Valuing Basic Pensions by Replication

Susan Thorp*, Konstantin Petrichev#

*University of Technology, Sydney, #Perpetual Wealth Management

22 May 2007

Abstract

Governments around the world are reviewing basic pension systems in the light of increasing demands on public funds. In Australia, Government policy aims to increase reliance on private retirement savings and reduce demand on the targeted Age Pension. Using a new analytical valuation method for retirement income streams (Milevsky and Robinson 2005) we value the Age Pension by calculating the amount of wealth needed to sustain an annual draw-down equivalent to the Australian basic pension, if pensioners were to be responsible for generating the income stream themselves. We account for both investment and longevity risk. A 65-year-old single retiree with average life-expectancy needs retirement wealth equivalent to 8.5 times average annual earnings to replicate the payments and insurance features of the public pension using standard draw-down products. Delaying retirement by 5 years reduces required savings by around 5%, but linking pension payments to earnings growth rather than price inflation increases required wealth by up to 25%. Commercial single life annuities can replicate the pension more cheaply than the draw-down plans we evaluate, but remain unpopular with retirees. We conclude that the basic pension is very valuable, representing a large notional transfer of wealth at retirement.
1 Introduction

Governments across the world are reviewing the structure of retirement saving systems in the light of aging populations and increasing public pension liabilities. In particular, the role of basic redistributive pensions that comprise the ‘first tier’ or ‘first pillar’ of retirement savings systems is being reassessed. Basic pensions are designed to ensure that the elderly reach a minimum level of welfare. According to Whitehouse’s (2007) survey of pension systems, redistributive public pensions are typically payments of between 20% and 40% of average earnings, targeted towards the more needy via income and assets testing. In countries with high rates of population aging, first pillar pensions are consuming an increasing part of public funds, a fact which has motivated changes to indexation systems and eligibility criteria. In developed countries such as Australia and the U.S., public policy has focussed on creating incentives for individuals to save for and fund their own retirements through earnings-related (second pillar) savings schemes, with the aim of reducing dependence on first pillar provisions and consequently relieving the fiscal burden.

In Australia, a large majority of current retirees relies on the targeted basic pension, the ‘Age Pension’, for income. Recent survey data show that nearly 70% of couple households and nearly 80% of single-person households comprising people over the age of 64 depend on government pensions as their primary source of income (Australian Bureau of Statistics 2006). As the population ages in the coming four decades, targeted first pillar pension payments, which now represent nearly 2.5% of Gross Domestic Product (GDP) are projected to increase to almost 4.4% of GDP (Commonwealth of Australia 2007). The reliance of elderly Australians on first-tier provision is expected to continue despite the introduction in 1992 of mandatory, earnings-related retirement savings under the Australian Superannuation Guarantee. The Superannuation Guarantee compels Australian workers to contribute at least 9% of income to privately managed and fully funded personal retirement savings accounts, but 15 years after inception, accumulations into superannuation accounts are still relatively modest, currently averaging less than $100,000\(^1\) at retirement (ASFA 2007) and projected to be less than $150,000 by 2020.\(^2\) As a result,

\(^1\)This amount is less than twice annual average earnings.
\(^2\)Kelly, Harding and Percival (2002) projects an average balance of $119,709 by 2020 in 1999 dollars,
most retirees will continue to depend on first-tier income support and the Age Pension will remain a large and increasing component of fiscal outlays.

Nevertheless, it is a stated policy aim of the Australian Government to encourage private saving and to reduce demand for the basic pension (Commonwealth of Australia, 2002). This raises the question: how much wealth would a retiree need to have saved by age 65 to generate an income stream with the same value and insurance features as the targeted public pension? Using a new analytical approximation to the retirement income problem (Milevsky and Robinson 2005), we calculate the amount of net retirement savings a 65 year old with average survival probability would need in order to replicate the real income stream offered by the Age Pension from the typical investment products of the Australian retirement incomes market.

