ASSET PRICE AND WEALTH DYNAMICS UNDER HETEROGENEOUS EXPECTATIONS

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Abstract. In order to characterize price and wealth dynamics under the interaction of heterogeneous agents with a CRRA utility, a discrete time stationary wealth dynamics model in terms of return and wealth proportions (among different types of agents) is established. Fundamentalists and chartists are the main heterogeneous agents in the model. It is found that the presence of heterogeneous agents can lead the stationary model to have multiple equilibria. The equilibrium is unstable when the chartists extrapolation rate is high and (locally) stable when the rate is low. The convergence to the equilibrium follows an optimal selection principle — the return and wealth proportion tend to one of the equilibria which has relative higher return. The model that is finally developed displays the essential characteristics of the standard asset price dynamics model assumed in continuous time finance in that the asset price is fluctuating around an geometrically growing trend.

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1. Introduction

A great deal of well established economic and finance theory is based on the assumption of investors homogeneity. In particular the paradigm of the representative agent assumes that all investors are homogeneous with regard to their preferences, their expectations and their investment strategies. The other paradigm of modern finance is the efficient market hypothesis assuming that the current price contain all available information and past prices cannot help in predicting future prices. However, as already argued by Keynes in the 1930s, agents do not have sufficient knowledge of the structure of the economy to form correct mathematical expectations that would be held by all agents. On the other hand, there is evidence that markets are not always efficient and real data show significant higher than expected autocorrelation returns. As a result, there is a growing dissatisfaction with (i) models of asset price dynamics based on the representative agent paradigm, as expressed for example by Kirman [17], and (ii) the extreme informational assumptions of rational expectations.

Research into the dynamics of financial asset prices resulting from the interaction of heterogeneous agents having different attitudes to risk and having different expectations about the future evolution of prices has flourished in recent years, e.g. Brock and Hommes [3], [4], [5], Bullard [6], Bullard and Duffy [7], Chiarella [8], Chiarella and He [9], [11], [10], Day and Huang [12], Franke and Nesemann [15], Franke and Sethi [16], Lux [22], [23], [24], Lux and Marchesi [25], [26] and Sethi [28].

In their framework, Brock and Hommes [4], [5] have introduced the concept of an adaptively rational equilibrium, where agents base decisions upon predictions of future values of endogenous variables whose actual values are determined by equilibrium equations. A key aspect of these models is that they exhibit expectations feedback. Agents adapt their beliefs over time by choosing from different predictors or expectations functions, based upon their past performance as measured by realized profits. The resulting dynamical system is capable of generating the entire “zoo” of complex behaviour from local stability to high order cycles and chaos as various key parameters of the model change.

The model of Brock and Hommes [5] has been extended in Chiarella and He [9], [11] by allowing agents to have different risk attitudes and different expectation formation schemes for both first and second moments of the price distribution under Walrasian and market maker scenarios. It is found that (see Chiarella and He [10] also for a general analysis on the dynamics of heterogeneous expectations and learning) the introduction of heterogeneity has a double-edged effect on the asset prices. On the one hand, the steady state can be stabilised under some balanced heterogeneous learning processes in the sense that although every individual forecasting rule may lead to divergence from the equilibrium, these may “cancel out” in the aggregate and the actual dynamics with learning may thus be locally stable. On the other hand, heterogeneity is a source of instability in
the market. A small group of traders with expectation functions involving significant divergence is in fact sufficient to destabilize the whole system, changes the dynamics dramatically and lead to periodic or even chaotic fluctuations in prices.

The asset pricing framework of Brock and Hommes [5] is based on the assumption that all the agents know the fundamental price equilibrium and the agents’ optimal demands on the risk asset are independent of their wealth (as a result of the underlying CARA utility functions). In general, these assumptions are unrealistic. Levy and Levy [18] and Levy, Levy and Solomon [19], [20] consider recently a more realistic model where investors optimal decisions depend on their wealth (resulting from the underlying CRRA utility function) and both price and wealth processes are thus growing. They compare stock price dynamics in models with homogeneous and heterogeneous expectations. Using numerical simulations, they conclude that the homogeneous expectation assumption leads to a highly inefficient market with periodic (and therefore predictable) booms and crashes while introduction of heterogeneous expectations leads to much more realistic dynamics. The above cited literature points to a need to develop a theoretical analysis of asset price and wealth dynamics with heterogeneous agents and that is the aim of this paper.

The contributions of the paper are as follows. Following the framework of Brock and Hommes [5] and Levy and Levy [18], Section 2 outlines the model of interaction of portfolio decisions and wealth dynamics with heterogeneous agents. A growth equilibrium model in both price and wealth is first set up. In order to characterize the interactions of heterogeneous agents in financial markets (typically fundamentalists, trend chasers and contrarians) and conduct a theoretical analysis, two stationary models in terms of return and wealth proportions (among different types of agents) are then developed. As a special case of the general heterogeneous model, Section 3 studies the return dynamics under homogeneous expectations. In the following section, various different models of two players with fundamentalists and trend traders are then considered. It is found that the presence of heterogeneous agents can lead the stationary model to have multiple equilibria for asset return and wealth distribution. The existence of different types of equilibrium arises as a function of parameters of expectation schemes are analyzed. Stability analysis implies that, depended on the extrapolation rate of the trend traders, the return dynamics tends to be unstable for a high extrapolation rate and stable for a low extrapolation rate. The convergence to equilibrium follows an optimal selection principle — the return and wealth proportion tend to one of the equilibria which has a relatively higher return. Also, numerical simulations are employed to study the types of time series that can emerge under a noisy dividend yield process. All the mathematical proofs are placed in appendices.
2. Wealth Models with Heterogeneous Expectations

This section is devoted to some generalizations of the simple asset pricing model established by Brock and Hommes [5] and Levy and Levy [18]. The key characteristic of this modelling framework is the heterogeneity of the economic agents. The heterogeneity is expressed in terms of different views on expectations of the distribution of future returns on the risky asset. Our hypothetical stock market consists of two investment options: a stock (or index of stocks) and a bond. The bond is assumed to be a risk free asset and the stock is a risky asset. The models considered in the following are discrete time models, in which investors are allowed to revise their portfolios at each time interval.

In this section, following the standard portfolio optimization approach, a growth model in terms of price and wealth is established. Then, instead of price and wealth, the return on the risky asset and the wealth proportions (among heterogeneous investors) are used as state variables in order to reduce the growth model to a stationary model.

2.1. Growth Model. Denote

\[ p_t : \text{Price (ex dividend) per share of the risky asset at time } t; \]
\[ y_t : \text{Dividend at time } t; \]
\[ R : \text{Risk free return with risk free rate } r = R - 1; \]
\[ N : \text{Total number of shares of the risky asset;} \]
\[ H : \text{Total number of investors;} \]
\[ N_{i,t} : \text{Number of shares acquired by agent } i \text{ at time } t; \]
\[ W_{i,t} : \text{Wealth of agent } i \text{ at time } t; \]
\[ W_{i0} : \text{Initial wealth of agent } i; \]
\[ \pi_{i,t} : \text{Proportion of wealth of agent } i \text{ invested in the risky asset at time } t; \]
\[ \rho_t : \text{The return on the risky asset at period } t. \]

It is assumed that\(^1\) the all the investors use the same utility function \( U(W) = \log(W) \). Following the above notation, the return on the risky asset at period \( t \) is then given by

\[ \rho_t = \frac{p_t - p_{t-1} + y_t}{p_{t-1}}. \] \hspace{1cm} (2.1)

\(^1\)The analysis conducted in this paper can be generalized to the case where the utility functions can be selected differently for different agents with different risk coefficients, say, \( U_i(W) = (W^{\gamma_i} - 1)/\gamma_i \) with \( 0 < \gamma_i < 1 \). However, to make the following analysis more tractable and transparent, the assumption that all investors have the same utility function \( U(W) = \log(W) \) is maintained.
2.1.1. Portfolio Optimization Problem. Following the framework of Brock and Hommes [5] and Levy and Levy [18], the wealth of investor $i$ at time period $t+1$ is given by

$$W_{i,t+1} = (1 - \pi_{i,t})W_{i,t}R + \pi_{i,t}W_{i,t}(1 + \rho_{t+1})$$

$$= W_{i,t}[R + \pi_{i,t}(\rho_{t+1} - r)]. \quad (2.2)$$

As in Brock and Hommes [5] and Levy and Levy [18], a Walrasian scenario is used to derive the demand equation, i.e., each trader is viewed as a price taker and the market is viewed as finding (via the Walrasian auctioneer) the price $p_t$ that equates the sum of these demand schedules to the supply. That is, the agents treat the period $t$ price, $p_t$, as parametric when solving their optimisation problem to determine $\pi_{i,t}$. Denote by $F_t = \{p_t, p_{t-1}, \ldots; y_t, y_{t-1}, \ldots\}$ the information set formed at time $t$. Let $E_t, V_t$ be the conditional expectation and variance, respectively, based on $F_t$, and $E_{i,t}, V_{i,t}$ be the “beliefs” of investor $i$ about the conditional expectation and variance. Then it follows from (2.2) that

$$E_{i,t}(W_{i,t+1}) = W_{i,t}[r + \pi_{i,t}(E_{i,t}(\rho_{t+1}) - r)],$$

$$V_{i,t}(W_{i,t+1}) = W_{i,t}^2\pi_{i,t}^2 V_{i,t}(\rho_{t+1}). \quad (2.3)$$

Consider the investor $i$, who faces a given price $p_t$, has wealth $W_{i,t}$ and believes that the asset return is conditionally normally distributed with mean $E_{i,t}(\rho_{t+1})$ and variance $V_{i,t}(\rho_{t+1})$. This investor chooses a proportion $\pi_{i,t}$ of his/her wealth to be invested in the risky asset so as to maximize the expected utility of the wealth at $t + 1$, as given by

$$\max_{\pi_{i,t}} E_{i,t}[U(W_{i,t+1})]. \quad (2.4)$$

It follows that the optimum investment proportion at time $t$, $\pi_{i,t}$ is given by

$$\pi_{i,t} = \frac{E_{i,t}(\rho_{t+1}) - r}{V_{i,t}(\rho_{t+1})}. \quad (2.5)$$

To introduce the heterogeneous beliefs, it is assumed that

$$E_{i,t}(\rho_{t+1}) = f_i(\rho_{t-1}, \ldots; \rho_{t-L_i}), V_{i,t}(\rho_{t+1}) = g_i(\rho_{t-1}, \ldots; \rho_{t-L_i})$$

with integers $L_i; f_i; g_i$ are some deterministic functions which can be differ across investors and $i = 1, \ldots, H$. Under this assumption, both $E_{i,t}(\rho_{t+1})$ and $V_{i,t}(\rho_{t+1})$ are functions of the past prices up to $t - 1$, which in turn implies the optimum wealth proportion $\pi_{i,t}$, defined by (2.5), is a function of the history of the prices $(p_{t-1}, p_{t-2}, \ldots)^3$.

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$^2$See appendix A.

$^3$In Levy and Levy [18], the hypothetical price $p_t$ is included in the above conditional expectations on the return and variance. In this case, the market clearing price is solved implicitly and is much more involved mathematically. The approach adopted here is the standard one in deriving the price via the Walrasian scenario and also keeps the mathematical analysis tractable. A similar approach has been adopted in Brock and Hommes [4], [5] and Chiarella and He [9], [11].
2.1.2. Market Clearing Equilibrium Price. The optimum proportion of investment in the risky asset, \( \pi_{i,t} \), determines the number of shares, \( N_{i,t} \), the investor \( i \) wishes to hold at price \( p_t \):

\[
N_{i,t} = \frac{\pi_{i,t} W_{i,t}}{p_t}.
\]

The number of shares investor \( i \) wishes to hold is his/her personal demand. Summing the demands of all investors gives us the aggregate demand. The total number of shares in the market, denoted by \( N \), is assumed to be fixed and hence the market clearing price \( p_t \) is determined by

\[
\sum_{i=1}^{H} N_{i,t} = \frac{\sum_{i=1}^{H} \pi_{i,t} W_{i,t}}{p_t} = N, \tag{2.6}
\]

i.e.,

\[
\sum_{i=1}^{H} \pi_{i,t} W_{i,t} = N p_t. \tag{2.7}
\]

Thus, in this model, as in real markets, the equilibrium price \( p_t \) and the wealth of the investors, \( W_t \equiv (W_{1,t}, \ldots, W_{H,t}) \), are determined simultaneously.

