, regardless of its price, the randomness assumption of total supply value seems reasonable if it is not more plausible. The assumption } \bar{y}_i > 0 \text{ implies a positive supply on average.}\]
\[ E(\ln \theta_k | v_k) = E(\ln g_k | v_k) - E[(y_k - \bar{y}_k)(d/b)] = \ln v_k - (2\gamma_k \alpha_k I_k)^{-1}. \]

The conditional precision of \( \ln \theta_k \) is
\[
\xi_k = \left[ \text{var}(\ln \theta_k | v_k) \right]^{-1} = [(d/b)^2 \eta_k^{-1} + (\gamma_k \alpha_k I_k)^{-1}]^{-1}.
\]
Thus, \( \theta_k | v_k \sim LN(\ln v_k - (2\gamma_k \alpha_k I_k)^{-1}, \xi_k^{-1}) \). With public information and signal \( \theta_k \), we can apply Lemma 1 twice to compute the uninformed investors’ posterior beliefs on future values:
\[
v'_k | (\theta_k, h_k) \sim LN(\ln \overline{\pi}_k^U, (\lambda_k^U)^{-1}), \quad i \in \Omega_k^U,
\]
where
\[
\ln \overline{\pi}_k^U \equiv \left[ \rho_k \ln \overline{v}_k + \gamma_k (1 - \alpha_k) I_k \ln h_k + \xi_k \ln \theta_k + \xi_k (2\gamma_k \alpha_k I_k)^{-1} \right] / \lambda_k^U,
\]
\[
\lambda_k^U \equiv \rho_k + \gamma_k (1 - \alpha_k) I_k + \xi_k.
\]

### 2.2 Investors’ Optimal Portfolios

Turning to the investors’ demand for assets, it is assumed that they have an identical utility function with Constant Relative Risk Aversion (CRRA) that
\[
u(w) = w^{1-\delta} / (1-\delta), \quad \delta > 1,
\]
where \( w \) is an investor’s wealth. Let investor \( i \)’s initial wealth be \( m^i \). If he/she allocates a friction, \( x_k^i \), of his/her initial wealth to asset \( k \) at date 0,\(^9\) his/her wealth at date 1 is equal to
\[
w^i = m^i \{ 1 + \sum_{k=1}^{K} x_k^i [\exp(r_k) - 1] \}, \quad \text{for investor } i,
\]
where \( r_k \equiv \ln(v_k / p_k) \) is defined as the rate of continuously compounded return of asset \( k \). Substituting it into (14) yields

\(^9\) The risk-free asset is indexed by subscript \( k = 0 \).
\[ u(w') = (1 - \delta)^{-1} (m')^{-\delta} \exp \left[ (1 - \delta) \ln \left( 1 + \sum_{k=1}^{K} x_k' \exp(r_k) - 1 \right) \right]. \]  

(16)

Since \( p_k \) is close to \( v_k \), the rate of return, \( r_k \), is a small number. Hence, using the Taylor’s theorem to expand the above exponent around zero, we have:

\[
\ln \left[ 1 + \sum_{k=1}^{K} x_k' \exp(r_k) - 1 \right] = \sum_{k=1}^{K} x_k' r_k + \mathbf{r}^T \mathbf{H}(\omega \mathbf{r}) \mathbf{r} / 2 ,
\]

where \( \mathbf{r}^T \mathbf{H}(\omega \mathbf{r}) \mathbf{r} / 2 \) is the remainder, \( 0 < \omega < 1 \) is a constant, \( \mathbf{H} \) is the associated Hessian matrix, and \( \mathbf{r} \equiv (r_1, ..., r_K)^T \).

**Lemma 2:** For a portfolio with \( K_0 \) non-zero weights \( (x_k' \neq 0, k = 1, ..., K) \), the error term of the above Taylor’s expansion has a bound:

\[
\left| \mathbf{r}^T \mathbf{H}(\omega \mathbf{r}) \mathbf{r} / 2 \right| \leq \| \mathbf{r} \|_2^2 / (8K_0) ,
\]

where the matrix 2-norm \( \| M \|_2 \) for a real matrix \( \mathbf{M} \) is defined as the square root of the maximum eigenvalue of \( \mathbf{M}^T \mathbf{M} \).

Lemma 2 ensures us that the remainder of the Taylor’ expansion is a second-order small amount so that we can ignore it in order to obtain a closed-form solution in the analysis. In other words, the utility function (16) can be approximated by

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10 In the text, we assume the risk-free rate is equal to zero to simplify notations. If it is positive, i.e., \( r_0 > 0 \), then \( w' = m' \left\{ \exp(r_0) + \sum_{k=1}^{K} x_k' [\exp(r_k) - \exp(r_0)] \right\} \). Accordingly, the Taylor’s expansion around \( r_0 \) is \( \ln \left\{ \exp(r_0) + \sum_{k=1}^{K} x_k' [\exp(r_k) - \exp(r_0)] \right\} \approx \exp(-r_0) \sum_{k=1}^{K} x_k' (r_k - r_0) \). Since \( \exp(-r_0) \) is a given constant and does not affect investors’ decisions, the analysis for the case of \( r_0 > 0 \) is the same after replacing \( r_k \) by \( r_k - r_0 \).
\[ u(w') = (1 - \delta)^{-1}(m')^{-\delta} \exp\left[ (1 - \delta) \sum_{k=1}^{K} x'_k \ln(v_k / p_k) \right]. \]

All investors maximize the expected utility to determine their trading strategies. The difference between their investment strategies stems from their difference in posterior expected utility since they have different information sets. The conditional expected utility can be written as:

\[ E[u(w') | I'] = (1 - \delta)^{-1}(m')^{-\delta} \prod_{k=1}^{K} \exp\left[ (1 - \delta)x'_k \ln(\bar{v}_k / p_k) + (x'_k)^2 (1 - \delta)^2 / 2 \lambda'_k \right], \quad (17) \]

where \( I' \) is the information set of investor \( i \) and \( \ln(\bar{v}_k / p_k) \) equals \( \ln(\bar{v}_k / p_k) \) or \( \ln(\bar{v}_k / p_k) \) depending on whether investor \( i \) is informed of asset \( k \) or not. From (17), maximizing the expected utility is equivalent to solve the following problem,

\[ \max_{x'_1, \ldots, x'_K} \sum_{k=1}^{K} [x'_k \ln(\bar{v}_k / p_k) - (x'_k)^2 (\delta - 1)/2 \lambda'_k]. \quad (18) \]

The assumption that \( \delta > 1 \) ensures that the first order condition is not only necessary but also sufficient. Therefore, investor’s optimal portfolio is given by

\[ x'_k = \lambda'_k \ln(\bar{v}_k / p_k) / (\delta - 1), \quad k = 1, \ldots, K, \quad x'_0 = 1 - \sum_{k=1}^{K} x'_k. \quad (19) \]

This optimal portfolio is well behaved and has the properties as expected. Since \( \ln(\bar{v}_k) \) and \( \lambda'_k \) are the investor \( i \)’s posterior expected value and precision of \( \ln v_k \), respectively, (19) shows that an investor invests proportionately more in an asset as the expectation of the logarithm of its future value increases and/or the uncertainty becomes smaller. The investment also increases when the expected rate of return (conditional on the investor’s information), \( \ln(\bar{v}_k / p_k) \), is higher. However, when the price of the asset rises, the share of asset \( k \) in the investment portfolio declines. Also, as \( \delta \) increases, that is the investor is more risk-averse, the investor reduces his/her
investment weights across all risky assets and increases his/her holding of the risk-free asset. As indicated earlier, investors have different portfolios and investment strategies due to their difference in posterior beliefs. More specifically, informed investors believe that $\lambda_i$ and $\ln(\bar{v}_i)$ in (19) are determined by (7) and (8), respectively, while uninformed investors believe that they are determined by (12) and (13), respectively.

2.3 Rational Expectations Equilibrium

We consider rational expectations equilibrium in this model so that each investor's expectation is self-fulfilling in equilibrium. The market clearing condition requires the aggregate demand for each asset equal its aggregate supply, i.e.,

$$\left[ \sum_{i=1}^{K} x_i^m + \sum_{j=K+1}^{N} x_j^m \right] = y_k, \quad k = 1, \ldots, K. \quad (20)$$

Recalling (7), (8), (12) and (13), we can obtain the equilibrium prices after inserting optimal portfolios in (19) into (20). This is summarized in Proposition 1 below.