Earlier studies of draw-down in retirement (Huang et al. 2004, Milevsky and Robinson 2000) show that for an infinite horizon and lognormally distributed investment returns, the stochastic present value of a desired spending plan is reciprocal Gamma distributed. Hence they evaluate the ex ante feasibility of a specified spending and investment plan by computing the probability that the stochastic present value of the plan in question exceeds initial wealth. This is the probability of ‘retirement ruin’ or the likelihood of running out of money. In the more general case, when time horizons are finite and stochastic, the probability of running out of resources before the end of life can be approximated by matching the first two moments of the stochastic present value of the retirement spending plan with the first two moments of the reciprocal gamma distribution. The result is an analytical approximation to the probability of retirement ruin, given random returns and uncertain lifetimes. By fixing the probability of ruin at an arbitrarily low level, we can use this moment-matching method to back out the minimum retirement accumulation that a retiree with typical mortality and investment opportunities needs to generate a specified real income stream over the whole retirement. We apply this method to value the Australian Age Pension, inferring the amount of savings necessary to generate a pension-equivalent income stream. We estimate that for a 65-year-old retiree, pension eligibility represents a substantial transfer of wealth.

which we scale up by 20% to get an estimate in 2007 dollars.
Given a small allowance for regulatory risk (a 1% probability that the pension scheme ‘fails’), we estimate the retiree needs close to $380,000 at age 65 to ensure the $14,000 p.a. real income now paid by the pension until the end of life. This amount of savings is almost 6.8 times average earnings and more than four times the average retirement savings balance of current 60-65 year olds. Even if pension age is delayed to 70 years, the amount of wealth needed to fund an equivalent real payment is close to $355,000 or nearly 6.4 times average earnings.

Some OECD countries have first-tier pension schemes that are adjusted in line with inflation and others are linked to wages growth. Schemes linked with wages growth ensure that pensioners maintain relativity with wage earners as productivity increases, whereas price-linked schemes shrink coverage to smaller sections of the population as the economy grows. Consequently by de-coupling productivity growth and pension obligations, governments can gradually shrink the size of pension liabilities over time. Whitehouse (2007) cites the example of the U.K. basic pension scheme which was indexed to prices rather than wages in 1981. At the time of the change, the pension represented 23.7% of average earnings, but two decades later was worth less than 16% of average earnings. By contrast, recipients of the Australian Age Pension enjoy an option over price and wage increases: the pension is adjusted with inflation, but also cannot fall below 25% of Male Total Average Weekly Earnings (MTAWE). When we incorporate an historical rate of wages growth into the pension path, the accumulation required to replicate it increases further. Historical rates of increase in MTAWE have been above rates of increase for price inflation in most quarters over the past few decades, and based on this higher rate of increase we estimate that a retiree would need an additional $90,000 or close to 8.5 times average earnings to replicate the pension path.

The most efficient pension-replicating investment strategy is a balanced or growth portfolio with around 50-70% allocated to equities or property securities. The best portfolio allocation may vary by age, gender and risk tolerance, but more aggressive and more conservative investments generally require higher initial wealth to be secure and sustainable. Further, we find that at current prices, commercially provided life annuities offer a level of indexed income comparable to the pension at a lower cost than self-insurance, but
curiously remain very unpopular with retirees. Overall, if current rates of superannuation accumulation are any guide, our analysis indicates it is unlikely that average Australian retirees will ever save sufficient wealth to self-insure a pension-equivalent income stream against longevity and investment risk, using products now available in the retirement incomes market.

Section 2 of the paper sets out the main features of the targeted age pension and its interaction with the second-pillar superannuation scheme. We describe the method for calculating the stochastic present value of a spending plan in Section 3, and Section 4 outlines parameter choices. In Section 5 we compute the wealth required by men and women of retirement age to replicate a pension payment with a high level of certainty and Section 6 concludes.