The optimum demand of the investors are functions of the price and their wealth. As observed in the financial market, both the price and the wealth are growing processes in general. Therefore the above model is regarded as a growth model in both the price and the wealth. Using numerical simulations, a simple price dynamic is demonstrated in Levy and Levy [18].\(^4\) However, to make possible theoretical analysis of the dynamics, a stationary model in terms of the proportion of the wealth among the investors and the return of the stock price is needed and this task is undertaken in the following subsection.

2.2. Stationary Model. The above growth model is rendered stationary by formulating it in terms of the relative proportions of the wealth among the investors and the return of the stock, instead of the wealth \( W_t \) and the stock price \( p_t \).

Let \( w_{i,t} \) be the proportion of wealth of the investor \( i \) relative to the total wealth of all the investors at \( t \), that is,

\[
w_{i,t} = \frac{W_{i,t}}{W_t} \quad \text{where} \quad W_t = \sum_{i=1}^{H} W_{i,t}. \tag{2.8}
\]

\(^4\)In Levy and Levy [18], the optimal proportion of wealth is determined by \( E_{i,t}[U(W_{i,t+1})] \), which is the average of the past utilities of the \( i \)-th investor using the past \( L \) observed returns; i.e. \( E_{i,t}[U(W_{i,t+1})] = (1/L) \sum_{k=1}^{L} U(W_{i,t}, p_{t-1-k}) \)
Then
\[
\begin{align*}
w_{i,t+1} &= \frac{W_{i,t+1}}{W_{t+1}} = \frac{W_{i,t}[R + (\rho_{t+1} - r)\pi_{i,t}]}{W_{t+1}} \\cr
&= \frac{w_{i,t}[R + (\rho_{t+1} - r)\pi_{i,t}]}{W_{t+1}/W_t}.
\end{align*}
\] (2.9)

Based on the growth model, a stationary model can be derived (and all the proofs can be found in appendices).

**Proposition 2.1.** The wealth proportions evolve according to
\[
w_{i,t} = \frac{w_{i,t-1}[R + (\rho_t - r)\pi_{i,t-1}]}{\sum_{j=1}^{H} w_{j,t-1}[R + (\rho_j - r)\pi_{j,t-1}]} (i = 1, \ldots, H),
\] (2.10)

where the return \( \rho_t \) is given by
\[
\rho_t = r + \frac{\sum_{i=1}^{H} w_{i,t-1}[(1 + r)(\pi_{i,t-1} - \pi_{i,t}) - \alpha_t(\pi_{i,t-1} - 1)]}{\sum_{i=1}^{H} \pi_{i,t-1}w_{i,t-1}(\pi_{i,t} - 1)}
\] (2.11)

with
\[
\alpha_t = \frac{y_t}{p_{t-1}} \quad \text{and} \quad \pi_{i,t} = \frac{E_{i,t}(\rho_{t+1}) - r}{V_{i,t}(\rho_{t+1})} (i = 1, \ldots, H).
\]

Assume the dividend yield \( \alpha_t \) follows a normal distribution with mean \( \alpha_o \) and variance \( \sigma_o^2 \), then the system (2.10)-(2.11) together forms a stationary model in the return and wealth proportion.

**Homogeneous Stationary Model.** As a simple application of the above derivation, assume that all the investors have homogeneous expectations on both mean and variance, that is
\[
E_{i,t}(\rho_{t+1}) \equiv E_t(\rho_{t+1}), \quad V_{i,t}(\rho_{t+1}) \equiv V_t(\rho_{t+1}) \quad \text{for} \quad i = 1, \ldots, H.
\] (2.12)

Then, under the assumption (2.12), the optimum investment proportion at time \( t \) is the same for all the investors, that is,
\[
\pi_t \equiv \frac{[E_t(\rho_{t+1}) - r]}{V_t(\rho_{t+1})} \quad \text{for} \quad i = 1, \ldots, H.
\]

Therefore, the resulting dynamics of \( (\rho_t, w_{i,t}) \) in (2.10) and (2.11) have the form
\[
\begin{align*}
\rho_t &= r + \frac{(1 + r)(\pi_{t-1} - \pi_t) - \alpha_t(\pi_{t-1} - 1)}{\pi_{t-1}(\pi_t - 1)} \\
\end{align*}
\] (2.13)
\[
\begin{align*}
w_{i,t} &= w_{i,t-1}(= w_{i,0}).
\end{align*}
\] (2.14)

The above analysis leads to the following corollary.

**Corollary 2.2.** Under the homogeneous expectations (2.12), the wealth proportion of each investor stays at his/her initial level while the return dynamics is governed by (2.13).
2.3. Fixed Proportions Model. Now suppose all the investors can be grouped in terms of their conditional expectations of mean and variance. That is, within a group, all the investors follow the same expectation schemes on the conditional mean and variance of the return \( \rho_{t+1} \) and hence the optimum wealth proportion \( \bar{\pi}_{i,t} \) invested in the risky asset for the investors are the same. Assume all the investors can be grouped as \( h \) types (or groups) and the group \( j \) has \( \ell_j \) investors for \( j = 1, \cdots, h \). Then \( \ell_1 + \cdots + \ell_h = H \). Denote \( n_j \) as the fixed proportion of the number of investors in the group \( j \) relative to the total number of the investors, \( H \), that is, \( n_j = \ell_j / H \), so that \( n_1 + \cdots + n_h = 1 \).

For the investors within the group \( j \), their optimum wealth proportions are the same, denoted \( \bar{\pi}_{j,t} \). Let \( \bar{W}_{j,t} \) be the average wealth of each investor within the group \( j \) (so that \( \ell_j \bar{W}_{j,t} \) gives the total wealth of the group \( j \)). Then the market clearing equilibrium price equation (2.7) can be rewritten as:

\[
\sum_{j=1}^{h} n_j \bar{\pi}_{j,t} \bar{W}_{j,t} = N p_t / H. \tag{2.15}
\]

Let \( \bar{W}_t = \sum_{j=1}^{h} \bar{W}_{j,t} \). Then

\[
W_t = \sum_{i=1}^{H} W_{i,t} = \sum_{j=1}^{h} \ell_j \bar{W}_{j,t}.
\]

Denote \( \bar{w}_{j,t} \) the average wealth proportion of the group \( j \) relative to the total average wealth \( \bar{W}_t \), i.e., \( \bar{w}_{j,t} = \bar{W}_{j,t} / \bar{W}_t \) (so that \( \sum_{j=1}^{h} \bar{w}_{j,t} = 1 \)). Then a stationary model with fixed proportions (among different groups) in terms of return and average wealth proportions can be obtained, as described by:

**Proposition 2.3.** The dynamics of the average wealth proportions evolve according to

\[
\bar{w}_{i,t} = \frac{\bar{w}_{i,t-1}[R + (\rho_t - r)\bar{\pi}_{i,t-1}]}{\sum_{j=1}^{h} \bar{w}_{j,t-1}[R + (\rho_t - r)\bar{\pi}_{j,t-1}]} \quad (i = 1, \cdots, h) \tag{2.16}
\]

with return \( \rho_t \) given by

\[
\rho_t = r + \frac{\sum_{i=1}^{h} n_i \bar{w}_{i,t-1}(1 + r)(\bar{\pi}_{i,t-1} - \bar{\pi}_{i,t-1})}{\sum_{i=1}^{h} n_i \bar{\pi}_{i,t-1}\bar{w}_{i,t-1}(\bar{\pi}_{i,t} - 1)}.
\]

Recall that the average wealth proportions \( \bar{\pi}_{i,t} \) are calculated according to (2.5). Assume the dividend yield \( \alpha_t \) follows a normal distribution with mean \( \alpha_o \) and variance \( \sigma_\alpha^2 \), then the system (2.16)-(2.17) together forms a stationary model.

This section has established two stationary dynamic models in terms of return and wealth proportions. However, because of the highly nonlinear nature of these models, the rest of this paper concentrates on a discussion of two simple models, the first is the homogeneous model and the second is the heterogeneous model with two agents.
3. Dynamics of homogeneous stationary model

In this section, as a special case of the heterogeneous model, the wealth dynamics with homogeneous investors is studied. The existence of steady-state is considered first. Then stability is analyzed using numerical simulations.

When agents are somewhat uncertain about the dynamics of the economic system in which they are to play out their roles, then they need to engage in some learning schemes. Among various learning schemes, the properties of least-squares learning processes under homogeneous expectations have been studied extensively (see, for example, Balasko and Royer [1], Bray [2], Evans and Honkapohja [13], Evans and Ramey [14], Lucas [21] and Marce and Sargent [27]). As in Balasko and Royer [1] (see also Chiarella and He [10] on the learning dynamics with both homogeneous and heterogeneous beliefs), it is assumed that agents’ expectations follow the finite least-squares learning processes in the following discussion.

Under homogeneous expectations, it is assumed that all agents hold the same view about the conditional return and variance of the return distributions. In particular it is here assumed that the common view can be expresses mathematically as\(^5\)

\[
E_t(\rho_{t+1}) = r + \delta + d\tilde{\rho}_t, \tag{3.1}
\]

\[
V_t(\rho_{t+1}) = \sigma^2 [1 + b [1 - (1 + \nu_t)^{-\xi}]], \tag{3.2}
\]

where \(\delta, \xi, \sigma^2\) and \(b\) are positive constants, \(d \in \mathbb{R}\) and

\[
\tilde{\rho}_t = \frac{1}{L} \sum_{k=1}^{L} \rho_{t-k}, \quad \nu_t = \frac{1}{L} \sum_{k=1}^{L} [\rho_{t-k} - \tilde{\rho}_t]^2 \tag{3.3}
\]

with the integer \(L \geq 1\) and constant \(0 \leq \xi\).

Equation (3.1) and (3.2) indicate that the agents common view on the conditional return distribution is influenced by the behaviour of returns over the last \(L\) time periods. Thus equation (3.1) states that the excess conditional mean (from the risk-free rate \(r\)) is calculated as a constant risk premium, measured by the parameter \(\delta\), plus an extrapolation of the average return. Equation (3.2) assumes that the conditional variance is an increasing bounded concave function of the historical variance. The concavity of this function indicates that agents are cautious about overreacting to large historical volatility. Depending on the parameter \(d\), the investor is called fundamentalist when \(d = 0\) and chartist when

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\(^5\)Assumption (3.1) can be written in a more general form, say, \(E_t(\rho_{t+1}) = f(r, d, \tilde{\rho}_t)\). In particular, a bounded function in \(\tilde{\rho}_t\) may be a more realistic choice. However, to make the analysis simple, a linear function is assumed here. As a result, an unrealistically low return may drive the price to zero. Thus further analysis on a suitable form of the function \(f\) seems necessary. A justification on (3.2) can be founded in Chiarella and He [11] and Franke and Sethi [16].
\(d \neq 0\). Chartists are further categorized as trend followers \((d > 0)\) or contrarians \((d < 0)\).

Under the homogeneous expectations (3.1) and (3.2), the corresponding stationary model is given by (2.13)-(2.14). Depending on the extrapolation rate \(d\), the system can have multiple equilibria.

**Theorem 3.1.** Let \(\tilde{\delta} = \delta / \sigma^2, \tilde{d} = d / \sigma^2\). Assume (3.1) and (3.2) hold and \(\alpha_t = \alpha_0 > 0, \tilde{\delta} < 1\).

- If \(d = 0\), then the system (2.13) has a steady-state
  \[
  \rho_o^* = r + \frac{\alpha_0}{1 - \tilde{\delta}},
  \]  
  which satisfies \(\rho_o^* > r + \alpha_0\).