**PROPOSITION 1:** There exists a partially revealing rational expectations equilibrium in which,

$$\ln p_k = a \ln \bar{v}_k + b \ln g_k + c \ln h_k - dy_k + e \bar{y}_k + f, \quad k = 1, \ldots, K \quad (21a)$$

or

$$p_k = (\bar{v}_k)^a (g_k)^b (h_k)^c \exp[-dy_k + e \bar{y}_k + f], \quad (21b)$$

where

$$a = \frac{\rho_k}{C_k}, \quad b = \frac{\mu_k \gamma_k I_k + (1 - \mu_k) \xi_k}{C_k},$$

$$c = \gamma_k (1 - \alpha_k) I_k / C_k, \quad d = (\delta - 1) \frac{1 + (1 - \mu_k) \xi_k / \mu_k \gamma_k \alpha_k I_k}{mC_k},$$

$$e = (\delta - 1)(1 - \mu_k) \xi_k / m \mu_k \gamma_k \alpha_k I_k C_k, \quad f = \frac{\mu_k + (1 - \mu_k) \xi_k / \gamma_k \alpha_k I_k}{2C_k},$$
\[ C_k = \rho_k + \gamma_k (1 - \alpha_k)I_k + \mu_k\gamma_k \alpha_k I_k + (1 - \mu_k)\xi_k, \]

\[ \xi_k = [(\delta - 1)^2 (m\mu_k\gamma_k \alpha_k I_k)^{-2} \eta_k^{-1} + (\gamma_k \alpha_k I_k)^{-1}]^{-1}. \]

In the above proposition, we have defined \( \mu_k \) as the fraction of initial wealth possessed by informed traders who receive the private signals of stock \( k \); that is

\[ \mu_k = \sum_{i \in I_k} m_i / m, \text{ and } m \text{ as the aggregate initial wealth in the market; that is,} \]

\[ m = \sum_i m_i. \]

3. THE EXPECTED RATES OF RETURN AND CAPM FORMULA UNDER ASYMMETRIC INFORMATION

3.1 The Expected Rates of Return

Like other rational expectations models, the equilibrium of our modeled market depends on a particular realization of random signals and asset supplies. Conditional on his/her information, investor \( i \)'s expected rate of return to holding asset \( k \) is

\[ \overline{\eta}_i^k = E(r_k | g_k, h_k, p_k) = \ln(\overline{\eta}_i^k / p_k) \quad \text{if } i \in \Omega_i^k \]

\[ \overline{\eta}_u^k = E(r_k | h_k, p_k) = \ln(\overline{\eta}_u^k / p_k) \quad \text{if } i \in \Omega_u^k. \] (22)

The corresponding precisions are \( \lambda^i_k \) and \( \lambda^U_k \), respectively, for informed and uninformed investors. As (22) illustrated, an investor with the private information of asset \( k \) believes that the rate of return follows distribution \( N(\overline{\eta}_i^k, (\lambda^i_k)^{-1}) \) while an uninformed investor believes the distribution is \( N(\overline{\eta}_u^k, (\lambda^U_k)^{-1}) \). Now consider an investor who is informed about \( k^* \) risky assets but is uninformed about the remaining
$K - k^*$ assets, where $0 \leq k^* \leq K$.\textsuperscript{11} Thus, if $k^* = K$, he/she is a completely informed investor since he/she receives private signals of all stocks and if $k^* = 0$ he/she is completely uninformed of all risky assets. Without loss of generality, let us assume that the investor is informed about the first $k^*$ assets. Based on his/her posterior, the investor perceives a portfolio frontier $F(k^*)$, determined by

$$r_k | (g_k, h_k, p_k) \sim N(\bar{r}_k^I, (\lambda_k^I)^{-1}) \ (k = 1, 2, ..., k^*),$$

$$r_k | (h_k, p_k) \sim N(\bar{r}_k^U, (\lambda_k^U)^{-1}) \ (k = k^* + 1, ..., K).$$

Figure 1 illustrates two particular portfolio frontiers, where $F(K)$ is perceived by a completely informed investor and $F(0)$ is perceived by a completely uninformed investor. Since the former has more information and a greater precision of estimation on every asset than the latter, $F(K)$ must be on the left of $F(0)$. Other portfolio frontiers (there are at most $\sum_{k=1}^{K-1} K!/[k!(K-k)!]$ such frontiers) position between these two frontiers. Remember that before trading and receiving signals, all investors have the same belief, so that they have the common prior portfolio frontier $FP$, which positions on the right of $F(0)$ in Figure 1 because investors have a better estimation on the rates of return after they observe the public signals and the equilibrium prices.

Figure 1 is about here

For an investment portfolio $\{x_0, x_1, ..., x_K\}$ we have analyzed, the share of risky investment is $\sum_{k=1}^{K} x_k$ and the share of the risk-free asset in the portfolio is

\textsuperscript{11} Slightly abusing terminology, we will also call this agent investor $k^*$. But there are up to $K!/ [k^*!(K - k^*)!]$ different types of investors, who can be called investor $k^*$, depending on which particular $k^*$ stocks they are informed of.
1 − \sum_{k=1}^{K} x_k. In the analysis below, we will call \{t'_k\} investor i’s portfolio on risky asset, where \( t'_k \equiv x'_k / \sum_{j=1}^{K} x'_j \) for \( k = 1, 2, \ldots, K \). According to (19), investor \( k^* \) in equilibrium chooses risk portfolio \( \{t^{k*}_k\} \) with weights given by

\[
t^{k*}_k = \begin{cases} 
\sigma_k \bar{r}^I_k / \sum_{k=1}^{K} \sigma_k \bar{r}^I_k & 1 \leq k \leq k^* \\
\sigma_k \bar{r}^U_k / \sum_{k=k^*+1}^{K} \sigma_k \bar{r}^U_k & k^* < k \leq K 
\end{cases}
\]

(23)

where \( \sigma_k^* \equiv \left( \sum_{k=1}^{k^*} \sigma_k^* \bar{r}^I_k + \sum_{k=k^*+1}^{K} \sigma_k \bar{r}^U_k \right)^{-1} \) is the normalization coefficient.

Apparently, \( \{t^{k*}_k\} \) is the tangency portfolio with respect to the portfolio frontier of investor \( k^* \) (see Ingersoll (1987), ch. 4). We thus have the following proposition.

**PROPOSITION 2**: The optimal portfolio determined by (19) is mean-variance efficient, conditional on the investor’s information. Moreover, the investor conditionally expects the risk premium of each asset satisfying the following relationship with the conditional expected return of his/her risk investment portfolio \( \{t^{k*}_k\} \) that

\[
\bar{r}^I_k = \sigma_{kk} \bar{r}^I_k / \sigma_{kk^*} \quad \text{for } 1 \leq k \leq k^* \quad (24a)
\]

\[
\bar{r}^U_k = \sigma_{kk^*} \bar{r}^I_k / \sigma_{kk^*} \quad \text{for } k^* + 1 \leq k \leq K \quad (24b)
\]

---

12 This shows the approximation we made for the utility function does not affect the mean-variance efficiency of optimal investment decisions.

13 The risk-free rate is assumed to be zero and the expected rate represents the risk premium in this paper. If the risk-free rate is positive, \( \bar{r}^I_k \), \( \bar{r}^U_k \) and \( \bar{r}^* \) in (24) should be replaced by \( \bar{r}^I_k - r_o \), \( \bar{r}^U_k - r_o \) and \( \bar{r}^* - r_o \) respectively. A similar operation is also applicable to the modified CAPM formula in Proposition 4 and equation (31) in Proposition 4P.
where \( \bar{r}^{k*} \equiv \sum_{k=1}^{K} t_k^r \bar{r}^I_k + \sum_{k=1}^{K} t_k^U \bar{r}^U_k \) is the expected rate of return on \( \{t_k^r\} \), \( \sigma_{k*} \) is the covariance between stock \( k \) and \( \{t_k^r\} \), and \( \sigma_{k*} \) is the variance of \( \{t_k^r\} \), based on the investor’s posterior belief.

Proposition 2 shows that investors optimally allocate their funds between the risk-free asset and their tangency portfolios—that is the same as predicted by the standard CAPM. However, the asymmetric information induces investors to have different posterior beliefs and consequently different tangency portfolios. This is the key difference from the standard CAPM where all investors and, in turn, the market have the same information and tangency portfolio. More importantly, the standard CAPM formula specifies the relationship between the unconditional (or average over time) expected risk premium of a particular asset and the unconditional expected risk premium of the market portfolio, from the market perception. So, to modify the standard CAPM formula, we first have to answer the question: what is the market belief, given a realization of all random parameters in the model?\(^{14}\) Second, we need to compute the unconditional expected rate of return on all individual risk assets, based on the market belief. This unconditional expected rate is common to all investors and is also observable to outsiders, particularly, observable to econometricians for their empirical tests of the modified CAPM formula under asymmetric information. Third, we have to identify a portfolio, which captures the supply and information characteristics of the market so that its unconditional expected rate of return can act as an index to gauge the unconditional expected rate of return.