2 Features of the Australian targeted pension

Retirement income provision in Australia comprises three ‘pillars’. The first pillar is the Age Pension, available to all residents over age 65 who fall within regulated asset and income boundaries. The second pillar is the mandatory Superannuation Guarantee, through which at least 9% of employees’ incomes are paid into privately managed and fully funded personal superannuation accounts, and the third pillar is voluntary private savings (Bateman, Kingston and Piggott 2001). The fact that Australia does not have a PAYGO social security scheme, along with the fact that the majority of Australian retirees rely on both first- and second-pillar provisions, creates a useful natural experiment in the interaction between targeted government retirement support and mandatory private savings.

The first-pillar Age Pension was originally designed as a safety net and is paid out of general Federal Government revenues. Compared with safety-net provisions in other rich developed nations, the Age Pension (around $14,000 p.a.) is relatively generous, but still much lower than payouts from typical employment-linked public pension systems, such as the United States Social Security system (Bateman Kingston and Piggott 2001). The

3Eligibility for women is currently at a lower age (62.5) but is rising towards 65 over the next few years.
Age Pension represents a modest replacement rate, but has the advantage of maintaining real value via indexing against average weekly earnings and consumer price inflation, and offering insurance against longevity by continuing until the end of life.

A gradual maturing in the second-pillar Superannuation Guarantee has seen a remarkable change in retirement savings patterns in Australia. During the 1980s, approximately 30% of private-sector workers were covered by superannuation schemes, with total superannuation assets of 9% of GDP by the mid-1980s (Edey and Simon 1998). However by 2006, total superannuation assets had reached 96% of GDP, with around 90% of the workforce covered by (mainly defined contribution) superannuation schemes. Despite this rapid increase in the past two decades, by 2020 savings at retirement for the average male worker are projected to be around $130,000 and within the Age Pension means tests. Consequently, while superannuation accumulations are expected to reduce dependence on the Age Pension for many individuals, at the aggregate level the combined effects of an aging population, moderate superannuation accumulations and relatively generous means-testing will mean increasing pressure on Federal Government revenues into the middle of this century.

The current Age Pension for a single pensioner who owns their own home is $525.10 per fortnight, or $13652.60 p.a. In addition, many pensioners are entitled to additional allowances for pharmaceuticals, utilities, telephone, rent assistance and allowances for living in remote areas. Pension eligibility also means access to a Pensioner Concession Card, which along with providing reduced cost medicines under the Pharmaceutical Benefit Scheme (PBS), introduces a range of reductions on state and local government charges such as reduced transport fares and lower motor vehicle registration. Table 1 below lists the current maximum rates of additional allowances for a single pensioner.

Apart from Rent Assistance, where payments depend on the living arrangements and rental costs, and Remote Area Allowance, which depends on zones of residence, eligibility for any level of Age Pension entitles the pensioner to the allowances and concessions listed in the table. The move from outside to inside pension eligibility creates a substantial windfall and is highly valued by retirees and their advisors. A single-home owning pen-
sioner not living in a remote area is automatically $343 p.a. better off through allowances, setting aside the value of further benefits accessed by holding a pensioner concession card. Allowances are adjusted in line with CPI changes once or twice a year, but do not rise in line with the general level of earnings as the base single pension does. In the analysis below we study the case of a single home-owning pensioner whose annual benefit is $14,000.

The base single pension is recalculated in March and September each year in line with the CPI, but is also increased to ensure that it does not fall below 25% of Male Total Average Weekly Earnings (MTAWE). Pensioners hold an option on the general level of wages and prices in the economy so that the relative as well as real value of payments is maintained. The adjustment in the base pension is

\[
\frac{P_t}{P_{t-1}} = \max \left( (1 + h_t), (1 + n_t) \right)
\]

where \( h \) is the rate of increase in the CPI and \( n \) is the rate of increase in MTAWE. This relative-income protection has been very valuable over the past 15 years because earnings growth has exceeded inflation in most years. Figure 1 below graphs monthly paths for inflation and MTAWE, since 1989.\(^4\)

\(^4\)Monthly series on the CPI and MTAWE are linearly interpolated from quarterly data.
The average annualised monthly increase in the CPI over this period was 2.9% compared with 4.4% for MTAWE and 4.5% for the maximum of both the CPI and MTAWE.