- If \(d < 0\), then the system (2.13) has two equilibria
  \[
  \rho_+^* = \frac{1}{2} \left[ r + \frac{1 - \tilde{\delta}}{d} + \sqrt{\left[ r - \frac{1 - \tilde{\delta}}{d} \right]^2 - \frac{4\alpha_0}{d}} \right],
  \]
  \[
  \rho_-^* = \frac{1}{2} \left[ r + \frac{1 - \tilde{\delta}}{d} - \sqrt{\left[ r - \frac{1 - \tilde{\delta}}{d} \right]^2 - \frac{4\alpha_0}{d}} \right],
  \]
  which satisfy
  (1) \(\rho_-^* < 0\) (of course, \(\rho_-^*\) is an undesired equilibrium for investors) and
  \(\rho_+^*\) is a decreasing function of \(\tilde{d}\);
  (2) \(r < \rho_+^* \leq \rho_o^*\) and \(\rho_+^*\) is an increasing function of \(\tilde{d}\) satisfying \(\rho_+^* \to \rho_o^*\) as \(\tilde{d} \to 0^+\).

- If \(d > 0\), then \(\rho_+^*\) defined by (3.5) and (3.6) are also the equilibria of the system (2.13) if and only if
  \[
  \left[ r - \frac{1 - \tilde{\delta}}{d} \right]^2 \geq \frac{4\alpha_0}{d}.
  \]

Furthermore, whenever exist, they satisfy
(1) \(0 < \rho_o^* \leq \rho_+^* \) for \(\tilde{d} \leq d_{oL} \equiv \frac{(1 - \tilde{\delta})^2}{r [1 - \tilde{\delta}^2 + 2\alpha_o]}\) and \(\rho_-^* \to \rho_o^* \) as \(\tilde{d} \to 0^+\);
(2) \(0 < \rho_+^* < \rho_-^* \) for \(\tilde{d} > d_{oL} \equiv \frac{(1 - \tilde{\delta})^2}{r [1 - \tilde{\delta}] + \alpha_o}\);
(3) \(\rho_+^*\) increases in \(\tilde{d}\) for \(0 < \tilde{d} < d_{oL}\) and decreases for \(\tilde{d} \geq d_{oL}\);
(4) \(\rho_-^*\) decreases in \(\tilde{d}\) for \(0 < \tilde{d} < d_{oL}\) and increases for \(\tilde{d} \geq d_{oL}\).

Theorem 3.1 characterizes various situations on the existence of steady-states when the dividend yield is assumed to be a constant \(\alpha_o\) and the risk premium \(\tilde{\delta}\) is less than the constant variance \(\sigma^2\). The existence and number of equilibrium depends on the extrapolation rate \(\tilde{d}\) and is indicated in Fig.3.2.

When there is no extrapolation, all investors are fundamentalists — believing that the excess conditional mean for the risky asset is given by the risk premium \(\tilde{\delta}\). In this case, the steady-state \(\rho_o^*\) indicated in (3.4) is the only one, which can be called a fundamental equilibrium. Moreover, the expression of \(\rho_o^*\) illustrates
that the excess return is higher than the dividend yield, also the higher the risk
premium $\delta$, the higher is the steady-state return.

When the extrapolation rate $\tilde{d} < 0$, all investors act as contrarians — believing
that the difference of excess conditional mean and the risk premium $[E_t(\rho_{t+1}) - r] - \delta$ is proportional negatively to the moving returns over the last $L$ time
periods. In this case, $\rho^*_+ \rho^*_-$ is the only positive steady-state and it can be called non-
fundamental (contrarian) equilibrium. More importantly, this non-fundamental
equilibrium satisfies $\rho^*_+ < \rho^*_-$, which indicates that, with the same risk premium,
the non-fundamental (contrarian) equilibrium of return is below the fundamental
equilibrium.

When the extrapolation rate $\tilde{d} > 0$, all investors act as trend chasers —
believing that the difference of excess conditional mean and the risk premium $[E_t(\rho_{t+1}) - r] - \delta$ is proportional positively to the moving returns over the last $L$
time periods. In this case, under the condition (3.7), the return has two positive
equilibria, which are called non-fundamental (trend chasing) equilibria. Furthermore,
the two non-fundamental (trend chasing) equilibria are above (below) the fundamental
equilibrium $\rho^*_o$ when $\tilde{d}$ is small (large).

The local stability of equilibrium of the homogeneous model depends on the lag
length $L$ and the parameters such as the risk premium $\delta$, extrapolation rate $\tilde{d}$ and
the variance $\sigma^2$, in general and is determined by the corresponding characteristic
equation.

**Proposition 3.2.** Let $\delta = \delta/\sigma^2$, $\tilde{d} = d/\sigma^2$. Assume (3.1) and (3.2) hold and
$\alpha_t = \alpha_o > 0$, $\delta < 1$. Then the equilibrium return of the homogeneous system
(2.13), say $x_o$, is locally asymptotically stable if and only if all the eigenvalues $\lambda_i$
of the polynomial

$$\Gamma_L(\lambda) \equiv \lambda^{L+1} - \frac{\tilde{d}}{L} \left[(C - A)\lambda^L + C\lambda^{L-1} + C\lambda^{L-2} + \cdots + C\lambda + A\right]$$  (3.8)

satisfy $|\lambda_i| < 1$ for $i = 1, \cdots, L + 1$, in which

$$A = \frac{1 + r}{\pi_o(\pi_o - 1)}, \quad C = \frac{\alpha_o}{(\pi_o - 1)^2}, \quad \pi_o = \delta + \tilde{d} x_o.$$  (3.9)

For realistic lag length $L$ (e.g. $L \geq 4$), a theoretical analysis of the stability
of various fundamental and non-fundamental equilibria becomes analytically
intractable. Here, a numerical approach is employed in the following discussion.

As in Levy and Levy [18], in the following simulations, the time period between
each trade is one year. Using the figures that existed in the United States during
the 1926-94 period, as reported by Ibbotson Associates, the annual risk-free
interest rate, $r = 3.7\%$, corresponds to the average rate during that period. The
initial history of rates of return on the stock consists of a distribution with a mean
of 12.2\% and a standard derivation of 20.4\%. A dividend yield of $\alpha_o = 4.7\%$ cor-
responds to the historical average yield on the $S&P500$. The initial share price
is $p_o = $4.00.
With a selection \( L = 15 \), Fig. 3.1(a) shows the time series of returns \( \rho_t \) for \( \tilde{d} = 0, 1, -1 \), respectively, without noise, while Fig. 3.1(b)-(d) show the corresponding time series with a noise dividend yield process \( \alpha_t = \alpha_o + \mathcal{N}(0, q) \), where \( \mathcal{N}(0, q) \) is a normal distribution with mean 0 and standard derivation \( q \).

\[ \text{Figure 3.1. Time series of homogeneous model without noise (a) and with noise (b)-(d)} \]

When \( d = 0 \), it can be shown that\(^6\) the fundamental equilibrium \( \rho^*_0 \) is locally asymptotically stable. Numerical simulations on the original nonlinear system also demonstrate the convergence of the fundamental equilibrium with different initial values, suggesting that it is globally asymptotically stable, as indicated by Fig. 3.1(a). For \( \tilde{d} = 0.5 \), in terms of price, it grows geometrically at the rate of \( \rho^*_0 = 13.1\% \).

For \( \tilde{d} = -1 \), it is found that \( \rho^*_- = -0.5766 \) and \( \rho^*_+ = 0.1136 \). The solutions \( \rho_t \) with various initial values converge to the unique positive non-fundamental equilibrium \( \rho^*_+ = 0.1136 (< \rho^*_0) \), as shown in Fig. 3.1(a). This convergence holds for \( \tilde{d} \in (-1.5, 0) \). However, for \( \tilde{d} < -1.6 \), the non-fundamental equilibrium becomes unstable.

For \( \tilde{d} = 1 \), the two positive non-fundamental equilibria are \( \rho^*_- = 0.1873 \) and \( \rho^*_+ = 0.3497 \). In this case, the solutions \( \rho_t \) with various initial values converge to the \( \rho^*_+ (> \rho^*_0 = 0.1136) \), indicated in Fig. 3.1(a). This convergence holds for \( \tilde{d} = 1 \), it can be shown that\(^6\) the fundamental equilibrium \( \rho^*_0 \) is locally asymptotically stable. Numerical simulations on the original nonlinear system also demonstrate the convergence of the fundamental equilibrium with different initial values, suggesting that it is globally asymptotically stable, as indicated by Fig. 3.1(a). For \( \tilde{d} = 0.5 \), in terms of price, it grows geometrically at the rate of \( \rho^*_0 = 13.1\% \).

For \( \tilde{d} = -1 \), it is found that \( \rho^*_- = -0.5766 \) and \( \rho^*_+ = 0.1136 \). The solutions \( \rho_t \) with various initial values converge to the unique positive non-fundamental equilibrium \( \rho^*_+ = 0.1136 (< \rho^*_0) \), as shown in Fig. 3.1(a). This convergence holds for \( \tilde{d} \in (-1.5, 0) \). However, for \( \tilde{d} < -1.6 \), the non-fundamental equilibrium becomes unstable.

For \( \tilde{d} = 1 \), the two positive non-fundamental equilibria are \( \rho^*_- = 0.1873 \) and \( \rho^*_+ = 0.3497 \). In this case, the solutions \( \rho_t \) with various initial values converge to the \( \rho^*_+ (> \rho^*_0 = 0.1136) \), indicated in Fig. 3.1(a). This convergence holds for

\(^6\)When \( b = 0 \), then \( V_t (\rho_{t+1}) \) is constant and one can see that the optimum proportion \( \pi_t \) is a constant and so is \( \rho_t \) (in this case, \( \rho^*_0 \) is globally stable). When \( b > 0 \), the equation (2.13) corresponds to an \( L + 2 \)-dimensional system, whose linearized system has all zeros as eigenvalues. This indicates local asymptotic stability of \( \rho^*_0 \).
\( \bar{d} \in (0, 1.1) \). However, for \( \bar{d} > 1.2 \), the two non-fundamental equilibria become unstable.

Fig. 3.1(b)-(d) show the time series of the return with stochastic dividend process and \( \bar{d} = -1, 0, 1 \), respectively. Here the standard deviation of the dividend yield is \( q = 0.01 \). These figures suggest that the noise affects the dynamics significantly when all the investors are trend followers, compared with the case when all are fundamentalists or all are contrarians.

In the above simulations, \( L = 15 \) is used. When \( L = 1 \), numerical simulations indicate that the above convergence (without the noise process) holds for \( \bar{d} \in (-0.11, 0.24) \). Compared with the stability interval \((-1.5, 1.1)\) for \( L = 15 \), it suggests that an increase in lag length \( L \) leads to an increase of convergence interval and also has a stabilizing role on the dynamics.

The existence and stability of various equilibrium in the homogeneous model can be summarised in Fig. 3.2.

![Figure 3.2. Existence and stability of homogeneous model](image)

**Figure 3.2.** Existence and stability of homogeneous model

### 4. A Two Heterogeneous Investors Model

To study the price and wealth dynamics with heterogeneous investors, this section focuses on the simplest possible case where there are only two investors involved. Understanding of the dynamics of the interaction in this simplest case is essential to future study of the more general case. The investors are classified as either fundamentalists or chartists.

For \( i = 1, 2 \), assume

\[
E_{i,t}(\rho_{t+1}) = r + \delta_i + d_i \rho_{i,t},
\]

\[
V_{i,t}(\rho_{t+1}) = \sigma^2[1 + b_i[1 - (1 + v_{i,t})^{-\gamma_i}]],
\]

where \( \delta_i \) and \( v_{i,t} \) are the expected returns and the volatility coefficients for investor \( i \), respectively. The parameter \( \gamma_i \) controls the degree of heterogeneity. If \( \gamma_i = 1 \), the investors are identical; if \( \gamma_i \to 0 \), the two investors become completely dissimilar.