\(^{14}\) Lintner (1969), and Easley and O’Hara (2004) suggest a weighted average of individual beliefs as the market belief. Our treatment, however, is more thorough although the results are quite similar.
return to each individual asset, as the market portfolio does in the standard CAPM formula.

To find out the market belief, we should note that informed investors have posterior belief \( N(\bar{r}_k^I, (\lambda_k^I)^{-1}) \) and possess an amount of initial wealth \( \mu_k m \) while the uninformed have posterior \( N(\bar{r}_k^U, (\lambda_k^U)^{-1}) \) and possess \((1 - \mu_k) m\). At the aggregate level, the market does not perceive that the rate of return to holding asset \( k \) is normally distributed since there are two heterogeneous sources of variation, i.e., two distinct beliefs, informed and uninformed. Rather, the distribution is a mixture of two normal distributions,

\[
r_k^M \sim q_k N(\bar{r}_k^I, (\lambda_k^I)^{-1}) + (1 - q_k) N(\bar{r}_k^U, (\lambda_k^U)^{-1}), \quad (k = 1, 2, \ldots, K),
\]

where \( q_k = \mu_k \lambda_k^I / [\mu_k \lambda_k^I + (1 - \mu_k) \lambda_k^U] \) depends on the distribution of wealth and the relative magnitude of the informed investors’ posterior precision to the uninformed investors’ posterior precision.\(^{15}\) For given \( \bar{r}_k^I \) and \( \bar{r}_k^U \), it is not hard to find that the mean and precision of \( r_k^M \) are equal to\(^{16}\)

\[
\bar{r}_k^M = E(r_k^M | \bar{r}_k^I, \bar{r}_k^U) = [\lambda_k^I \mu_k \bar{r}_k^I + \lambda_k^U (1 - \mu_k) \bar{r}_k^U] / \lambda_k^M,
\]

\[
\lambda_k^M = [\text{var}(r_k^M | \bar{r}_k^I, \bar{r}_k^U)]^{-1} = \lambda_k^I \mu_k + \lambda_k^U (1 - \mu_k).
\]

Hence, the market perceives a market portfolio frontier \( F(M) \), falling between \( F(0) \) and \( F(K) \) as illustrated in Figure 1.

\(^{15}\) Finite mixture distributions are widely used in statistics when there exist more than one heterogeneous sources of variation; see Titterington et al. (1985), McLachlan and Peel (2000) for overviews.

\(^{16}\) We use the notations \( E(r_k^M | \bar{r}_k^I, \bar{r}_k^U) \) and \( \text{var}(r_k^M | \bar{r}_k^I, \bar{r}_k^U) \) to emphasize the mean and variance rely on the values of \( \bar{r}_k^I \) and \( \bar{r}_k^U \), and in turn rely on a particular realization of random signals.
To calculate market’s unconditional expectation on the rate of return, we decompose the task into two steps. First, we indicate in Lemma 3 below that the rate of return \( r^M_k \) in (25), perceived by the market, has the same unconditional mean and variance as the actual rate of return, \( r_k \). This means that in long-term, the market has a correct estimate of the actual rate of return. Second, we give the closed form of the mean and variance in Proposition 3.

**Lemma 3:** The unconditional mean and variance of the rate of return \( r^M_k \), defined by (25), are equal to, respectively, the unconditional mean and variance of the actual rate of return, \( r_k \); that is, \( E(r^M_k) = E(r_k) \equiv \bar{r}_k \), \( \text{var}(r^M_k) = \text{var}(r_k) \equiv \lambda^{-1}_k \), where the expectation and variance are taken over the full space for all random variables.

It is worth noticing that \( \lambda^{-1}_k = (\lambda_k^M)^{-1} + [q_k \text{var}(\bar{r}_k^I) + (1 - q_k) \text{var}(\bar{r}_k^U)] \), as shown in the proof of the lemma. This means that the total variance of the rate of return \( \var(r^M_k) \) is determined by two factors. The first factor, \( (\lambda_k^M)^{-1} \), is the market’s conditional belief based on the collected signals while the second factor, 
\[ [q_k \text{var}(\bar{r}_k^I) + (1 - q_k) \text{var}(\bar{r}_k^U)] \], represents the average quality of the estimated rates of return by informed and uninformed investors. The equation implies that 
\[ 0 \leq \lambda_k / \lambda_k^M \leq 1 \], which is a useful condition for our analysis information asymmetry below.

**Proposition 3:** The unconditional expected value and variance of the rate of return on asset \( k \), \( r_k \) (\( k = 1, \ldots, K \)), are given by
\[
\bar{r}_k = E[\ln(v_k / p_k)] = \frac{\bar{y}_k (\delta - 1)}{m[\rho_k + \gamma_k (1-\alpha_k)I_k + \mu_k \alpha_k I_k + (1 - \mu_k)\bar{y}_k]}, \tag{28}
\]

\[
\text{var}(r_k) \equiv \text{var}[\ln(v_k / p_k)] = a^2 / \rho_k + b^2 (\gamma_k \alpha_k I_k)^{-1} + c^2 [\gamma_k (1 - \alpha_k)I_k]^{-1} + d^2 \eta_k^{-1}. \tag{29}
\]

3.2 The CAPM Formula under Asymmetric Information

We are now ready to extend the standard CAPM to accommodate asymmetric information. Obviously, the average market portfolio is \(\{\bar{r}_k^M\}\), where \(\bar{r}_k^M = \bar{z}^M \bar{y}_k\) and \(\bar{z}^M = (\sum_{j=1}^{K} \bar{y}_j)^{-1}\) is the normalizing coefficient. Then the Information-adjusted Average Market Portfolio (IAMP), \(\{\bar{r}_k^{R}\}\), can be defined as the average market portfolio \(\{\bar{r}_k^M\}\) adjusted for information weight, \(\bar{z}_k / \lambda_k^M\); that is, \(\bar{r}_k^{R} \equiv \bar{z}^R \bar{y}_k \lambda_k / \lambda_k^M\), where \(\bar{z}^R \equiv \left(\sum_{j=1}^{K} \bar{y}_j \lambda_j / \lambda_j^M\right)^{-1}\) is the normalizing coefficient.\(^{17}\) Apparently, the information adjustment weight, \(\bar{z}_k / \lambda_k^M\), is the proportion of the variation of the rate of return, from market perception, which cannot be explained by the collected information while \(1 - \bar{z}_k / \lambda_k^M\) is the proportion of the variation which is explained by the collected information. The latter is equivalent to the coefficient of determination in regression analysis, so we call \(\bar{z}_k / \lambda_k^M\) the coefficient of indetermination, a quantity indicating the uncertainty and relative risk after deducting the effect of the collected information. Thus, the definition of IAMP captures the aggregate characteristics of the market: the average market portfolio and the excess relative risk of all assets after taking into account of the received signals about future values of assets. Furthermore,

\(^{17}\) As will be seen below information adjustment is actually risk adjustment so we use superscript/subscript \(R\) (for risk) to identify the IAMP.
the IAMP plays the similar role as the market portfolio in the standard CAPM formula and the CAPM formula has to be amended as the following.

**PROPOSITION 4**: Under asymmetric information, the following CAPM formula holds:

\[
\bar{r}_k = \sigma_{kr} \bar{r}^R / \sigma_{RR} = \beta_k \bar{r}^R , \quad k = 1, 2, \ldots, K ,
\]

where, \( \bar{r}^R = \sum_{k=1}^{K} \frac{\bar{r}_k}{\bar{r}_k} \) is the market’s unconditional expected rate of return on the IAMP, \( \sigma_{kr} = \bar{r}_k / \lambda_k \) is the covariance between asset \( k \) and the IAMP, \( \sigma_{RR} = \sum_{k=1}^{K} \frac{(\bar{r}_k)^2}{\lambda_k} \) is the variance of the IAMP, and \( \beta_k = \sigma_{kr} / \sigma_{RR} . \) Moreover, the IAMP is mean-variance efficient with respect to mean \( \bar{r}_k \) and precision \( \lambda_k . \)

Proposition 4 postulates a security market line, specifying the relationship between the expected rate of return to a risky security and its risk under asymmetric information. However, different from the standard security market line, the benchmark here is the IAMP rather than the market portfolio—the slope of the security market line is equal to the risk premium of the IAMP and the beta should be calculated with respect to the IAMP. This is not surprising. In the standard CAPM, investors are symmetric and hold the exactly same portfolio in equilibrium, and consequently the rate of return on the market portfolio is the index for the performance of each individual asset. When information is asymmetric, investors choose their portfolios according to (19) to maximize their utility and these portfolios deviate from the market portfolio. In turn, the portfolio determining the expected rate of return, averaged over time, is also deviates from the average market portfolio. The deviation is captured by the coefficient of indetermination, \( \xi / \lambda^M_k . \) Since \( \xi / \lambda^M_k \) depends on the distributions of private and public signals across all investors, the
IAMP and betas of all securities also depend on the information distribution. It is obvious that when \( \frac{\bar{\lambda}_k}{\lambda^M_k} \) is constant across all assets; that is, all assets have the identical excess relative risk after deducting the effect of the collected information, the IAMP collapses to the average market portfolio and the CAPM formula in (30) reduces to the standard CAPM, if the average market portfolio is understood as the conventional “market portfolio”.