Means-testing of the Age Pension creates other option-like features over the wealth of the retired. The means tests begin to reduce the pension at a fixed levels of income and/or wealth and the pension declines linearly to zero as income and/or wealth increase. Means test boundaries are reviewed in line with changes in the CPI. Since the means tests may interact with each other, the pensioner is entitled to the least payment from either test, or zero.

The assets test begins to reduce the pension when wealth (excluding the pensioner’s home) reaches $161500 (A1), and payments reach zero when wealth reaches $338500 (A2).\footnote{From September 2007 this rate of reduction will be halved, reducing the pension payment by $39 dollars per thousand increase in wealth rather than by the current $78 per thousand. Pension payments will reach zero at wealth of around $511600.}

For income receipts, pension payments begin to reduce when the individual receives $128 per fortnight or $3328 p.a., and reach zero when income is $1455.25 per fortnight or $37837 p.a. Legislation ‘deems’ fixed rates of return for income from investments for the purposes of social security means testing, rather then relying on individuals to estimate...
actual investment returns for the year ahead. For financial assets to a value of $38400, income is deemed to accrue at a rate of 3.5% p.a. For all financial assets above that value, income is deemed to accrue at 5.5% p.a. The deeming rules allow us to translate the income test into an asset test-equivalent form by assuming that all income comes from financial asset returns. To generate an income of $3328 p.a., the income level when the Age Pension begins to reduce, the individual must hold $74473 (Y1) of financial assets. Similarly, to generate an income of $37837 p.a. when the pension reaches zero under the income test, an individual needs $701909 (Y2) of financial assets.

Under current legislation, the income test binds earlier than the assets test, at \( W = Y1(\$74473) \), but the assets test, once it begins to bind at \( W = A1 (\$161500) \), becomes the binding constraint until \( P(t) = 0 \), at \( W = A2 (\$338500 \text{ or } \$511600 \text{ under the taper to begin in September 2007}) \). Combining the two constraints gives us

\[
P(W(t)) = \begin{cases} 
P0 & \text{if } W(t) \leq Y1 \\
\left( \frac{Y2}{Y2-Y1} - \frac{1}{Y2-Y1}W(t) \right) P0 & \text{if } Y1 < W(t) \leq A1 \\
\left( \frac{A2}{A2-A1} - \frac{1}{A2-A1}W(t) \right) P0 & \text{if } A1 < W(t) \leq A2 \\
0 & \text{if } A2 < W(t)
\end{cases}
\]

which is piecewise linear, with changing slope at \( W = Y1 \) and \( W = A1 \) and where \( P0 \) is the full payment. The combined effect is shown by the dark line in Figure 2 below.
This option structure provides a payoff in higher pension payment as wealth falls below the means-test boundary so that an optimising retiree will trade off the marginal advantages of pension eligibility against the costs of lower wealth.\(^6\) The taper may encourage higher rates of consumption early in retirement. For the remaining analysis, however, we do not study the pension taper since the majority of retirees receive the full pension.

3 Stochastic present value of retirement wealth

First consider the problem of a retiree who plans to consume one dollar each year from an initial retirement wealth \(W_0 = w\).\(^7\) The retiree invests wealth in a portfolio returning a continuously compounded risk-free rate of return \(\mu\). Wealth invested this way has a dynamic path given by the ordinary differential equation

\[
dW_t = (\mu W_t - 1) \, dt, \quad W_0 = w, \quad W_t \geq 0,
\]

which has a solution

\[
W_t = (w - \frac{1}{\mu}) e^{\mu t} + \frac{1}{\mu}. \tag{2}
\]

We are interested in finding the time \(t^*\) at which the investor’s wealth is used up, so that

\[
W_t = \begin{cases} 
(w - \frac{1}{\mu}) e^{\mu t} + \frac{1}{\mu}, & t < t^* \\
0, & t \geq t^*
\end{cases}
\]

Solving for \(t^*\),

\[
t^* = \frac{1}{\mu} \ln \left[ \frac{1}{1 - \mu w} \right]. \tag{3}
\]

A retiree invested in a risk-free portfolio knows if and when her wealth will expire, and for large enough combinations of investment return and initial wealth \((\mu w \geq 1)\) she will never reach zero wealth while consuming only a dollar per year.