The equilibrium values are determined by solving the system of equations

\[
\begin{align*}
E_{1,t}(\rho_{t+1}) &= E_{2,t}(\rho_{t+1}) = r + \delta + d \rho_{t}, \\
V_{1,t}(\rho_{t+1}) &= V_{2,t}(\rho_{t+1}) = \sigma^2[1 + b[1 - (1 + v_{t})^{-\gamma}]].
\end{align*}
\]
where $\delta_i$, $d_i$, $\xi_i, \sigma^2$ and $b_i$ are constants and

$$
\rho_{i,t} = \frac{1}{L_i} \sum_{k=1}^{L_i} \rho_{t-k}, \quad v_{i,t} = \frac{1}{L_i} \sum_{k=1}^{L_i} [\rho_{t-k} - \rho_{i,t}]^2
$$

(4.3)

with the integers $L_i \geq 1$.

Denote $w_t = w_{1,t} - w_{2,t}$. Then

$$
w_{1,t} = \frac{1 + w_t}{2}, \quad w_{2,t} = \frac{1 - w_t}{2}.
$$

(4.4)

Assume that the dividend yield $\alpha_t = \alpha_o$. It then follows from Proposition 2.1 that the dynamics of $(w_t, \rho_t)$ are determined by

$$
w_t = \frac{[1 + w_{t-1}] [R + (\rho_t - r) \pi_{1,t-1}] - [1 - w_{t-1}] [R + (\rho_t - r) \pi_{2,t-1}]}{[1 + w_{t-1}] [R + (\rho_t - r) \pi_{1,t-1}] + [1 - w_{t-1}] [R + (\rho_t - r) \pi_{2,t-1}]},
$$

(4.5)

$$
\rho_t = r + \frac{[1 + w_{t-1}] [(1 + r)(\pi_{1,t-1} - \pi_{1,t}) - \alpha_t \pi_{1,t-1}]}{[1 + w_{t-1}] [(\pi_{1,t-1} - 1) + \pi_{2,t-1} [1 - w_{t-1}] (\pi_{2,t-1} - 1)]} + \frac{[1 - w_{t-1}] [(1 + r)(\pi_{2,t-1} - \pi_{2,t}) - \alpha_t \pi_{2,t-1}]}{[1 + w_{t-1}] [(\pi_{1,t-1} - 1) + \pi_{2,t-1} [1 - w_{t-1}] (\pi_{2,t-1} - 1)]},
$$

(4.6)

where

$$
\pi_{i,t} = \frac{\delta_i + d_i \rho_t}{\sigma^2 [1 + b_i [1 - (1 + v_{i,t})^{-\xi}]], \quad i = 1, 2.
$$

(4.7)

Depending on the selections of the parameters $d_1$ and $d_2$, the system (4.5) and (4.6) characterise an interaction among fundamentalists and chartists. The following discussion is devoted to study of four simple cases: fundamentalists versus fundamentalists (with different views on the risk premia), fundamentalists versus trend follower, fundamentalist versus contrarian and two trend traders.

4.1. **Fundamentalists with different risk premia.** For $i = 1, 2$, when $d_i = 0$, trader $i$ is called a fundamentalist or informed trader who believes that the conditional excess return of the stock from the risk free rate $r$ is given by a risk premium $\delta_i$. Here it is assumed that both investors are fundamentalists, but with different views on the risk premia $\delta_i$.

The following notations are used throughout this section:

$$
\bar{\delta}_i = \frac{\delta_i}{\sigma^2} \quad \bar{d}_i = \frac{d_i}{\sigma^2} \quad \text{for} \quad i = 1, 2.
$$

**Theorem 4.1.** Assume (4.1) and (4.2) hold and $\bar{\delta}_1 < 1, \bar{d}_1 = 0$ for $i = 1, 2$. Then the system (4.5) and (4.6) has two steady-states $E_i(w^*_i, \rho^*_{o,i})$ for $i = 1, 2$ with

$$
w^*_1 = 1, w^*_2 = -1, \rho^*_{o,i} = r + \frac{\alpha_o}{1 - \delta_i} \quad i = 1, 2.
$$

In particular, if $\bar{\delta}_1 = \bar{\delta}_2 = \bar{\delta}$, then $E_3(w^*_3, \rho^*_3)$ is also a steady-state, where $w^*_3 \equiv w_o = w_{1,o} - w_{2,o}$ and $\rho^*_o = r + \frac{\alpha_o}{1 - \delta}$. 
Theorem 4.1 implies that, at the equilibrium, the whole wealth go to only one of the investors and there is no coexistence. Furthermore, \( \rho_{0,1}^* < \rho_{0,2}^* \) iff \( \bar{\delta}_1 < \bar{\delta}_2 \). To study the local stability of the equilibrium, the following Proposition, which is hold in general case of two heterogeneous investors model, will be used.

**Proposition 4.2.** Assume (4.1) and (4.2) hold. Let \( L = \max \{L_1, L_2\} \) and \( E(w^*, \rho^*) \) be an equilibrium of the system (4.5) and (4.6) with \( w^* = \pm 1 \). Then the characteristic polynomial of the linearized system of (4.5) and (4.6) is given by \( \Gamma_{L_i}(\lambda) \) when \( w^* = 1 \) and \( \Gamma_{L_2}(\lambda) \) when \( w^* = -1 \), where

\[
\Gamma_{L_i}(\lambda) \equiv (\lambda - \beta)\lambda^{L_i-1} - \frac{d_i}{L_i} [(C_i - A_i)\lambda^{L_i-1} + C_i\lambda^{L_i-2} + \cdots + C_i\lambda + A_i]
\]

and

\[
A_i = \frac{1 + r}{\pi_{i,o} (\pi_{i,o} - 1)}, \quad C_i = \frac{\alpha_o}{(\pi_{i,o} - 1)^2}, \quad \pi_{i,o} = \bar{\delta}_i + d_i \rho^*, \quad \beta = \frac{(1 + r) + (\rho^* - r)\pi_{2,o}}{(1 + r) + (\rho^* - r)\pi_{1,o}}
\]

for \( i = 1, 2 \).

It is in general not a easy task to obtain explicit conditions, in terms of the parameters and lags of the system, on the local stability of the fixed equilibrium. However, Proposition 4.2 has two significant consequences:

- Noting that \( \beta \) is an eigenvalue of \( L_1(\lambda) \) while \( 1/\beta \) is an eigenvalue of \( P_{-1}(\lambda) \). Thus, if all of the fixed equilibria can be classified as two groups by either \( w^* = 1 \) or \( w^* = -1 \) (which is indeed the case for all the two heterogeneous traders model considered in this section), then the local stability of one of the equilibrium from one group implies the instability of all the equilibrium from the other group. This property is called *mutually exclusive stability principle*.

- Noting that the polynomials in the big brackets in \( L_{\pm}(\lambda) \) has the exactly same form as the one for the homogeneous model in (3.8) of Proposition 3.2. Therefore, when the equilibrium (equilibria) of one group is unstable, the local stability of the equilibrium (equilibria) of the other group is totally determined by the corresponding homogeneous model where both traders follow the homogeneous expectations. This property is called *stability decomposition principle*.

Applying these principles to Theorem 4.1, one obtains the following corollary.

**Corollary 4.3.** Under the assumptions of Theorem 4.1, a necessary condition for \( E_1 \) to be stable is \( \delta_1 > \delta_2 \), or equivalently, \( \rho_{0,1}^* > \rho_{0,2}^* \). Furthermore, when this
condition is satisfied, the stability of \( E_1 \) is equivalent to the stability of the (homogeneous) fundamental equilibrium where trader 2 follows the same expectations of trader 1.

As in the last section, let \( r = 3.7\% \), \( \alpha = 4.7\% \) be selected in numerical simulations. It shows that all the solution \( \rho_t \) converge to \( \rho^*_0,1 (\rho^*_0,2) \) whenever \( \delta_1 > (\leq) \delta_2 \). For a fixed \( \delta_1 \), the existence of multiple equilibria and the stability switching property can be illustrated in Fig.4.1. It demonstrates an optimal selection principle in the model of two fundamentalists with different risk premia — the investor with the higher risk premium will be the winner. However, when both investors have the same risk premium, investors are homogeneous in their conditional expectations on mean and variance of the return and, of course, the wealth proportions of both investors will stay at their initial level. Further simulations demonstrate that there is not much difference for different lag lengths \( L_1 \) and \( L_2 \) in this model.

\[ \rho_t \]

\[ \rho^*_0,1 \]

\[ \rho^*_0,2 \]

\[ \tilde{\delta}_2 = \tilde{\delta}_1 \]

\[ \tilde{\delta}_2 \]

**Figure 4.1.** Two fundamentalists model—Existence of multiple equilibria and stability (the solid curve indicates the locally stable equilibrium and the dotted curve indicates the unstable equilibrium).

### 4.2. Fundanentalist versus contrarian

In this subsection, it is assumed that \( d_1 = 0 \) and \( d_2 \leq 0 \). That is trader 1 is a fundamentalist (or informed trader) while trader 2 is a contrarian who extrapolates both the excess return of the stock from the risk-free rate \( r \) and variance by using history of past \( L_2 \) observed returns.

**Theorem 4.4.** Assume (4.1) and (4.2) hold and \( \tilde{\delta}_1 = 0, \tilde{\delta}_2 < 0 \) and \( \tilde{\delta}_i < 1 \) for \( i = 1, 2 \). Then the system (4.5)-(4.6) has three steady-states \( E_1 (\rho^*_1, w^*_1) \) and
\( E_{2,\pm}(\rho_{1,\pm}^*, w_2^*) \) with
\[
\begin{align*}
  w_1^* &= 1, \quad w_2^* = -1, \quad (4.12) \\
  \rho_1^* &= r + \frac{\alpha_o}{1 - \delta_1}, \quad (4.13) \\
  \rho_{2,\pm}^* &= \frac{1}{2} \left[ r + \frac{1 - \delta_2}{d_2} \pm \sqrt{\left( r + \frac{1 - \delta_2}{d_2} \right)^2 - \frac{4\alpha_o}{d_2}} \right], \quad (4.14)
\end{align*}
\]
Moreover,
- \( \rho_1^* > r + \alpha_o, \rho_{2,\pm}^* < 0 \) and \( r < \rho_{2,\pm}^* \leq \rho_2^* \equiv r + \alpha_o/(1 - \delta_2) \) with \( \rho_{2,\pm}^* \to \rho_2^* \) as \( d_2 \to 0^- \);
- If \( \delta_1 \geq \delta_2 \) then \( \rho_1^* \geq \rho_{2,\pm}^* \);
- If \( \delta_1 < \delta_2 \), then \( \rho_1^* \geq \rho_{2,\pm}^* \) iff
\[
\frac{1}{1 - \delta_1} \left[ r + \frac{\alpha_o}{1 - \delta_1} \right] \geq - \frac{1}{d_2} \left[ 1 - \frac{1 - \delta_2}{1 - \delta_1} \right]. \quad (4.15)
\]
Furthermore, if \( (\delta_1 - \delta_2)/d_2 = r + \alpha/(1 - \delta_2) \), then \( E_4(\rho_1^*, w_4^*) \) is also an equilibrium, where \( \rho_1^* = (\delta_1 - \delta_2)/d_2 \) and \( w_4^* = w_o \equiv w_{1.o} - w_{2.o} \).

Similarly, applying the principles to Theorem 4.4, one obtains the following necessary conditions for the stability.

**Corollary 4.5.** Under the assumptions of Theorem 4.4, a necessary condition for \( E_1(E_{2,\pm}) \) to be stable is \( \delta_1 > (\leq) \delta_2 + d_2 \rho_{2,\pm}^* \). Furthermore, when this condition is satisfied, the stability of \( E_1(E_{2,\pm}) \) is equivalent to the stability of the homogeneous fundamental (non-fundamental) equilibrium where trader 2 (trader 1) follows the same expectations of trader 1 (trader 2).