The importance of IAMP can also be seen from its mean-variance efficiency with respect to the belief \((\bar{\kappa}_k, \lambda_k)\), as indicated in Proposition 4. Easley and O’Hara (2004) shows that the average market portfolio is mean-variance efficient with respect to mean \( \bar{\kappa}_k \) and precision \( \lambda^M_k \) although the return in their model is defined as return per share. It is not hard to demonstrate that the average market portfolio in our model has the similar property. However, the mean \( \bar{\kappa}_k \) and precision \( \lambda^M_k \) do not constitute a well-defined belief because it is the pair \((\bar{\kappa}_k, \lambda_k)\) rather than the pair \((\bar{\kappa}_k, \lambda^M_k)\) that forms the mean and precision of random variable \( r_k \) or \( r^M_k \).

3.3 Implications to Empirical Testing

Proposition 4 demonstrates the danger of blindly applying the classic CAPM formula when investors have diverse beliefs on risky assets. To understand this, we first state the following proposition in parallel with Proposition 4.
PROPOSITION 4P. With conditional market belief (26)-(27) there is a relationship between the conditional expected risk premium of asset $k$ and the conditional risk premium of the market portfolio,\(^{18}\)

$$\bar{r}_k^M = \frac{\sigma_{km}\bar{r}_k^M}{\sigma_{MM}}, \quad k = 1, 2, \ldots, K, \quad (31)$$

where $\bar{r}_k^M$ is the expected rate of return on the market portfolio conditional on market belief (26)-(27), $\sigma_{km} = \frac{t_k^M}{\lambda_k^M}$ is the covariance between asset $k$ and the market portfolio, and $\sigma_{MM} = \sum_{k=1}^{K} \frac{(t_k^M)^2}{\lambda_k^M}$ is the variance of the market portfolio.

An important feature of (31) is that it holds at each of individual time points and relies on a particular realization of random supply and signals. However, any relationship holding only at individual time points like (31) is not testable via regression analysis because regression analysis is based on averaging models. For a single-period CAPM to be testable it has to be a model that is about the expectation of rate of return over time (see, e.g., Campbell et al, 1997). If we take expectation operation over (31), we obtain

$$\bar{E}\bar{r}_k^M = E\left(\frac{\sigma_{km}\bar{r}_k^M}{\sigma_{MM}}\right).$$

An important fact is that both $\sigma_{km}$ and $\sigma_{MM}$ in the equation are random variables, \textit{ex ante}, since they depend on the market portfolio, $t_k^M \equiv y_k / Y \ (k = 1, \ldots, K)$, which is a collection of random variables. Thus, $\sigma_{km}$ and $\sigma_{MM}$ are correlated and they are also correlated with $\bar{r}_k^M$. So, $E\left(\frac{\sigma_{km}\bar{r}_k^M}{\sigma_{MM}}\right) \neq E(\sigma_{km})\bar{r}_k^M / E(\sigma_{MM})$, which implies that the equation below

\(^{18}\)The unobservable market portfolio is defined as $\{t_k^M\}$, where $t_k^M = y_k / Y$ for $k = 1, 2, \ldots, K$ and

\[ Y = \sum_{k=1}^{K} y_k \]
\[
\bar{r}_k = \frac{\sigma_{km}}{\sigma_{MM}} \bar{r}_k
\]

with \( \sigma_{km} = \frac{\bar{r}_k}{\lambda_k} \), and  
\( \sigma_{MM} = \sum_{k=1}^{K} \frac{(\bar{r}_k^M)^2}{\lambda_k^M} \), is incorrect in general. Equation (32) is valid only when \( \frac{\bar{r}_k}{\lambda_k^M} \) is constant across all assets and in turn the IAMP reduces to the average market portfolio.

An implication of this analysis in practice is that when econometricians wish to test the CAPM formula under the circumstance of asymmetric information by averaging historical return data, they should actually test (30), which has adjusted for asymmetric information, rather than test (31) or (32). This may partially explain why the classical CAPM cannot be successfully tested by econometricians since in practice information is typically asymmetric. An implication of this analysis in practice is that when econometricians wish to test the CAPM formula under the circumstance of asymmetric information by averaging historical return data, they should actually test (30), which has adjusted for asymmetric information, rather than test (31) or (32). This may partially explain why the classical CAPM cannot be successfully tested by econometricians since in practice information is typically asymmetric. In other words, the value of information should be reflected in regression models, as Easley, et al. (2002), and Biais et al. (2003) do. These empirical studies have demonstrated that information asymmetry does induce a greater (unconditional) expected risk premium.

In fact, Proposition 4 suggests some alternative empirical tests to the tests conducted by Easley et al. (2002) and Biais et al. (2003) to examine the effects of asymmetric information on asset prices and returns. With a properly constructed IAMP, econometricians can directly run regressions of the expected rate of return with respect to its relative risk. In this sense, our model has a straightforward empirical implication on the capital asset pricing.

4. FURTHER DISCUSSION ON THE EXPECTED RATE OF RETURN, INVESTMENT STRATEGY AND THE VALUE OF INFORMATION

19 By saying so, we do not intend to undermine the existing findings about the CAPM’s empirical failure. In contrast, we try to provide an additional reason for the classic CAPM’s disappointing performance in empirical tests.
4.1 Some Properties of the Unconditional Expected Rate of Return

In our model, parameters $\alpha_k$ ($k = 1, 2, \ldots, K$) are most important because they measure the extent of information asymmetry among investors. When $\alpha_k = 0$, all information about the asset $k$ is public and there is no information asymmetry but when $\alpha_k = 1$, all signals of asset $k$ are private and the information asymmetry is at its maximum. An increase in $\alpha_k$ implies that some public signals change to private signals, keeping the total number of signals unchanged. Therefore, uninformed investors have less information and face greater uncertainty but informed investors’ information sets remain unchanged. To compensate the greater risk the uninformed investors bear, the market has to pay them with a higher risk premium. The proposition below summarizes this result together with other results of comparative statics analysis.

PROPOSITION 5: The unconditional expected rates of return have the following properties that

$$\frac{\partial \bar{r}_k}{\partial \alpha_k} > 0, \quad \frac{\partial \bar{r}_k}{\partial \delta} > 0, \quad \frac{\partial \bar{r}_k}{\partial \mu_k} < 0,$$

$$\frac{\partial \bar{r}_k}{\partial \gamma_k} > 0, \quad \frac{\partial \bar{r}_k}{\partial I_k} < 0,$$

$$\frac{\partial \bar{r}_k}{\partial \rho_k} < 0, \quad \frac{\partial \bar{r}_k}{\partial \gamma_k} < 0$$

for all $k = 1, \ldots, K$.

In addition to the effect of $\alpha_k$, the proposition indicates that the risk premium of all assets increases in the parameter of investors’ risk preferences, $\delta$. Investors are less risk-averse if this parameter is smaller. When it tends to unity, investors are virtually risk neutral since the optimization problem in (18) becomes linear and consequently no agents in the market require a risk premium. The aggregate initial
wealth has a negative effect on the expected rates of return. When the market is injected with more funds, investors are more buoyant and buy more risk assets accordingly since an investor’s investment portfolio is independent of his/her initial wealth (see (19)). The greater demand in turn pushes the rates of return fall. This prediction has an obvious empirical implication. The higher expected rates of return in a particular market drag funds from other markets towards it. Upon the arrivals of extra funds, the rates of return in this market gradually decline and converge to the average level of other markets. It is worth to notice that the effects caused by \( \delta \) and \( m \) are market-wide but not asset specific. In fact, \( m/(\delta - 1) \) acting as one factor affects the equilibrium and we can consider it as preference-adjusted aggregate initial wealth. The effect of the stock supply is asset specific because it is assumed that all random variables are independent from each other in the model. Actually, (28) shows that the expected rate of return, \( \bar{r}_k \), is proportionate to the mean of stock supply, \( \bar{y}_k \).

