MODELLING VOLATILITY SURFACES WITH GARCH

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WHY GARCH?

• stylised facts about asset price returns

  • spot markets

    • changes not normally distributed

      • relatively more very small changes and more very large changes—leptokurtic

    • changes are hard to predict

      • very noisy movements around changing mean

    • but not independent or identically distributed

      • large changes followed by large changes of either sign—volatility clustering

      • risk is partly predictable—large amount of mean reversion, with small amount of persistence

• options markets

  • historical vol underestimates option prices

  • at-the-money vol ‘underestimates’ away-from-the-money vol—‘smiles’ and now ‘smirks’ in equities

  • suggest that some other source of ‘risk’ is being priced by market (implicit, not explicit)
• explaining stylised facts—GARCH and Stochastic Vol?

• spot markets
  • enormous and growing literature on GARCH-type models suggests that they can explain much of the stylised facts
  • more recent interest in stochastic vol models
    • but estimation much more demanding
    • => much less empirical evidence

• options markets
  • some emerging evidence for GARCH-type models
  • less research for stochastic vol
  • have been lots of implementation problems
• implementation problems

• GARCH
  • what is ‘risk-neutralised’ or ‘no-arbitrage’ probability measure
  • how estimate parameters
  • how calculate option prices
    • Monte Carlo—*not* American
  • these problems have now been solved

• stochastic volatility
  • what is ‘risk-neutralised’ or ‘no-arbitrage’ probability measure
  • how estimate parameters
    • discrete data, continuous specification
  • how calculate option prices
    • Monte Carlo—*not* American
    • finite difference
GARCH OPTION PRICING MODEL

• assumed process for spot prices

\[
\ln \left( \frac{S_{t+1}}{S_t} \right) = r + \lambda \sqrt{h_{t+1}} - \frac{1}{2} h_{t+1} \varepsilon_{t+1} + \sqrt{h_{t+1}} \nu_{t+1}
\]

GARCH \hspace{1cm} h_{t+2} = \beta_0 + \beta_1 h_{t+1} + \beta_2 h_{t+1} \nu_{t+1}^2

NGARCH \hspace{1cm} h_{t+2} = \beta_0 + \beta_1 h_{t+1} + \beta_2 h_{t+1} (\nu_{t+1} - c)^2

GJR \hspace{1cm} h_{t+2} = \beta_0 + \beta_1 h_{t+1} + \beta_2 h_{t+1} \nu_{t+1}^2 + \beta_3 h_{t+1} \max(0, -\nu_{t+1})^2

where the innovation \( \nu_{t+1} \) is iid standard normal and

\[
\text{Var}\left[ \ln(S_{t+1}/S_t) \right] = h_{t+1}
\]

• risk-neutralised/arbitrage-free process

\[
\ln \left( \frac{S_{t+1}}{S_t} \right) = r - \frac{1}{2} h_{t+1} \varepsilon_{t+1} + \sqrt{h_{t+1}} \varepsilon_{t+1}
\]

GARCH \hspace{1cm} h_{t+2} = \beta_0 + \beta_1 h_{t+1} + \beta_2 h_{t+1} (\varepsilon_{t+1} - \lambda)^2

NGARCH \hspace{1cm} h_{t+2} = \beta_0 + \beta_1 h_{t+1} + \beta_2 h_{t+1} (\varepsilon_{t+1} - (c + \lambda))^2

GJR \hspace{1cm} h_{t+2} = \beta_0 + \beta_1 h_{t+1} + \beta_2 h_{t+1} (\varepsilon_{t+1} - \lambda)^2 + \beta_3 h_{t+1} \max(0, -\varepsilon_{t+1} + \lambda)^2

where the innovation \( \nu_{t+1} \) has been replaced by \( \varepsilon_{t+1} - \lambda \)

• can show that this process is arbitrage free—what about \( \lambda \)?

A HELPFUL RE-PARAMETERISATION

• some restrictions on the parameters of the GARCH process are required

  • ensure that conditional variance is non-negative

    \[
    \text{GARCH} \quad h_{1p} > 0, \beta_0 > 0, \beta_1 > 0, \beta_2 > 0
    \]

    \[
    \text{NGARCH} \quad h_{1p} > 0, \beta_0 > 0, \beta_1 > 0, \beta_2 > 0
    \]

    \[
    \text{GJR} \quad h_{1p} > 0, \beta_0 > 0, \beta_1 > 0, \beta_2 > 0, \beta_3 > 0
    \]

  • ensure that unconditional variance is bounded

    \[
    \text{GARCH} \quad \beta_1 + \beta_2 < 1
    \]

    \[
    \text{NGARCH} \quad \beta_1 + \beta_2 (1 + c^2) < 1
    \]

    \[
    \text{GJR} \quad \beta_1 + \beta_2 + \beta_3 / 2 < 1
    \]

• unfortunately parameters NOT independent of observation frequency

  • \( \beta_1 \) and \( \beta_2 \) depend on frequency, but don't vary much across assets