In this model, there are only two positive equilibria, one is the fundamental equilibrium \( E_1 \) and the other is the non-fundamental equilibrium \( E_{2,+} \). Let \( r = 0.037 \) and \( \alpha = 0.047 \) be selected as before and \( \delta_1 = 0.5, \delta_2 = 0.6 \). Then \( \rho_1^* > \rho_{2,+}^* \) for \( \delta_2 < -0.7634 \) and \( \rho_1^* < \rho_{2,+}^* \) for \( 0 > \delta_2 > -0.7634 \). Numerical simulations results show that \( E_1 \) is stable for \( \delta_2 < -0.7634 \) and \( E_{2,+} \) is stable for \( 0 > \delta_2 > -0.7634 \). In other words, the dynamics of the return \( \rho_t \) depends on the relative levels of these two equilibria and follows a similar optimal selection principle: \( E_1 \) (\( E_{2,+} \)) is stable whenever \( \rho_1^* \geq (\leq) \rho_{2,+}^* \).

In the case of \( \delta_1 \geq \delta_2 \), that is the fundamentalist has a higher risk premium compared with the contrarian, the return and wealth proportion \((\rho_t, w_t)\) converge eventually to the fundamental equilibrium. However, when the fundamentalist has a lower risk premium than the contrarian, the dynamics of \((\rho_t, w_t)\) depends on the extrapolation rate of the contrarian, indicated by the condition (4.15). Note that \( \rho_{2,+}^* \) is an increasing function for \( \delta_2 < 0 \). The fundamental equilibrium is stable for a large range of values of the extrapolation rate (in terms of the absolute value), while the contrarian equilibrium is stable for a small range of values of the extrapolation rate. In general, the existence of multiple positive
equilibria and their stability (in terms of the return $\rho_t$) can be characterized by Fig. 4.2.

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{fig1.png}
\caption{Fundamentalist and contrarian model—Existence of multiple positive equilibria and their stability (the solid curve indicates the locally stable equilibrium and the dotted curve indicates the unstable equilibrium).}
\end{subfigure}
\end{figure}

4.3. **Fundamentalist versus trend followers.** In this part, it is assumed that $d_1 = 0$ and $d_2 > 0$. That is trader 1 is a fundamentalist while trader 2 is a trend follower.

**Theorem 4.6.** Assume (4.1) and (4.2) hold and $\bar{\alpha}_i = 0$, $\bar{\alpha}_2 > 0$ and $\bar{\delta}_1 < 1$ for $i = 1, 2$. Then $E_1(\rho^*_1, w^*_1)$ (defined as in (4.12) and (4.13)) is a steady-state of the system (4.5)-(4.6). Also, $E_{2,+}$ and $E_{2,-}$, defined by (4.12) and (4.14), are equilibria iff

$$[r - \frac{1 - \bar{\delta}_2}{d_2}]^2 \geq \frac{4\alpha_o}{d_2}. \quad (4.16)$$

Furthermore, when (4.16) is satisfied,

- $0 < \rho^*_2$;
- if $\bar{\delta}_1 \leq \bar{\alpha}_2$, then $\rho^*_{2,+} > \rho^*_{2,-} > \rho^*_1$ iff
  $$0 < \bar{\alpha}_2 < \bar{\delta}_2 \equiv \frac{(1 - \bar{\delta}_1)(1 - \bar{\alpha}_2)}{r(1 - \bar{\delta}_1) + 2\alpha_o}. \quad (4.17)$$
- if $\bar{\delta}_1 > \bar{\alpha}_2$, then $\rho^*_{2,+} > \rho^*_{2,-} > \rho^*_1$ iff
  $$\frac{(\bar{\delta}_1 - \bar{\alpha}_2)(1 - \bar{\delta}_1)}{r(1 - \bar{\delta}_1) + \alpha_o} \equiv \bar{\delta}_2 < \bar{\delta}_2 < \bar{\delta}_2 \equiv \frac{(1 - \bar{\delta}_1)(1 - \bar{\delta}_2)}{r(1 - \bar{\delta}_1) + 2\alpha_o}. \quad (4.18)$$
- if $(\bar{\delta}_1 - \bar{\delta}_2)/\bar{\alpha}_2 = r + \alpha/(1 - \bar{\delta}_1)$, then $E_4(\rho^*_1, w^*_1)$ is also an equilibrium, where $\rho^*_1 = (\bar{\delta}_1 - \bar{\delta}_2)/\bar{\delta}_2$ and $w^*_1 = w_o = w_{1,0} - w_{2,0}$.

Let $r = 0.037$ and $\alpha = 0.047$ be selected as before and $\bar{\delta}_2 = 0.5$. Then (4.16) is satisfied for $\bar{\delta}_2 \in (0,1.11)$. Consider two cases: (i) let $\bar{\delta}_1 = 0.4(\leq \bar{\delta}_2)$, then $d_{2U} = 2.58$. In this case, Theorem 4.6 indicates that $0 < \rho^*_2 < \rho^*_{2,-}$ for
\( d \in (0, 1.11); \) (ii) let \( \bar{\delta}_1 = 0.6(> \bar{\delta}_2) \), then \( d_{2L} = 0.64725 \) and \( d_{2U} = 1.838 \). It follows from Theorem 4.6 that \( \rho_1^* > \rho_{2,-}^* > 0 \) for \( d \in (0, d_{2L}) \) and \( 0 < \rho_1^* < \rho_{2,-}^* \) for \( \bar{d} \in (d_{2L}, 1.11) \). In general, numerical simulations on the stability of the equilibria can be summarised as following:

- The two non-fundamental equilibria exist for either small or large extrapolation rate and they are unstable for large extrapolation rate;
- For small extrapolation rate \( \bar{d}_2 \) (say \( \bar{d}_2 \in (0, 1.11) \) discussed above), either \( E_1 \) or \( E_{2,-} \) is stable and the stability switching of \( (\rho_t, w_t) \) follows a \((\text{quasi}) \) optimal selection principle: \( (\rho_t, w_t) \) converge to \( E_1 \) \((E_{2,-}) \) whenever \( \rho_1^* > (<) \rho_{2,-}^* \) and \( E_{2,+} \) is always unstable.

The stability of the various equilibria is illustrated in Fig. 4.3.

**Figure 4.3.** Stability of fundamentalist and trend follower model

### 4.4. Two trend traders.

In this part, it is assumed that \( d_i \neq 0 \) for \( i = 1, 2 \). That is trader \( i \) is a trend trader: trend follower when \( d_i > 0 \) and contrarian when \( d_i < 0 \).

**Theorem 4.7.** Assume \( \bar{d}_i \neq 0 \) and \( \bar{\delta}_i < 1 \) for \( i = 1, 2 \). Let

\[
\Delta_i \equiv r \frac{1 - \bar{\delta}_i^2}{d_i^2} - \frac{4\alpha_o}{d_i^2} \quad (i = 1, 2). \tag{4.19}
\]

If \( \Delta_i > 0 \) for \( i = 1, 2 \). Then the system (4.5)-(4.6) has four steady-states \( E_i(\rho_i^*, w_i^*) \) \((i = 1, 2) \) with

\[
w_1^* = 1, \quad w_2^* = -1, \tag{4.20}
\]

\[
\rho_{i,\pm}^* = \frac{1}{2} \left[ r \frac{1 - \bar{\delta}_i^2}{d_i^2} \pm \sqrt{\left( r \frac{1 - \bar{\delta}_i^2}{d_i^2} - \frac{4\alpha_o}{d_i^2} \right)^2} \right]. \tag{4.21}
\]
Furthermore, if

\[ \Delta = \frac{\tilde{\delta}_1/d_1 - \tilde{\delta}_2/d_2}{1/d_1 - 1/d_2} \]

satisfies \((r - \Delta)(1 - \Delta) + \alpha_o = 0\), then \(E_3(\rho_3^s, w_3^s)\) is also an equilibrium, where \(\rho_3^s = \Delta\) and \(w_3^s = w_o \equiv w_{1,o} - w_{2,o}\).

![Figure 4.4. Stability of models with two contrarians (a), two trend followers (b) and contrarian and trend follower (c)](image)

The two trend traders can be either contrarian or trend follower or both.

- **Two contrarians**: In this case, \(\tilde{d}_i < 0\) for \(i = 1, 2\) and \(\Delta_i > 0\) is satisfied. Then the system has only two positive non-fundamental equilibria: \(E_{1,+}\) and \(E_{2,+}\). Also, \(\rho_{i,+}^s\) is an increasing function of \(\tilde{d}_i\) for \(i = 1, 2\). Thus there are two cases — either \(\rho_{i,+}^s \geq \rho_{2,+}^s\) or \(\rho_{i,+}^s < \rho_{2,+}^s\). Say, for example, let

\[ r = 0.037, \alpha_o = 0.047, \tilde{\delta}_1 = 0.4, \tilde{\delta}_2 = 0.5, \tilde{d}_1 = -1. \]

Then \(\rho_{1,+}^s \geq \rho_{2,+}^s\) for \(\tilde{d}_2 \leq -2\) and \(\rho_{1,+}^s < \rho_{2,+}^s\) for \(\tilde{d}_2 \in (-1.9, 0)\). Numerical simulations on the stability can be summarized in Table 1 and the stability switching behaviour is indicated in Fig. 4.4(a). Essentially, the stability follows the optimal selection principle.

| \(\tilde{d}_2\) | \((\rho_{1,+}^s, \rho_{2,+}^s)\) | \((\rho_t, w_t)\) | \(
\begin{array}{c|c|c}
\hline
\tilde{d}_2 & (\rho_{1,+}^s, \rho_{2,+}^s) & (\rho_t, w_t) \\
\hline
\tilde{d}_2 < -2 & (\rho_{1,+}^s \geq \rho_{2,+}^s) & E_{1,+} \\
\tilde{d}_2 \in (-1.9, -1.6) & (\rho_{1,+}^s < \rho_{2,+}^s) & \text{both unstable} \\
\tilde{d}_2 \in (-1.6, 0) & (\rho_{1,+}^s < \rho_{2,+}^s) & E_{2,+} \\
\hline
\end{array}
\) |

**Table 1. Stability of two contrarians model**
• **Two trend followers:** In this case, $\tilde{d}_i > 0$ for $i = 1, 2$. When $\Delta_i > 0$, $E_{i, \pm}$ exist. As in the case of homogeneous model, $E_{i,+}$ is always unstable and, for small extrapolation rates $\tilde{d}_i$, the local stability switches between $E_{1,-}$ and $E_{2,-}$. Let

$$r = 0.037, \alpha_o = 0.047, \delta_1 = 0.4, \delta_2 = 0.5.$$  

Then $E_{1,\pm}$ exist for $\tilde{d}_1 \in (0, 1.56)$ and $E_{2,\pm}$ exist for $\tilde{d}_2 \in (0, 1.11)$. Let $\tilde{d}_1 = 1$ be fixed. Then $\rho_{2,-} < \rho_{1,-}^*$ for $\tilde{d}_2 \in (0, 0.28)$ and $\rho_{2,-}^* > \rho_{1,-}^*$ for $\tilde{d}_2 \in (0.28, 1.11)$. Numerical simulations on the stability can be summarized in Table 2 and the stability switching behaviour is demonstrated in Fig. 4.4(b). Essentially, the stability follows the *quasi-optimal selection principle.*

<table>
<thead>
<tr>
<th>$\tilde{d}_2$</th>
<th>$(\rho_{1,-}^{<em>}, \rho_{2,-}^{</em>})$</th>
<th>$(\rho_t, w_t) \to$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{d}_2 \in (0, 0.28)$</td>
<td>$\rho_{1,-}^{<em>} &gt; \rho_{2,-}^{</em>}$</td>
<td>$E_{1,-}$</td>
</tr>
<tr>
<td>$\tilde{d}_2 \in (0.28, 1.11)$</td>
<td>$\rho_{1,-}^{<em>} &lt; \rho_{2,-}^{</em>}$</td>
<td>$E_{2,-}$</td>
</tr>
</tbody>
</table>

**Table 2. Stability of two trend followers model**

• **Contrarian and trend follower:** In this case, assume that $\tilde{d}_1 < 0$ and $\tilde{d}_2 > 0$. When $\Delta_2 > 0$, $E_{2,\pm}$ exist and the system has three positive non-fundamental equilibria $E_{1,+}$, $E_{2,\pm}$. As in the case of the homogeneous model, $E_{2,+}$ is always unstable and, for small extrapolation rates $\tilde{d}_i$ (in terms of the absolute value), the local stability switches between $E_{1,+}$ and $E_{2,-}$. Let $r = 0.037, \alpha_o = 0.047$. Then, for different $(\delta_1, \delta_2)$ and $(\tilde{d}_1, \tilde{d}_2)$, the numerical simulations on the stability are summarized in Table 3. The stability switching behaviour is indicated in Fig. 4.4(c). It also follows the *(quasi-)* optimal selection principle.