However, different from other rational expectations models, the supply here is defined as the total supply value of an asset rather than the number of shares per-capita. The greater is the supply the higher is the expected rate of return. If the mean of supply is equal to zero, then there is no trading for risk sharing purpose and the risk premium is also zero.

The remaining effects in Proposition 5 are related to information. Among them \( \partial \bar{r}_k / \partial \alpha_k > 0 \), \( \partial \bar{r}_k / \partial \mu_k < 0 \), \( \partial \bar{r}_k / \partial I_k < 0 \) can be interpreted similarly to their counterparts in Easley and O’Hara (2004) although we analyze the rates of return while they analyze returns per share. The intuition for the effect of \( \alpha_k \) has been...

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20 This effect does not exist in other rational expectations models with a normal-CARA framework since normal distributions accompanied with constant absolute risk aversion utility generate a demand system independent of initial wealth.
outlined in the beginning of this subsection and we discuss the effects of changes in other two parameters. The definition of $\mu_k$ is the ratio of total initial wealth possessed by investors who are informed of stock $k$ to the aggregate initial wealth. So, a rise in $\mu_k$ can be the result of the following two causes or their combination: (a) the fraction of informed investors in the market becomes larger, keeping each investor’s initial wealth unchanged; (b) each informed investor’s initial wealth relative to uninformed investors become greater. Since a stock is less risky to informed investors than to uninformed investors, an informed investor holds a greater proportion of the stock than an uninformed does. Thus, on average, the demand for the stock increases as more investors are informed and/or informed investors have more funds. In turn, the risk premium declines. Moreover, when more funds are “informed” of the stock, the information revealed by the equilibrium price is more accurate so that the “uninformed” funds also require a lower risk premium. The explanation of $\partial R_k / \partial I_k < 0$ is straightforward: When more information is available, either private or public, the asset is less risky so the risky premium falls.

Turning to $\partial R_k / \partial \rho_k < 0$ and $\partial R_k / \partial \gamma_k < 0$, they reflect the effects of prior belief and signals on the expected rate of return, respectively. However, there are substantial differences between a normal model and a lognormal model in these effects. The standard deviation of a normal distribution is independent of its mean. The risk of a stock is usually measured by the standard deviation of its price in a normal model. But such a measurement is not sensible when two stocks are compared. Consider two stock prices which have the same standard deviation but one has a higher mean than the other. Investors definitely consider the stock with a higher price mean is less risky although their risk measures, measured by the standard deviation of price, are indifferent. So, a better measurement of stock risk for a normal
model is the coefficient of variation—the ratio of the standard deviation of a random variable to its mean. For a lognormally distributed variable, it is the coefficient of variation rather than the standard deviation that is independent of the mean. So lognormal models facilitate risk comparison across stocks. In the model, \( \rho_k \) and \( \gamma_k \) are not precisions of prior and signals. Instead, they are decreasing functions of the coefficients of variation for the prior price distribution and signal distributions respectively.\(^{21}\) Hence, investors’ risk, measured by the coefficients of variation, decreases when \( \rho_k \) and \( \gamma_k \) increase and so does the equilibrium risk premium.

4.2 Further Results on Investment Strategy

Propositions 2 and 4P demonstrate that an investor’s risk portfolio \( \{t^*_k\} \) is the tangency portfolio of frontier \( F(k^*) \) while the market portfolio is the tangency portfolio of frontier \( F(M) \). Since \( F(k^*) \) generally differs from \( F(M) \) when information is asymmetric, \( \{t^*_k\} \) and the market portfolio do not overlap.

**Corollary 1:** The market portfolio differs from an investor’s tangency portfolio and is not mean-variance efficient to the investor, conditional on his/her posterior.

An important prediction of the standard CAPM is that investors should allocate their funds between the market portfolio and the risk-free asset when investors have a mean-variance utility function or the rates of return are normally distributed.

\(^{21}\) More accurately, the coefficient of variation of the prior price distribution is equal to 
\[\exp(\rho_k^{-1}) - 1\]^{1/2}. The same is true to the signal distributions.
distributed. However, under asymmetric information, this is no longer the case. More precisely, we propose the proposition below.

**PROPOSITION 6**: Allocating funds between the market portfolio (or the average market portfolio or the IAMP) and the riskless asset would not maximize investors’ conditional or unconditional expected utility when information is asymmetric.

The argument for the proposition goes as follows.\(^{22}\) Corollary 1 shows that when information is asymmetric, the market portfolio differs from the tangency portfolio of investor \(k^*\). Because the tangency portfolio is generated by maximizing expected utility, conditional on the investor’s posterior, allocating wealth between the market portfolio and the risk-free asset definitely brings the investor lower conditional expected utility. Furthermore, on average (taking expectation over all random variables), the unconditional expected utility is also smaller. This argument is apparently applicable to the average market portfolio and the IAMP.

Proposition 6 clearly indicates that the standard CAPM is invalid as a principle in investment decision making under asymmetric information. Even if he/she is completely uninformed, an investor cannot maximize his/her utility by allocating his/her funds between the market portfolio and the risk-free asset. The reason behind this is that an investor’s conditional expected utility depends on his/she posterior. Suppose that an investor believed that the conditional expected rates of return and precisions were those specified by (26) and (27), then he/she would agree that the market portfolio has a conditional mean return \(\bar{r}_M\) and precision

\(^{22}\) A more rigorous proof is straightforward and therefore is omitted.
\[ \hat{\lambda}^M = \left[ \sum_{k=1}^{K} \left( \lambda_k^M \right)^2 / \lambda_k^M \right]^{-1}. \]

In this case, allocating wealth between the market portfolio and the risk-free would maximize his/her conditional expected utility. Unfortunately, there is no such an investor since the market belief (26) and (27) is a weighted average of the informed and uninformed investors’ beliefs so that according to Proposition 6 the market portfolio is never efficient to any investors.

Since an investor’s expected utility is belief dependent, it is natural to ask: If an investor ignores his/her information and blindly takes the average market belief, \( \bar{r}_k \) and \( \lambda_k \) \((k = 1, 2, \ldots, K)\) as his/she belief, how should he/she allocate the wealth and what expected utility can he/she obtain? The following proposition answers the question.

**PROPOSITION 7:** If an investor takes the average market belief, \( \bar{r}_k \) and \( \lambda_k \) \((k = 1, 2, \ldots, K)\), he/she can maximize his/her expected utility by optimally allocate the wealth between the IAMP and the risk-free asset; i.e., his/her optimal portfolio is

\[ x_k^R = \frac{\bar{r}_k}{m_x^R}, \quad k = 1, \ldots, K, \quad x_0^R = 1 - \sum_{k=1}^{K} x_k^R. \]  

(33)

By doing so, the investor’s expected utility, on average, is lower in comparison with what he/she can achieve if he/she actively utilizes his/her information and adopts the optimal investment strategy (19).

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23 It is clear that the market belief (26)-(27) involves private signals of all stocks. Therefore, no investors except for completely informed investors have the knowledge of the market belief. It can be shown that if a completely informed investor ignored his/her own information and blindly takes the market belief (26)-(27) as his/her belief, he/she would achieve a lower expected utility level.
With a similar argument, it is can be easily shown that if an investor blindly takes mean $\pi_k$ and precision $\lambda_k^m$ as his/her belief, the best trading strategy is to allocate resources between the average market portfolio and the risk-free asset. But again, such a strategy results in a lower average expected utility level than trading according investor’s information and investing rule (19).

4.3 The Value of Information

The importance of information in investment is obvious. To investigate the effect of information further, let us consider what happens if an investor is informed of more risk assets. As illustrated in the proof of Proposition 7, investor $k^* - 1$, who has informed of $k^* - 1$ assets, has the following conditional expected utility after dropping off some positive coefficients,

$$\bar{u}(k^* - 1) = -\prod_{k=1}^{k^* - 1} \exp[-0.5\lambda_k^j(\bar{\pi}_k^j)^2] \prod_{k^* + 1}^{k^*} \exp[-0.5\lambda_k^j(\bar{\pi}_k^j)^2].$$

Applying expectation operation to it, we obtain the average of conditional expected utility $\bar{u}(k^* - 1) = E[\bar{u}(k^* - 1)]$. Now, suppose this investor receives the private information of one more stock, other things being equal. Then his/her unconditional expected utility changes to $\bar{u}(k^*)$. It can be verified that $\bar{u}(k^*) > \bar{u}(k^* - 1)$. In other words, we have the proposition below, indicating the benefit of information to an investor.