  • \( \beta_0 \) can be difficult to understand, best to recast as an annualised steady state or unconditional vol

    \[
    \text{GARCH} \quad \sigma_s = \sqrt{365 \times \beta_0 / (1 - \beta_1 - \beta_2)}
    \]

    \[
    \text{NGARCH} \quad \sigma_s = \sqrt{365 \times \beta_0 / (1 - \beta_1 - \beta_2 (1 + c^2))}
    \]

    \[
    \text{GJR} \quad \sigma_s = \sqrt{365 \times \beta_0 / (1 - \beta_1 - \beta_2 - \beta_3 / 2)}
    \]

    for daily data

  • annualise initial vol

    \[
    \sigma_{10} = \sqrt{365 \times h_{1p}}
    \]

    for daily data
ESTIMATING THE PARAMETERS

• can estimate from spot data using econometric techniques (maximum likelihood)
  • need quite a few observations—at least 500 for daily frequency
  • log-likelihood surface can be problematic—often need simplex to refine initial values and BHHH (or BFGS) to estimate
  • non-normal conditional densities can be easily handled, or can use QML to get robust standard errors
  • many econometric packages available. My preference is RATS (http://www.estima.com). Comes with GARCH estimation routines which I wrote.

• can also imply out from observed option prices, once you have a procedure to price options when underlying is a GARCH process
  • need range of strikes (and/or maturities)—at least as many as number of parameters
  • pick parameters that minimise distance from observed prices/vols
  • optimisation can be tricky/slow

• ideally both approaches should give same results
  • spot data yields underlying process
  • option data yields risk-neutralised/arbitrage-free
  • simply related via change of measure
Moneyness Stock Strike Actual Implied Lattice Implied Difference

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<th>Stock</th>
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Implied NGARCH model

\[
\sigma_{10} = 15.18\%, \sigma_s = 13.87\%, \beta_1 = 0.7983, \beta_2 = 0.0503, c = 1.6425
\]
PICKING THE RIGHT PRICING ALGORITHM

• no closed form solutions available, even for Europeans—key is volatility clustering => path dependent vol, so closed forms unlikely


  • model is *not* a standard GARCH formulation—volatility innovation is not scaled by conditional variance

  • need to *numerically* solve a bivariate system of difference equations to get coefficients of the characteristic function for price of underlying

  • then need to *numerically* integrate real part of a function of this complex valued characteristic function to invert it to find European option values

• analytical approximations (JC Duan, G Gauthier and JG Simonato (1999), "An Analytical Approximation for the GARCH Option Pricing Model", *Journal of Computational Finance*, Vol 2, pp75-166)

  • can deduce approximations to moments of a GARCH process for a given maturity—very messy formula (corrections from authors)—moments can be used to approximate the distribution of terminal value of an asset

  • can use this to value European claims on terminal values

• Monte Carlo

  • European only, but may be useful for path dependent payouts

  • control variates—depend on option being valued

    • BS evaluated at $\sqrt{365 \times h_{10}}$, $\sqrt{365 \times \sigma_s^2}$ or at BS implied vol from GARCH option value

• Markov chain approximation with sparse matrix tricks (JC Duan and JG Simonato (forthcoming), “American Option Pricing under GARCH by a Markov Chain Approximation”, *Journal of Economic Dynamics and Control*)

  • American and European, but not as efficient as lattice


  • American and European
  
  • efficient procedure also handles stochastic volatility models
  
  • can be modified for use with path dependent payouts

• once have prices for range of parameter values, strikes, maturities, etc, can use a neural network for interpolation (See M. Hanke (1997), "Neural Network Approximation of Option Pricing Formulas for Analytically Intractable Option Pricing Models", *Journal of Computational Intelligence in Finance*, Vol 5, pp20-27)

• recommended methods?

  • European, terminal distribution only—Duan et al analytical approximation

  • European, heavy path dependencies or multiple assets—Monte Carlo

  • Americans or other types of Europeans—Ritchken and Trevor lattice

• GARCH provides accurate, efficient approximation to stochastic volatility models (bivariate diffusions)—see Ritchken and Trevor paper for quality of approximation
GARCH IMPLIED VOLATILITY SURFACES

• smiles, smirks and grimaces controlled by parameters of GARCH equation

• following charts show actual fitted volatility surfaces (large) with initial volatility 20% below and above steady state volatility (pair of small charts)
AUD Options on OTC

Moneyness

Maturity

0.09

0.14

0.09

0.14

0.07

0.12

0.07

0.12

0.10

0.15

0.10

0.15
HSI Options on HKSE

Moneyness

Maturity

BS Implied Vol

0.30
0.25
0.20
0.15
0.35
0.30
0.25
0.20
0.15
0.9
1
1.1
10 30 50 70 90

0.9
1
1.1
10 30 50 70 90

0.9
1
1.1
10 30 50 70 90
WHAT REMAINS TO BE DONE?

• does GARCH explain the smile/smirk—preliminary evidence suggests that it does
  • strike price bias in BS
  • maturity bias in BS
• key is dynamics of volatility clustering
• providing parameters prove to be stable, likely to provide superior hedging results
• more detailed, rigorous testing to be done over range of instruments and markets, especially on hedging

• can use GARCH not just for pricing and hedging normal options
  • exotics that trade off the same underlying
  • calculating the risk neutral probability distribution