<table>
<thead>
<tr>
<th>$(\delta_1, \delta_2)$</th>
<th>$(\tilde{d}_1, \tilde{d}_2)$</th>
<th>$(\rho_{1,+}^{<em>}, \rho_{2,-}^{</em>})$</th>
<th>$(\rho_t, w_t) \to$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0.4, 0.5)$</td>
<td>$(-1, 1)$</td>
<td>$\rho_{1,+}^{<em>} &lt; \rho_{2,-}^{</em>}$</td>
<td>$E_{2,-}$</td>
</tr>
<tr>
<td>$(0.6, 0.5)$</td>
<td>$(-1, 0.1)$</td>
<td>$\rho_{1,+}^{<em>} &lt; \rho_{2,-}^{</em>}$</td>
<td>$E_{2,-}$</td>
</tr>
<tr>
<td>$(0.6, 0.5)$</td>
<td>$(-0.5, 0.1)$</td>
<td>$\rho_{1,+}^{<em>} &gt; \rho_{2,-}^{</em>}$</td>
<td>$E_{1,+}$</td>
</tr>
<tr>
<td>$(0.6, 0.5)$</td>
<td>$(-0.4, 0.1)$</td>
<td>$\rho_{1,+}^{<em>} &gt; \rho_{2,-}^{</em>}$</td>
<td>$E_{1,+}$</td>
</tr>
</tbody>
</table>

**Table 3. Stability of contrarian and trend follower model**
To sum up, for the two trend traders model, there exist multiple equilibria. When the extrapolation rate is large, none of the equilibria are stable. However, when the extrapolation rate is small, the stability of the system follows the (quasi-) optimal selection principle — the one with relatively higher return equilibrium tends to dominate the market in long run.

**Noise effect.** When the deterministic system converges to one of the non-fundamental equilibria, a normally distributed noise dividend yield process can cause the return distribution to display non-normality.

<table>
<thead>
<tr>
<th>Fig.</th>
<th>((\delta_1, \delta_2))</th>
<th>((\bar{d}_1, \bar{d}_2))</th>
<th>Mean</th>
<th>Std Dev</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>((0.5, 0.5))</td>
<td>((0, -1))</td>
<td>0.1256</td>
<td>0.0198</td>
<td>-0.1523</td>
<td>0.3742</td>
</tr>
<tr>
<td>(b)</td>
<td>((0.6, 0.5))</td>
<td>((0, 0.5))</td>
<td>0.15</td>
<td>0.0244</td>
<td>0.1496</td>
<td>0.0940</td>
</tr>
<tr>
<td>(c)</td>
<td>((0.4, 0.5))</td>
<td>((-1, -1))</td>
<td>0.1048</td>
<td>0.0406</td>
<td>0.0863</td>
<td>0.4471</td>
</tr>
<tr>
<td>(d)</td>
<td>((0.4, 0.5))</td>
<td>((1, 1.05))</td>
<td>0.1972</td>
<td>0.0451</td>
<td>0.2868</td>
<td>0.1319</td>
</tr>
<tr>
<td>(e)</td>
<td>((0.5, 0.4))</td>
<td>((-1, 1))</td>
<td>0.1385</td>
<td>0.0232</td>
<td>0.1581</td>
<td>0.2819</td>
</tr>
</tbody>
</table>

**Table 4. Time series statistics for two trend traders model**

Consider the effect of some two traders model under a normal dividend yield noise process with mean 0 and standard derivation \(q = 0.01\). Without the noise process, the system is stable. Table 4 lists various models with different selection of risk premia and extrapolation rates. It also summaries the corresponding statistics of the time series. Fig. 4.5 show the empirical distribution, comparing with normal distribution, of time series. The statistics indicate a certain degree of excess volatility and non-normality of the returns.

5. Conclusion

We have developed a model of heterogeneous agents trading a risky asset. The agents all have the same attitude to risk but they differ in their expectation of the conditional mean and variance of asset return. Following the framework of Brock and Hommes [5] and Levy and Levy [18], the heterogeneous agents have been incorporated in an equilibrium model in both price and wealth. The resulting dynamical system for price and wealth turns out to be non-stationary. In order to characterize the interactions of heterogeneous agents in financial markets (typically fundamentalists, trend chasers and contrarians), two stationary models in terms of return and wealth proportions (among different types of agents) have been developed.
Figure 4.5. Historical distributions of returns (with solid lines) and the corresponding normal distributions (with dotted lines)

It is found that the presence of heterogeneous agents leads the stationary model to have multiple equilibria for asset return and wealth distribution. As far as (local asymptotic) stability is concerned, the extrapolation rates of the chartists play an important role: the return dynamics tends to be unstable for a high extrapolation rate and stable for a low extrapolation rate. The convergence to equilibrium follows the optimal selection principle — the return and wealth proportion tend to one of the equilibria which has a relatively higher return. Numerical simulations have been employed to study the effects of the dynamics under a noisy dividend yield process. It has been demonstrated that, when one of the equilibria is locally asymptotically stable without the dividend noisy process, adding the noisy processes lead to a certain degree of excess volatility and non-normality of the returns.

An extension of the current paper to a model with adaptive beliefs is more realistic. Brock and Hommes [4], [5] introduce Adaptive Belief Systems to study
the price dynamics with heterogeneous expectations agents. A performance measure can be defined by the realized wealth return on the proportion invested in the risky asset. Based on their realized performance measure, the fractions of different types at the end of period \( t \) will be determined by the fitness function and the investors are allowed to change their expectations from one group to another from time to time so that the fractions \( n_{ij} \) become functions of time \( t \), that is, \( n_{ij,t} \). Consequently, an evolutionary dynamics across predictor choice is coupled to the dynamics of the endogeneous variables. This extension is an interesting problem which is left to future research work.

**APPENDIX A. ONE PERIOD INTERTEMPORAL OPTIMISATION**

The optimum investment proportion at time \( t \), \( \pi_{i,t} \) is determined by the maximization of the expected utility of the wealth at \( t + 1 \), which is given by

\[
\max_{\pi_{i,t}} E_{i,t}[U(W_{i,t+1})].
\]

To solve this problem, one needs to work out the evolution of \( U(W(t)) \).

Assume the wealth \( W(t) \) follows a continuous stochastic differential equation

\[
dW(t) = \mu(W)dt + \sigma(W)dz(t), \tag{A.1}
\]

where \( z(t) \) is a Wiener process. Let \( X = U(W) \) is an invertible differentiable function with inverse function \( W = G(X) \). Following from Ito’s Lemma,

\[
dX = [U'(W)\mu(W) + \frac{1}{2}\sigma^2(W)U''(W)]dt + \sigma(W)U'(W,t)dz, \tag{A.2}
\]

which can be written as

\[
dX = \mu(X)dt + \sigma(X)dz, \tag{A.3}
\]

where

\[
\mu(X) = U'(G(X))\mu(G(X)) + \frac{1}{2}\sigma^2(G(X))U''(G(X)), \tag{A.4}
\]

\[
\sigma(X) = \sigma(G(X))U'(G(X)). \tag{A.5}
\]

So,

\[
X(t + \Delta t) = X(t) + \mu(X(t))\Delta t + \sigma(X(t))\Delta z(t) \tag{A.6}
\]

and

\[
E_t[X(t + \Delta t)] = X(t) + \mu(X(t))\Delta t, \tag{A.7}
\]

\[
V_t[X(t + \Delta t)] = \sigma^2(X(t)).
\]

Rescaling the time unit \( \Delta t \) in equation (A.7),

\[
E_t[X_{t+1}] = X_t + \mu(X_t). \tag{A.8}
\]

Substituting (A.4) into (A.8), one has

\[
E_t[U(W_{t+1})] = U(W_t) + \mu_t(W_t)U'(W_t) + \frac{1}{2}\sigma_t^2(W_t)U''(W_t). \tag{A.9}
\]
Let 
\[ \rho_{t+1} = E_t(\rho_{t+1}) + \sqrt{V_t(\rho_{t+1})} \xi_t, \]  
(A.10)
where \( \xi_t \) is a Wiener process. It then follows from (2.2) that \(^7\)
\[ W_{t+1} - W_t = r(1 - \pi_t)W_t + \pi_t \rho_{t+1} W_t. \]  
(A.11)
Substituting (A.10) into (A.11), it leads to
\[ W_{t+1} - W_t = \mu_t(W) + \sigma_t(W) \xi_t, \]  
(A.12)
where \( \mu_t(W) = [r(1 - \pi_t) + \pi_t E_t(\rho_{t+1})]W_t \) and \( \sigma_t(W) = \pi_t \sqrt{V_t(\rho_{t+1})}W_t \). Applying (A.9) to (A.12), then
\[ E_t[U(W_{t+1})] = U(W_t) + [r(1 - \pi_t) + \pi_t E_t(\rho_{t+1})]W_t U'(W_t) + \frac{1}{2} \pi_t^2 V_t(\rho_{t+1}) W_t^2 U''(W_t). \]  
(A.13)
Thus the first order condition of the problem \( \max_{\pi_t} E_t[U(W_{t+1})] \) leads to the optimum solution
\[ \pi_t = -\frac{E_t(\rho_{t+1}) - r}{V_t(\rho_{t+1})} \frac{U'(W_t)}{U''(W_t)}. \]  
(A.14)
Let \( U(W) = \log W \). Applying (A.14), one obtain \( \pi_t = \frac{E_t(\rho_{t+1}) - r}{V_t(\rho_{t+1})} \), as claimed in (2.5). Next let \( U(W) = [W^\gamma - 1]/\gamma \) with constant \( 0 < \gamma < 1 \). Then (A.14) leads to \( \pi_t = \frac{1}{1 - \gamma} \frac{E_t(\rho_{t+1}) - r}{V_t(\rho_{t+1})} \).