**PROPOSITION 8:** If an investor is informed of more risk assets, his/her expected utility, on average, rises. More specifically, the relative increment of the unconditional expected utility is given by $1 - \lambda_{k^* - 1} / \lambda_{k^*}$ percentage points when the private information increases from observing private signals of $k^* - 1$ stocks to $k^*$ stocks.
From Proposition 8, it is clear that the benefit of extra private information increases in the difference between the informed and uninformed posterior precisions. The more accuracy of private information is over public information, the greater benefit can the investor obtains from receiving private signals. Since this benefit is independent of the amount of investment, it implies increasing returns to scale—the larger is the investor in terms his/her investment funds, the greater benefit he/she receives from information superiority. Thus, the model also explains why large institutional investors have more incentives to collect information and are willing to pay greater costs for the access of private information.

5. CONCLUDING REMARKS

This paper builds a bridge between the classic CAPM and the analysis of rational expectations equilibrium under asymmetric information. Because of systematic noise in the market, the equilibrium prices only partially reveal the information of the future values of risky assets. In other words, although investors can learn the information other investors possess from equilibrium prices, such learning is imperfect and in turn investors’ posterior beliefs are diverse. Therefore, the market portfolio is not mean-variance efficient to any individual investor because each investor has their own efficient frontier and tangency portfolio, and optimally allocates his/her wealth between the risk-free asset and the tangency portfolio.

However, as we have demonstrated, there is an information-adjusted average market portfolio, which plays the similar role as the market portfolio in the standard CAPM formula in the sense that it can determine the security market line under asymmetric information. Moreover, we can predict that the expected rate of return on
each individual asset, on average, equals the unconditional expected rate of return to the IAMP times the beta of the asset—the ratio of the covariance between the asset and the IAMP to the variance of the IAMP. This suggests that for econometric analysis, the best portfolio gauging the performance of a risk asset or a combination of risky assets is the IAMP rather than the market portfolio as traditionally believed.

Our analysis adopts the similar assumptions as the standard CAPM except for asymmetric information. Like the classic CAPM, the model can be extended to accommodate the cases such as short sales are disallowed, interested rates are different for lending and borrowing, investors have to pay taxes, etc. These extensions will definitely enrich the understanding of the effects of asymmetric information on asset pricing and make the model more close to the real world. They are interesting subjects for further studies.
Appendix A. A Brief Summary of Lognormal Distributions

We briefly summarize here some important properties of lognormal distributions relevant to our analysis. A random variable \( X \) is lognormally distributed, 
\[
X \sim LN(\mu, \tau^{-1}) \tag{1}
\]
if it has the following density function \((x > 0)\):
\[
g(x; \mu, \tau^{-1}) = (2\pi)^{-1/2} \tau^{1/2} x^{-1} \exp[-\tau (\ln x - \mu)^2 / 2],
\]
where \( \mu \in \mathbb{R} \) and \( \tau \in \mathbb{R}^+ \). The mean and variance are equal to \( \exp(\mu + \tau^{-1}/2) \) and \( \exp(2\mu + \tau^{-1})[\exp(\tau^{-1}) - 1] \), respectively. In this article, \( X \sim LN(\mu, \tau^{-1}) \) is also written as \( LN(\ln E(X) - (2\tau)^{-1}, \tau^{-1}) \), emphasizing that the mean of \( X \) is \( E(X) \).

**Lemma A1:** A random variable \( X \) has a lognormal distribution \( LN(\mu, \tau^{-1}) \) if and only if \( Y \equiv \ln(X) \) has a normal distribution \( N(\mu, \tau^{-1}) \), where \( \mu \) and \( \tau \) are the mean and precision of the normal distribution, respectively.

**Lemma A2:** Suppose random variable \( X \) has a lognormal distribution \( LN(\mu, \tau^{-1}) \), or re-parameterized as \( LN(\ln E(X) - (2\tau)^{-1}, \tau^{-1}) \). Then for any constant, \( c_0 \), there is
\[
E[\exp(c_0 \ln X)] = \exp(c_0 \mu + c_0^2 / 2\tau) = \exp\{c_0[\ln E(X) - (2\tau)^{-1}] + c_0^2 / 2\tau\}.
\]
The proofs of Lemmas A1 and A2 are straightforward and thus omitted here.

**Lemma A3:** Let \( X_1, \ldots, X_n \) denote a random sample of size \( n \) independently drawn from distribution \( LN(\ln v - (2\tau)^{-1}, \tau^{-1}) \), where \( v \) is an unknown parameter and \( \tau \) is a known parameter. Let \( \ln t(X_1, \ldots, X_n) \equiv \sum_{i=1}^{n} \ln X_i / n + (1 - n^{-1})(2\tau)^{-1} \), then
\[ t(X_1, \ldots, X_n) \sim LN(\ln v - (2n\tau)^{-1}, (n\tau)^{-1}) . \] In addition, for the collection of observations \( X_1, \ldots, X_n, t(X_1, \ldots, X_n) \) is a sufficient statistic for \( v \).

**Proof:** From Lemma A1, \( \ln X_i \sim N(\ln v - (2\tau)^{-1}, (\tau)^{-1}) \). Hence, we have

\[
\sum_{i=1}^{n} \ln X_i / n \sim N(\ln v - (2\tau)^{-1}, (n\tau)^{-1}) .
\]

Applying lemma A1 again we obtain that

\[
t(X_1, \ldots, X_n) \sim LN(\ln v - (2n\tau)^{-1}, (n\tau)^{-1}) .
\]

For the collection of observations \( X_1, \ldots, X_n \), the joint distribution is given by

\[
(2\pi)^{-n/2} \tau^{n/2} \left( \prod_{i=1}^{n} X_i^{-1} \right) \exp \left\{ -\tau \sum_{i=1}^{n} [\ln X_i - \ln v + (2\tau)^{-1}]^2 / 2 \right\}
\]

\[
= (2\pi)^{-n/2} \tau^{n/2} \left( \prod_{i=1}^{n} X_i^{-1} \right) \exp \left\{ -\tau \sum_{i=1}^{n} [\ln X_i - A]^2 / 2 \right\} \exp \left\{ -n\tau [\ln v - (A + (2\tau)^{-1}) / 2]^2 \right\} .
\]

where \( A = \sum_{i=1}^{n} \ln X_i / n \). From the factorization theorem of Neyman (see Hogg and Craig, 1995, p318), \( A \) is a sufficient statistic for \( v \), and so is its linear function \( t(X_1, \ldots, X_n) \).

**Appendix B. Proofs**

**Proof of Lemma 1:** Applying the Bayesian theorem, the posterior density of \( v \), given \( s \), is

\[
f(v | s) \propto v^{-1} \exp \left\{ -b_0 (\ln v - a_0)^2 / 2 - b_1 (\ln s - a_1 - \ln v)^2 / 2 \right\} ,
\]

which can be rewritten as

\[
f(v | s) \propto v^{-1} \exp \left\{ - (b_0 + b_1) \left[ \ln v - (a_0 b_0 + b_1 (\ln s - a_1)) / (b_0 + b_1) \right]^2 / 2 \right\} .
\]

Hence, \( v | s \sim LN([a_0 b_0 + b_1 (\ln s - a_1)] / (b_0 + b_1), (b_0 + b_1)^{-1}) .
\]

**Proof of Lemma 2:** In the proof, we apply the following lemma:
**Lemma A4:** Let $Q = [q_1, \ldots, q_n]^T$ and $D = \text{diag}\{q_1, \ldots, q_n\}$. If $q_i \neq 0$ ($i = 1, \ldots, n$), we have $\|D - QQ^T\|_2 \leq 1/(4n)$. (The proof is available upon request.)

Now we prove Lemma 2. Let $q_j = x_j' \exp(r_j) / \left\{1 + \sum_{k=1}^K x_k'[\exp(r_k) - 1]\right\}$ and $f(r_1, \ldots, r_K) = \ln\left\{1 + \sum_{k=1}^K x_k'[\exp(r_k) - 1]\right\}$. Applying the Taylor’s Theorem we obtain

$$f(r_1, \ldots, r_K) = \sum_{j=1}^K x_j' r_j + r^T H(\omega \mathbf{r}) \mathbf{r} / 2,$$

where $H(\mathbf{r}) = [\partial^2 f(r_1, \ldots, r_K) / \partial r_j \partial r_k, \partial^2 f(r_1, \ldots, r_K) / \partial r_j \partial r_k = -q_j q_k$ ($j \neq k$),

$$\partial^2 f(r_1, \ldots, r_K) / \partial r_j^2 = q_j - q_j^2.$$ Now define set $J \equiv \{k | x_k' \neq 0\}$. Note that $q_j \neq 0$ for all $j \in J$ and the number of elements in set $J$ is $K_0$. Without loss of generality, assume that $J = \{1, \ldots, K_0\}$. Let $H_0$ denote the sub-matrix containing only the first $K_0$ rows and the first $K_0$ columns of $H$. Then the entries of remaining rows and columns of $H$ are zero. Applying Lemma A4 to $H_0$, we have $\|H(\omega \mathbf{r})\|_2 \leq 1/(4K_0)$.