**Appendix B. Proofs of the Results**

**B.1. Proof of Proposition 2.1.**

*Proof.* The wealth proportion equation (2.10) follows directly from (2.8), (2.9) and
\[ \frac{W_t}{W_{t-1}} = \sum_{i=1}^{H} \frac{W_{i,t}}{W_{t-1}} = \sum_{i=1}^{H} w_{i,t-1}[R + (\rho_t - r)\pi_{i,t-1}]. \]  
(B.1)
Rewrite the market clearing equilibrium equation (2.7) as
\[ W_t \sum_{i=1}^{H} w_{i,t} \pi_{i,t} = N \rho_t. \]  
(B.2)
Replace \( t \) by \( t - 1 \) in (B.2),
\[ W_{t-1} \sum_{i=1}^{H} w_{i,t-1} \pi_{i,t-1} = N \rho_{t-1}. \]  
(B.3)
\(^7\)For the convenient, the index \( i \) is dropped from now on.
one obtains from (B.2) and (B.3) that

\[
\frac{W_t}{W_{t-1}} \cdot \frac{\sum_{i=1}^{H} \pi_{i,t} w_{i,t}}{\sum_{i=1}^{H} \pi_{i,t-1} w_{i,t-1}} = \frac{p_t}{p_{t-1}} = 1 + \rho_t - \alpha_t, \tag{B.4}
\]

where \(\alpha_t = y_t/p_{t-1}\) defines the dividend yield. So, from (B.4) and (B.1),

\[
\left( \sum_{i=1}^{H} w_{i,t-1} \left[ R + (\rho_t - r) \pi_{i,t-1} \right] \right) \left( \sum_{i=1}^{H} \pi_{i,t} w_{i,t} \right)
\]

\[
= \left[ (\rho_t - r) + (1 + r - \alpha_t) \right] \left( \sum_{i=1}^{H} \pi_{i,t-1} w_{i,t-1} \right). \tag{B.5}
\]

Note from (2.10) that \(w_{i,t}\) is the function of \(\rho_t\). To solve for \(\rho_t\) from (B.5) explicitly, one can see from (2.10) that

\[
\sum_{i=1}^{H} \pi_{i,t} w_{i,t} = \frac{\sum_{i=1}^{H} \pi_{i,t} w_{i,t-1} \left[ R + (\rho_t - r) \pi_{i,t-1} \right]}{\sum_{i=1}^{H} w_{i,t-1} \left[ R + (\rho_t - r) \pi_{i,t-1} \right]}.
\tag{B.6}
\]

Substituting equation (B.6) into (B.5),

\[
\sum_{i=1}^{H} w_{i,t-1} \pi_{i,t} \left[ R + (\rho_t - r) \pi_{i,t-1} \right]
\]

\[
= \left[ (\rho_t - r) + (1 + r - \alpha_t) \right] \left( \sum_{i=1}^{H} \pi_{i,t-1} w_{i,t-1} \right). \tag{B.7}
\]

Solving for \(\rho_t\) from (B.7) yields equation (2.11) for \(\rho_t\). \hfill \Box

B.2. Proof of Proposition 2.3.

**Proof.** Note that

\[
W_t = H \sum_{j=1}^{h} n_j \bar{W}_{j,t},
\]

\[
\bar{W}_{j,t+1} = \bar{W}_{j,t} \left[ R + (\rho_{t+1} - r) \bar{\pi}_{i,t} \right]
\]

and

\[
\frac{\bar{W}_{j,t}}{W_t} = \frac{\bar{W}_{j,t} \bar{W}_t}{\bar{W}_t} = \bar{\omega}_{j,t} \frac{\bar{W}_t}{H \sum_{k=1}^{h} n_k \bar{W}_k, t} = \frac{1}{H \sum_{k=1}^{h} n_k \bar{w}_{k,t}}.
\]

Hence the equilibrium equation (2.15) becomes

\[
\sum_{j=1}^{h} n_j \bar{\omega}_{j,t} \frac{\bar{w}_{j,t}}{\sum_{k=1}^{h} n_k \bar{w}_{k,t}} = N \frac{p_t}{W_t},
\]
that is,
\[
W_t \sum_{j=1}^{h} n_j \bar{\bar{\pi}}_{j,t} \bar{\bar{\mu}}_{j,t} = N \rho_t \sum_{j=1}^{h} n_j \bar{\bar{\mu}}_{j,t}. \tag{B.8}
\]

Also, from (2.10),
\[
\bar{\bar{\mu}}_{j,t} = \frac{\bar{\bar{\mu}}_{j,t-1} [R + (\rho_t - r) \bar{\bar{\pi}}_{j,t-1}]}{\sum_{k=1}^{h} \bar{\bar{\mu}}_{k,t-1} [R + (\rho_t - r) \bar{\bar{\pi}}_{k,t-1}]}.
\tag{B.9}
\]
Similar to the last part of the proof of Proposition 2.1, replacing \( t \) by \( t - 1 \) in equation (B.8) one obtains
\[
\frac{W_t}{W_{t-1}} \sum_{j=1}^{h} n_j \bar{\bar{\pi}}_{j,t} \bar{\bar{\mu}}_{j,t} = \frac{p_t \sum_{j=1}^{h} n_j \bar{\bar{\mu}}_{j,t}}{p_{t-1} \sum_{j=1}^{h} n_j \bar{\bar{\mu}}_{j,t-1}}.
\tag{B.10}
\]
Note that
\[
\frac{p_t}{p_{t-1}} = 1 + \rho_t - \alpha_t
\]
and
\[
\frac{W_t}{W_{t-1}} = \frac{\sum_{j=1}^{h} n_j \bar{\bar{\pi}}_{j,t} \bar{\bar{\mu}}_{j,t}}{\sum_{j=1}^{h} n_j \bar{\bar{\pi}}_{j,t-1} \bar{\bar{\mu}}_{j,t-1}} = \frac{\sum_{j=1}^{h} n_j \bar{\bar{\pi}}_{j,t} \bar{\bar{\mu}}_{j,t-1} [R + (\rho_t - r) \bar{\bar{\pi}}_{j,t-1}]}{\sum_{j=1}^{h} n_j \bar{\bar{\pi}}_{j,t-1} \bar{\bar{\mu}}_{j,t-1}}.
\tag{B.11}
\]
From (B.10) and (B.11)
\[
\sum_{j=1}^{h} n_j \bar{\bar{\pi}}_{j,t} \bar{\bar{\mu}}_{j,t-1} + \rho_t - r \sum_{j=1}^{h} n_j \bar{\bar{\pi}}_{j,t} \bar{\bar{\mu}}_{j,t}
\]
\[
= (1 + \rho_t - \alpha_t) \sum_{j=1}^{h} n_j \bar{\bar{\pi}}_{j,t} \sum_{j=1}^{h} n_j \bar{\bar{\pi}}_{j,t-1} \bar{\bar{\mu}}_{j,t-1}.
\tag{B.12}
\]
Also, using (B.9),
\[
\sum_{j=1}^{h} n_j \bar{\bar{\pi}}_{j,t} \bar{\bar{\mu}}_{j,t} = \frac{\sum_{j=1}^{h} n_j \bar{\bar{\pi}}_{j,t} \bar{\bar{\mu}}_{j,t} [R + (\rho_t - r) \bar{\bar{\pi}}_{j,t-1}]}{\sum_{k=1}^{h} \bar{\bar{\mu}}_{k,t-1} [R + (\rho_t - r) \bar{\bar{\pi}}_{k,t-1}]}.
\tag{B.13}
\]
\[
\sum_{j=1}^{h} n_j \bar{\bar{\mu}}_{j,t} = \frac{\sum_{j=1}^{h} n_j \bar{\bar{\mu}}_{j,t-1} [R + (\rho_t - r) \bar{\bar{\pi}}_{j,t-1}]}{\sum_{k=1}^{h} \bar{\bar{\mu}}_{k,t-1} [R + (\rho_t - r) \bar{\bar{\pi}}_{k,t-1}]}.
\tag{B.14}
\]
Substituting (B.13) and (B.14) into (B.12) and simplifying the corresponding expression lead to equation
\[
\sum_{i=1}^{h} n_i \bar{w}_{i,t-1} \bar{\pi}_{i,t}[R + (\rho_t - r)\bar{\pi}_{i,t-1}]
= [(\rho_t - r) + (1 + r - \alpha_t)] \left( \sum_{i=1}^{h} n_i \bar{\pi}_{i,t-1} \bar{w}_{i,t-1} \right).
\] (B.15)

Solving for \( \rho_t \) from (B.15), one obtains the equation (2.17) for return \( \rho_t \).

\[\Box\]

B.3. **Proof of Theorem 3.1.**

*Proof.* It follows from (2.13)-(2.14) and the notations introduced in Theorem 3.1 that the fixed point \( \rho^* \) satisfies
\[
(\rho^* - r)(1 - \bar{\delta} - \rho^* \bar{d}) = \alpha_o.
\]
If \( d = 0 \), then \( \rho^* = r + \frac{\alpha_o}{1 - \bar{\delta}} \). For \( d \neq 0 \), \( \rho^* \) satisfies
\[
(\rho^* - r)^2 + (r - \frac{1 - \bar{\delta}}{d})(\rho^* - r) + \frac{\alpha_o}{d} = 0,
\]
which has two solutions given by (3.5) and (3.6). Let \( A = [r - \frac{1 - \bar{\delta}}{d}]^2 - \frac{4\alpha_o}{d} \). Then \( \rho^*_\pm \in \mathbb{R} \) if \( A \geq 0 \).

Suppose \( \bar{d} < 0 \). Then \( A > 0 \), so \( \rho^*_\pm \in \mathbb{R} \). Also,
\[
\rho^*_+ > \frac{1}{2}[(r + (1 - \bar{\delta})/\bar{d}) + (r - (1 - \bar{\delta})/\bar{d})] = r,
\]
\[
\rho^*_- < \frac{1}{2}[(r + (1 - \bar{\delta})/\bar{d}) - (r - (1 - \bar{\delta})/\bar{d})] = (1 - \bar{\delta})/\bar{d} < 0.
\]
One can verify that, when \( \bar{d} < 0 \), \( \rho^*_- < 0 \) and \( \rho^*_+ \leq \rho^*_o \).

Assume now \( \bar{d} > 0 \) and (3.7) holds. Then \( \rho^*_\pm \in \mathbb{R} \). Also, it can be verified that \( \rho^*_o \leq \rho^*_+ \) iff \( r + 2\alpha_o/(1 - \bar{\delta}) \leq (1 - \bar{\delta})/\bar{d} \), that is, \( \bar{d} \leq d_{oL} \). \( \rho^*_+ < \rho^*_o \) iff \( r + 2\alpha_o/(1 - \bar{\delta}) > (1 - \bar{\delta})/\bar{d} \), that is \( \bar{d} \geq d_{oL} \). Furthermore, let \( f(\bar{d}) = \rho^*_+ \) and \( g(\bar{d}) = \rho^*_o \). It can be verified that, for \( \bar{d} < 0 \), \( f'(\bar{d}) > 0 \), \( g'(\bar{d}) < 0 \) and \( f(\bar{d}) \to \rho^*_o \) as \( \bar{d} \to 0^- \). While for \( \bar{d} > 0 \), \( f'(\bar{d}) > 0 \) iff \( \bar{d} > d_{oL} \) and \( g'(\bar{d}) < 0 \) iff \( \bar{d} < d_{oL} \). This completes the proof. \[\Box\]

B.4. **Proof of Proposition 3.2.**

*Proof.* For \( \alpha_t = \alpha_o \), rewrite equation (2.13) in the form
\[
\rho_t = F(\pi_{t-1}, \pi_t),
\] (B.16)
where
\[
F(u, v) = r + \frac{(1 + r)(u - v) - \alpha_o u}{u(v - 1)}.
\] (B.17)
For a fixed lag $L$, introducing a new set of variables:

$$x_{1,t} = \rho_t, \ x_{2,t} = x_{1,t-1} = \rho_{t-1}, \ \ldots, \ x_{L+1,t} = x_{L,t-1} = \rho_{t-(L+1)},$$

then the equation (2.13) is equivalent to an $L+1$ dimensional system:

$$\begin{cases}
    x_{1,t+1} = F(\pi_t, \pi_{t+1}), \\
    x_{2,t+1} = x_{1,t}, \\
    \vdots \\
    x_{L+1,t+1} = x_{L,t}.
\end{cases}$$

Note $\pi_t = \bar{\pi} + \bar{d} \bar{\rho}_t = \bar{\pi} + (\bar{d}/L)[x_{2,t} + x_{3,t} + \cdots + x_{L+1,t}]$. Then $\pi_{i,t+1} = \bar{\pi} + (\bar{d}/L)[x_{1,t} + x_{2,t} + \cdots + x_{L,t}]$ and

$$\frac{\partial \pi_t}{\partial x_{1,t}} = 0, \quad \frac{\partial \pi_t}{\partial x_{i,t}} = \bar{d}/L \quad \text{for} \quad i = 2, \ldots, L+1 \quad (B.18)$$

$$\frac{\partial \pi_{t+1}}{\partial x_{L+1,t}} = 0, \quad \frac{\partial \pi_{t+1}}{\partial x_{j,t}} = \bar{d}/L \quad \text{for} \quad j = 1, \ldots, L. \quad (B.19)$$

Let $x_o$ be a equilibrium of $(\pi)$ and denote $a_i = \partial F/\partial x_{i,t}|_{x_o,\ldots,x_o}$, evaluated at the equilibrium, for $i = 1, \ldots, L+1$. Then one can obtain

$$a_1 = -\frac{\bar{d}}{L} (C - A), \quad a_2 = \cdots = a_L = -\frac{\bar{d}}{L} C, \quad a_{L+1} = -\frac{\bar{d}}{L} A. \quad (B.20)$$

Then the Jacobian matrix of the system at the equilibrium $x_o$ is

$$\begin{pmatrix}
    a_1 & a_2 & \cdots & a_L & a_{L+1} \\
    1 & 0 & \cdots & 0 & 0 \\
    0 & 1 & \cdots & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & 1 & 0
\end{pmatrix}, \quad (B.21)$$

from which, the characteristic polynomial is derived. \qed

B.5. **Proof of Theorem 4.1.** To obtain all the steady-states of the system, it is more convenient to consider the system (2.10) and (B.5). Let $(w^*, \rho^*)$ be the fixed point of the system (4.5) and (B.5). Then

$$w^* = \frac{[1 + w^*][R + (\rho^* - r) \pi^*_1] - [1 - w^*][R + (\rho^* - r) \pi^*_2]}{[1 + w^*][R + (\rho^* - r) \pi^*_1] + [1 - w^*][R + (\rho^* - r) \pi^*_2]}, \quad (B.22)$$

$$\rho^* = r + \frac{\alpha_o}{1 - \sum_{j=1}^2 \pi_j^* w_j^*}, \quad (B.23)$$

where $w^*_1 = (1 + w^*)/2, \ w^*_2 = (1 - w^*)/2,$

$$\pi^*_i = \frac{\delta_i + d_i \rho^*}{\sigma^2} = \bar{\pi}_i + \bar{d}_i \rho^* \quad (i = 1, 2). \quad (B.24)$$
Consider the case \( d_i = 0 \) first. In this case,
\[
\rho^* = r + \frac{\alpha_o}{1 - (\delta_1[1 + w^*]) + \delta_2[1 - w^*])}/2, \tag{B.25}
\]
\[
w^* = \frac{[1 + w^*][R + (\rho^* - r)\delta_1] - [1 - w^*][R + (\rho^* - r)\delta_2]}{[1 + w^*][R + (\rho^* - r)\delta_1] + [1 - w^*][R + (\rho^* - r)\delta_2]}, \tag{B.26}
\]
It can be easily verified that \( w^* = \pm \) lead to the steady-states \( E_1 \) and \( E_2 \) as given by Theorem 4.1. Now, assume \( w^* \neq \pm 1 \). Then
\[
[1 + w^*][R + (\rho^* - r)\delta_1] + [1 - w^*][R + (\rho^* - r)\delta_2]
= 2[R + (\rho^* - r)\delta_1]
= 2[R + (\rho^* - r)\delta_2]
\]
leads to either \( \rho^* = r \) or \( \delta_1 = \delta_2 \). However, \( \rho^* = r \) implies \( \alpha_o = 0 \), which contradicts \( \alpha_o > 0 \). Therefore, under the condition \( \delta_1 = \delta_2 \), \( E_3 \) is also a steady-state of the system with \( w^* = w_{1,0} - w_{2,0} \). So the proof of Theorem 4.1 is completed.

B.6. **Proof of Proposition 4.2.**

*Proof.* The proof of Proposition 4.2 is similar to that of Proposition 3.2 and here is the main steps of the modification.

The system (4.5) and (4.6) can be write in the form
\[
\rho_{i+1} = F(\pi_{1,t}, \pi_{1,t+1}, \pi_{2,t}, \pi_{2,t+1}, w_t), \tag{B.27}
\]
\[
w_{t+1} = G(\pi_{1,t}, \pi_{1,t+1}, \pi_{2,t}, \pi_{2,t+1}, w_t), \tag{B.28}
\]
where \( F \) and \( G \) denote the corresponding functions of the right hand side of the system (4.5) and (4.6). Denote
\[
f_i(\pi_{i,t}, \pi_{i,t+1}) = r + \frac{(1 + r)(\pi_{i,t} - \pi_{i,t+1}) - \alpha_o \pi_{i,t}}{\pi_{i,t}(\pi_{i,t+1} - 1)}, \quad i = 1, 2. \tag{B.29}
\]
Then, evaluated at the equilibrium \( E(\rho^*, w^*) \) with \( w^* = 1 \),
\[
\frac{\partial F}{\partial \pi_{1,t}} = \frac{\partial f_1}{\partial \pi_{1,t}}, \quad \frac{\partial F}{\partial \pi_{1,t+1}} = \frac{\partial f_1}{\partial \pi_{1,t+1}},
\]
\[
\frac{\partial F}{\partial \pi_{2,t}} = \frac{\partial F}{\partial \pi_{2,t+1}} = 0, \quad \frac{\partial F}{\partial w_t} = \frac{\alpha_o \pi_{2,o} (\pi_{1,o} - \pi_{2,o})}{\pi_{1,o} (1 + r)^2},
\]
\[
\frac{\partial G}{\partial \pi_{1,t}} = \frac{\partial G}{\partial \pi_{1,t+1}} = \frac{\partial G}{\partial \pi_{2,t}} = \frac{\partial G}{\partial \pi_{2,t+1}} = 0,
\]
\[
\frac{\partial G}{\partial w_t} = \frac{(1 + r) + (\rho^* - r)\pi_{2,o}}{(1 + r) + (\rho^* - r)\pi_{1,o}} = \beta.
\]

Introduce a set of new variable
\[
x_{1,t} = \rho_t, \quad x_{2,t} = \rho_{t-1}, \cdots, x_{L+1,t} = \rho_{t-L}, \quad x_{L+2,t} = w_t \tag{B.30}
\]
so that the system (4.5) and (4.6) can be written equivalently as an $L + 2$-dimensional system

$$
x_{1,t+1} = F, \quad x_{2,t+1} = x_{1,t}, \ldots, x_{L+1,t+1} = x_{L,t}, \quad x_{L+2,t+1} = G. \quad (B.31)
$$

Denote $a_i = \partial F/\partial x_{i,t}$, $b_i = \partial G/\partial x_{i,t}$, evaluated at the equilibrium $E(\rho^*, w^*)$ with $w^* = 1$, then the Jacobian matrix of the system

$$
J_1 = \begin{pmatrix}
a_1 & a_2 & \cdots & a_L & a_{L+1} & a_{L+2} \\
1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 \\
b_1 & b_2 & \cdots & b_L & b_{L+1} & b_{L+2}
\end{pmatrix}. \quad (B.32)
$$

Based on the calculation in Proposition 3.2, one can verify that

$$
b_1 = b_2 = \cdots b_{L+1} = 0, \quad b_{L+2} = \beta,
$$

$$
a_1 = \frac{\tilde{d}_1}{L_1} [A_1 - C_1], \quad a_2 = \cdots = a_{L_1} = \frac{\tilde{d}_1}{L_1} C_1, \quad a_{L_1+1} = \frac{\tilde{d}_1}{L_1} A_1,
$$

$$
a_{L_1+2} = \cdots = a_{L+1} = 0, \quad a_{L+2} = \frac{\alpha_o \bar{\pi}_{1,0} - \bar{\pi}_{2,0}}{2\bar{\pi}_{1,0} \bar{\pi}_{2,0} - 1}.
$$

Thus,

$$
\det(\lambda I - J_1) = (\lambda - \beta)\lambda^{L-1} [\lambda^{L_1} - a_1 \lambda^{L_1} - \cdots - a_{L_1} \lambda - a_{L_1+1}],
$$

from which the characteristic polynomial follows. The case when $w^* = -1$ can be derived similarly. \qed

\textbf{B.7. Proof of Theorems 4.4, 4.6 and 4.7.} Now let $d_1 = 0$, it follows from (B.22)-(B.24) and the proof of Theorem 4.1 that the steady-state $(\rho^*, w^*)$ is determined by

$$
\rho^* = r + \frac{\alpha_o}{1 - (\tilde{d}_2 [1 + w^*] + (\tilde{d}_2 + \bar{d}_2 \rho^*) [1 - w^*])/2}, \quad (B.33)
$$

$$
w^* = \frac{[1 + w^*][R + (\rho^* - r) \tilde{d}_1] - [1 - w^*][R + (\rho^* - r) (\tilde{d}_2 + \bar{d}_2 \rho^*)]}{[1 + w^*][R + (\rho^* - r) \tilde{d}_1] + [1 - w^*][R + (\rho^* - r) (\tilde{d}_2 + \bar{d}_2 \rho^*)]}. \quad (B.34)
$$

One can see that $w^* = 1$ leads to the equilibrium $E_1$. Next, let $w^* = -1$ then $\rho^*$ satisfies

$$
(\rho^* - r) (1 - \tilde{d}_2 - \bar{d}_2 \rho^*) = \alpha_o, \quad (B.35)
$$

which has two real solutions if and only if $A_2 \equiv [r - (1 - \tilde{d}_2)/\bar{d}_2]^2 - 4\alpha_o/\bar{d}_2 \geq 0$. Then the same argument used for Theorem 3.1 leads to the existence of $E_{2, \pm}$, as defined in Theorems 4.4 when $d_2 < 0$ and 4.6 when $d_2 > 0$. 


For \( \delta_2 < 0 \), similar argument used in the proof of Theorem 4.4 leads to \( \rho_{2,+}^* > r \) and \( \rho_{2,-}^* < 0 \). Also, \( \rho_{2,+}^* \geq \rho_{2,-}^* \) is equivalent to

\[
\frac{1}{1 - \delta_1} \left[ r + \frac{a_o}{1 - \delta_1} \right] \geq -\frac{1}{d_2} \left[ 1 - \frac{1 - \delta_2}{1 - \delta_1} \right],
\]

which is always true for \( \delta_1 > \delta_2 \).

Similarly, \( a_o > 0 \) implies that \( \rho^* = r \) is not a steady-state. Now one can assume that \( w^* \neq \pm 1 \) so that equation (B.34) leads to

\[
[1 + w^*](R + (\rho^* - r)\delta_1) + [1 - w^*](R + (\rho^* - r)\delta_2) = 2[R + (\rho^* - r)(\delta_1 + \delta_2)],
\]

which implies \( \rho^* = (\delta_1 - \delta_2)/\delta_2 \). In this case, (B.33) leads to \( \rho^* = (\delta_1 - \delta_2)/\delta_2 = r + \alpha_o/(1 - \delta_1) \) while (B.34) holds for any \( w^* \in (-1, 1) \). Therefore, apart from the existence of \( E_1 \) when \( (\delta_1 - \delta_2)/\delta_2 = r + \alpha_o/(1 - \delta_1) \), the system has no other steady-state and this proves Theorem 4.4.

Now let \( d_2 > 0 \). Then, under the condition \( A_2 \geq 0 \), \( \rho_{2,\pm}^* > 0 \). Furthermore, \( \rho_{2,-}^* \geq \rho_{1}^* \int \iff \)

\[
r + \frac{2a_o}{1 - \delta_1} < \frac{1 - \delta_2}{2}
\]

and

\[
\frac{1}{1 - \delta_1} \left[ r + \frac{a_o}{1 - \delta_1} \right] \geq \frac{1}{\delta_2} \left[ 1 - \frac{\delta_2}{1 - \delta_1} - 1 \right].
\]

This leads to Theorem 4.6.

Theorem 4.7 can be proved similarly. When \( w^* = 1 \), \( \rho^* \) satisfies

\[
(\rho^* - r)(1 - \delta_1 - \delta_2) = \alpha_o,
\]

which leads to the existence of \( E_{1,\pm} \). While when \( w^* = -1 \), \( \rho^* \) is determined by (B.35), which in turn gives \( E_{2,\pm} \). When \( w^* \neq \pm 1 \), similar argument leads to no existence of any other equilibrium except \( E_3 \). This completes the proof.

References