Thus the remainder has a bound of $\|\mathbf{r}\|_2^2 / (8K_0)$.

**Proof of Proposition 1:** Substituting equation (19) into (20), we obtain

$$\mu_k \lambda^I_k \ln \bar{v}^I_k + (1 - \mu_k) \lambda^U_k \ln \bar{v}^U_k - (\delta - 1) \gamma_k / m = [\mu_k \lambda^I_k + (1 - \mu_k) \lambda^U_k] \ln p_k.$$ \hfill (B1)

Define $C_k \equiv \mu_k \lambda^I_k + (1 - \mu_k) \lambda^U_k = \rho_k + \gamma_k (1 - \alpha_k) I_k + \mu_k \gamma_k \alpha_k I_k + (1 - \mu_k) \xi_k$. Substituting

$$\ln \theta_k$$

in (10) into (B1) yields

$$\ln p_k = C_k^{-1} \{\rho_k \ln \bar{v} + [\mu_k \gamma_k \alpha_k I_k + (1 - \mu_k) \xi_k] \ln g_k + \gamma_k (1 - \alpha_k) I_k \ln h_k$$

$$- [(\delta - 1) / m + (1 - \mu_k) \xi_k (d / b)] y_k$$

$$+ (1 - \mu_k) \xi_k (d / b) \bar{y}_k + [\mu_k / 2 + (1 - \mu_k) \xi_k / (2 \gamma_k \alpha_k I_k)]\}.$$ \hfill (B2)
Comparing the coefficients of $\ln(g_k)$ and $y_k$ with their counterparts in (9), we obtain

$$\frac{d}{b} = -[(\delta - 1)/m + (1 - \mu_k)\xi_k (d/b)]$$

$$\mu_k \gamma_k \alpha_k I_k + (1 - \mu_k)\xi_k,$$

which in turn requires $d/b = (\delta - 1)/m \mu_k \gamma_k \alpha_k I_k$. Inserting it into (B2) yields the coefficients in the proposition.

**Proof of Proposition 2:** The mean-variance efficiency is obvious by recalling Ingersoll (1987), chapter 4. Since $\{t^*_k\}$ is the tangency portfolio and the necessary and sufficient condition for a portfolio being a tangency portfolio is (24), the validity of (24) is apparent.

**Proof of Lemma 3:** Let $f(x; \mu, \tau^{-1})$ be the density function of a normal distribution with mean $\mu$ and variance $\tau^{-1}$. Then, (25) shows that the density function of $r_k^M$ is

$$\pi(r_k; q_k, \bar{r}_k, (\lambda_k^1)^{-1}, \bar{U}_k, (\lambda_k^2)^{-1}) = q_k f(r_k; \bar{r}_k, (\lambda_k^1)^{-1}) + (1 - q_k) f(r_k; \bar{U}_k, (\lambda_k^2)^{-1}).$$

So,

$$E(r_k^M) = E \left[ q_k \int_{-\infty}^{+\infty} f(r; \bar{r}_k, (\lambda_k^1)^{-1}) dr + \int_{-\infty}^{+\infty} f(r; \bar{U}_k, (\lambda_k^2)^{-1}) dr \right] = E[q_k \bar{r}_k + (1 - q_k) \bar{U}_k] = E(r_k),$$

where the expectations are taken for all private signals $g_k$, public signals $h_k$, and supply $y_k$. For variance, we have

$$\text{var}(r_k^M) = E \left[ \int_{-\infty}^{+\infty} r^2 [q_k f(r; \bar{r}_k, (\lambda_k^1)^{-1}) + (1 - q_k) f(r; \bar{U}_k, (\lambda_k^2)^{-1})] dr \right] - [E(r_k^M)]^2$$

$$= q_k E[(\bar{r}_k)^2 + (\lambda_k^1)^{-1}] + (1 - q_k) E[(\bar{U}_k)^2 + (\lambda_k^2)^{-1}] - [E(r_k)]^2.$$

Since $q_k = \mu_k \lambda_k^1 / [\mu_k \lambda_k^1 + (1 - \mu_k) \lambda_k^2]$, the above equation can be rewritten as
\[
\text{var}(r_k^M) = (\lambda_k^M)^{-1} - [q_k \text{ var}(\bar{r}_k^I) + (1 - q_k) \text{ var}(\bar{r}_k^U)].
\]  \hfill (B3)

Note \( \text{var}(\bar{r}_k^I) = - (\lambda_k^I)^{-1} + \text{var}(r_k) \) and \( \text{var}(\bar{r}_k^U) = - (\lambda_k^U)^{-1} + \text{var}(r_k) \). Inserting these two equations into (B3) yields \( \text{var}(r_k^M) = \text{var}(r_k) \).

**Proof of Proposition 3:** Note that

\[
E[\ln(v_k / p_k)] = E(\ln v_k) - [a \ln \bar{v}_k + bE(\ln g_k) + cE(\ln h_k) - (d - e)\bar{v}_k + f]. \hfill (B4)
\]

It can be verified that

\[
E(\ln v_k) = \ln \bar{v}_k - (2\rho_k)^{-1}, \quad E(\ln g_k) = \ln \bar{v}_k - (2\rho_k)^{-1} - (2\gamma_k \alpha_k I_k)^{-1}
\]

\[
E(\ln h_k) = \ln \bar{v}_k - (2\rho_k)^{-1} - [2\gamma_k (1 - \alpha_k) I_k]^{-1}.
\]

Recalling the following two identities

\[
a + b + c = 1,
\]

\[
(2\rho_k)^{-1} = b(2\gamma_k \alpha_k I_k)^{-1} + c(2\gamma_k (1 - \alpha_k) I_k)^{-1} + (b + c)(2\rho_k)^{-1} - f,
\]

and substituting the coefficients in Proposition 1 into (B4), we obtain (28).

Now we compute \( \text{var}(r_k) \). First we note the following identity

\[
\text{var}(r_k) = E[\text{var}(r_k | v_k)] + \text{var}[E(r_k | v_k)]. \hfill (B5)
\]

Since \( \text{var}(\ln g_k | v_k) = (\gamma_k \alpha_k I_k)^{-1} \), \( \text{var}(\ln h_k | v_k) = [\gamma_k (1 - \alpha_k) I_k]^{-1} \) and \( \text{var}(v_k) = \eta_k^{-1} \), we have

\[
\text{var}(r_k | v_k) = \text{var}(b \ln g_k + c \ln h_k - dY_k | v_k)
\]

\[= b^2 (\gamma_k \alpha_k I_k)^{-1} + c^2 [\gamma_k (1 - \alpha_k) I_k]^{-1} + d^2 \eta_k^{-1}. \hfill (B6)
\]

Moreover, \( E[\text{var}(r_k | v_k)] = \text{var}(r_k | v_k) \) since \( \text{var}(r_k | v_k) \) is a constant. On the other hand, (21) shows

\[
E(r_k | v_k) = \ln v_k - [a \ln \bar{v}_k + bE(\ln g_k | v_k) + cE(\ln h_k | v_k) + (d - e)\bar{v}_k + f].
\]

CAPM-Asymmetry4.doc 38
Inserting \( E(\ln g_k \mid v_k) = \ln v_k - (2\gamma_k\alpha_k I_k)^{-1} \) and \( E(\ln h_k \mid v_k) = \ln v_k - [2\gamma_k(1-\alpha_k)I_k]\) into it, we obtain \( E(r_k \mid v_k) = \text{constant} + a \ln v_k \), which implies \( \text{var}[E(r_k \mid v_k)] = a^2 \rho_k^{-3} \). Substituting the last equation and (B6) into (B5) yields (29).

**Proof of Proposition 4:** Applying Lemma 3, we have

\[
\sigma_{kr} = E\left[ (r_k^M - \bar{r}_k^M)\left( \sum_{k=1}^K \bar{r}_k^R (r_k^M - \bar{r}_k^M) \right) \right] = \bar{r}_k^R / \kappa_k, \]

\[
\sigma_{rr} = E\left[ \left( \sum_{k=1}^K \bar{r}_k^R (r_k^M - \bar{r}_k^M) \right)^2 \right] = \sum_{k=1}^K (\bar{r}_k^R)^2 / \kappa_k. \]

From (28), we have

\[
m\bar{r}_k / (\delta - 1) = \bar{y}_k / \lambda_k^M. \tag{B7}\]

Substituting \( \bar{r}_k^R \) into it and then using \( \sigma_{kr} \), we obtain

\[
m\bar{r}_k / (\delta - 1) = \bar{r}_k^R / \kappa_k \bar{z}^R = \sigma_{kr} / \bar{z}^R. \]

Multiplying both sides of it by \( \bar{r}_k^R \) and then summing over all \( k \)’s yield

\[
m\bar{z}^R / (\delta - 1) = \left[ \sum_{k=1}^K (\bar{r}_k^R)^2 / \kappa_k \right] / \bar{z}^R = \sigma_{rr} / \bar{z}^R. \]

The combination of the last two equations gives (30).

**Proof of Proposition 4P:** By definition, \( \bar{\bar{r}}^M = \sum_{k=1}^K t_k^M \bar{r}_k^M \). Applying

\[\pi(r_k; q_k, \bar{r}_k^I, (\lambda_k^I)^{-1}, \bar{r}_k^U, (\lambda_k^U)^{-1}) \text{ given in the proof of Lemma 3, we get} \]

\[
\sigma_{km} = \int_{-\infty}^{\infty} (r_k^M - \bar{r}_k^M)\left( \sum_{k=1}^K t_k^M (r_k^M - \bar{r}_k^M) \right) \prod_{k=1}^K \pi(r_k; q_k, \bar{r}_k^I, (\lambda_k^I)^{-1}, \bar{r}_k^U, (\lambda_k^U)^{-1}) dr_k = t_k^M / \lambda_k^M, \]

\[
\sigma_{nm} = \int_{-\infty}^{\infty} \left( \sum_{k=1}^K t_k^M (r_k^M - \bar{r}_k^M) \right)^2 \prod_{k=1}^K \pi(r_k; q_k, \bar{r}_k^I, (\lambda_k^I)^{-1}, \bar{r}_k^U, (\lambda_k^U)^{-1}) dr_k = \sum_{k=1}^K (t_k^M)^2 / \lambda_k^M. \]
Substituting $\sigma_{\delta M}$ into (B1) yields $m\bar{r}_k^M/(\delta - 1) = \sigma_{\delta M}^2 Y$. Multiplying both sides of the equation by $t_k^M$ and summing over $k$ yield $m\bar{r}_k^M/(\delta - 1) = \sigma_{\delta M}^2 Y$. The last two equations then generate (31).

**Proof of Proposition 5:** From (28) we have $\bar{r}_k = y_k/(\delta - 1)/m\lambda_k^M$. But

$$\lambda_k^M = \rho_k + r_k^I I_k - (1 - \mu_k) \gamma_k \alpha_k I_k + (1 - \mu_k) \xi_k^I$$

$$= \rho_k + r_k^I I_k - (1 - \mu_k) \left[ \frac{(\delta - 1)^2 (m \mu_k)^{-2} \eta_k^{-1} \gamma_k \alpha_k I_k}{(\delta - 1)^2 (m \mu_k)^{-2} \eta_k^{-1} + \gamma_k \alpha_k I_k} \right].$$

So, $\lambda_k^M$ decreases in $\alpha_k$ and $\partial \bar{r}_k / \partial \alpha_k > 0$. Other partial derivatives can be obtained similarly and their derivations are omitted.

**Proof of Proposition 7:** When an investor takes the belief $r_k$ and $\lambda_k$, the optimal solution to (18) is

$$x_k^R = \lambda_k r_k / (\delta - 1), \quad k = 1, \ldots, K, \quad x_0^R = 1 - \sum_{k=1}^{K} x_k^R.$$  

Recalling (B7), we have $\bar{r}_k / (\delta - 1) = \bar{r}_k / m\lambda_k^M$. Substituting it into the above equations yields (33).

Substituting $x_k^R$ into (17) with proper replacement of $r_k$ and $\lambda_k$, we have expected utility (after dropping off some positive coefficients)

$$\bar{u}^R = -\prod_{k=1}^{K} \exp(-0.5 \lambda_k^R \bar{r}_k^2).$$

On the other hand, for investor $k^*$, his conditional expected utility, after substituting (19) into (17) and dropping off the same positive coefficients, is

$$\bar{u}(k^*) = -\prod_{k=1}^{K} \exp[-0.5 \lambda_k^U (\bar{r}_k^U)^2] \prod_{k=1}^{K} \exp[-0.5 \lambda_k^U (\bar{r}_k^U)^2].$$

Taking expectation operation, we get unconditional expected utility that
\[\pi(k^*) = E[\pi'(k^*)] = \prod_{k=1}^{K} E\left[\exp[-0.5\lambda'_k (k^* - 1)]\right] - \prod_{k=1}^{K} E\left[\exp[-0.5\lambda''_k (k^* - 1)]\right]\]

So, if we can prove that
\[
\exp(-0.5\lambda'_k \pi^2_k) > E\left[\exp[-0.5\lambda'_k (\pi'_k)^2]\right], \quad (B8)
\]
\[
\exp(-0.5\lambda''_k \pi^2_k) > E\left[\exp[-0.5\lambda''_k (\pi'_k)^2]\right], \quad (B9)
\]
we can immediately obtain \(\bar{\pi}R < \pi(k^*)\). To show (B8) holds, we first note that
\[\pi'_k \sim N(\pi'_k, \text{var}(\pi'_k)), \quad \text{where} \quad \text{var}(\pi'_k) = \lambda'_k^{-1} - (\lambda'_k)^{-1}.\]
Second, if \(X \sim N(\mu, \sigma^2)\), then
\[E[\exp(-d_0 X^2/(2\sigma^2))] = (1 + d_0)^{-1/2} \exp[-d_0 \mu^2/(2\sigma^2 (d_0 + 1))] \quad \text{for any} \quad d_0 > 0.\]
With these two results we get
\[E\left[\exp[-0.5\lambda'_k (\pi'_k)^2]\right] = (\lambda'_k / \lambda''_k)^{1/2} \exp(-0.5\lambda'_k \pi^2_k). \quad (B10)\]
Since \(\lambda'_k < \lambda''_k\), it is obvious that (B8) holds. We can prove (B9) similarly.

**Proof of Proposition 8:** Recalling (B10) and its counterpart
\[E\left[\exp[-0.5\lambda''_k (\pi'_k)^2]\right] = (\lambda'_k / \lambda''_k)^{1/2} \exp(-0.5\lambda'_k \pi^2_k),\]
we find that the relative increment in expected utility is
\[\frac{\pi'(k^*) - \pi'(k^*-1)}{|\pi'(k^*-1)|} = \frac{E\left[\exp[-0.5\lambda'_k (\pi'_k)^2]\right] - E\left[\exp[-0.5\lambda''_k (\pi'_k)^2]\right]}{E\left[\exp[-0.5\lambda''_k (\pi'_k)^2]\right]} = \frac{\lambda''_k - \lambda'_k}{\lambda''_k} > 0.\]
Thus, \(1 - \lambda'_k / \lambda''_k\) is the number of percentage points of relative increment in the expected utility.

**Appendix C (For referees only)**

**Proof of Lemma A4:** An eigenvalue of matrix \(D - QQ^T\), \(\lambda\), satisfies
\[|\lambda I - (D - QQ^T)| = 0, \quad \text{where} \quad I \quad \text{is an identity matrix. It is easy to verify that} \quad \lambda = q_i, \quad \text{is not} \quad \text{an eigenvalue of} \quad D - QQ^T. \quad \text{Hence we have} \quad |\lambda I - D + QQ^T| = 0.\]
\[ \| I - (D - \lambda I)^{-1}QQ^T \| = 0 \iff Q^T(D - \lambda I)^{-1}Q = 1 \iff \sum_{i=1}^{n} q_i^2 / (q_i - \lambda) = 1 \] (see Wilkinson, 1965, p54).

Note that the last equation implicitly defines a function \( \lambda = \lambda(q_1, \ldots, q_n) \). Define

\[ F = \sum_{i=1}^{n} q_i^2 / (q_i - \lambda) - 1 \]

Consider the following optimization problem:

\[ \max_{q_i (i=1, \ldots, n)} \{ \lambda(q_1, \ldots, q_n) \}^2 \]

The first-order condition, \( \partial \lambda / \partial q_i = (\partial F / \partial q_i) / (\partial F / \partial \lambda) = 0 \), \( i = 1, \ldots, n \), yields either \( q_i = 0 \) or \( q_i = 2\lambda \). Since \( q_i \neq 0 \) for all \( i = 1, \ldots, n \), we retain \( q_i = 2\lambda \) \( (i = 1, \ldots, n) \). Substituting this solution into \( \sum_{i=1}^{n} q_i^2 / (q_i - \lambda) = 1 \) results in \( \lambda = 1/(4n) \), i.e.

\[ \| D - QQ^T \|_2 \leq 1/(4n). \]
REFERENCES


Figure 1—Portfolio Frontiers and Tangency Portfolios
When Investors Have Diverse Information