\(^6\) A complete analysis of optimal consumption and investment paths with targeted pensions is outside the scope of this study and we leave it to future research.

\(^7\) In this section we follow Milevsky (2006, chapter 9 and appendix to chapter 9) and Huang et. al (2004).
If the retiree now invests in a risky portfolio of assets following a geometric Brownian motion with known drift and diffusion,

\[ dS_t = \mu S_t dt + \sigma dB_t \]  

(4)

where \( B_t \) is a standard Wiener process, then the solution to (4) is

\[ S_t = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}, S_0 = 1. \]  

(5)

If the retiree keeps consuming at a continuous rate of one dollar per year, the wealth process is now

\[ dW_t = dS_t - 1 dt = (\mu W_t - 1) dt + \sigma W_t dB_t, W_0 = w, \]  

(6)

and the solution to this stochastic differential equation is

\[ W_t = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t} \left[ w - \int_0^t e^{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s} ds \right], W_0 = w, \]  

(7)

or equivalently,

\[ W_t = S_t \left[ w - \int_0^t S_s^{-1} dt \right], W_0 = w. \]  

(8)

The crucial difference between the draw-down process (7) and the simple portfolio process (5) is that a standard geometric Brownian motion \( dS_t \) does not become negative, but the wealth process here with constant draw-down \( dW_t \) can become negative if the drift \( \mu W_t \) is small.

One way to evaluate a consumption plan for retirement is to determine the probability of exhausting wealth before the end of life. The probability of reaching ‘ruin’ \( (\hat{t}^*) \) is

\[ \phi (w) := \Pr \left[ \inf_{0 \leq s \leq \hat{t}^*} W_s \leq 0 | W_0 = w \right]. \]  

(9)

---

\( ^8 \)For some intuition on \( \int_0^t S_t^{-1} dt \), consider the discrete time analogue. A $1 draw-down discounted at a stochastic rate has a present value \( SPV = \sum_{t=0}^{T} \prod_{j=1}^{t} \left( 1 + R_j \right)^{-1} \). We compare this present value with initial wealth to determine the probability of net wealth reaching zero.
or the likelihood that the lowest value of the stochastic process goes to zero before the retiree reaches the terminal date, $T$.

A person’s tolerance for the probability of ruin is related to risk preferences: we could think of retirement utility as some general function where the level of (constant real) consumption is a positive argument and the probability of ruin is a negative argument. Further, Milevsky and Robinson (2005) propose that asking a retiree a straightforward question about willingness to tolerate possible ruin may be as good a guide to risk preferences as hypothetical surveys commonly used by financial advisors to assess risk tolerance. For the constant draw-down process here, this probability depends on whether (7) becomes zero during the course of retirement. Since the portfolio return $S_t$ is bounded away from zero, retirement wealth can go to zero only if the stochastic present value of the spending plan approaches initial wealth.

$$Z_T := \int_0^T e^{-\left(\frac{\mu - 1/2\sigma^2}{2}\right)t - \sigma B_t} dt \to w.$$  (10)

As $t$ increases, $Z_t$ increases monotonically, which means that if $W_t$ does becomes negative, it cannot recover (even very high returns cannot increase a zero wealth). As a result, the probability of ruin before a pre-determined time $T$ is

$$\phi (w) = \Pr[w \leq \int_0^T e^{-\left(\frac{\mu - 1/2\sigma^2}{2}\right)t - \sigma B_t} dt] = 1 - \Pr[Z_T < w],$$  (11)

or the likelihood that the stochastic present value of the spending plan exceeds initial wealth. If the time horizon is infinite, Huang et al. (2004) prove that the ruin probability has a closed form analytic solution. But here we look at the case of a limited lifetime, $T < \infty$, and more specifically at the case of an uncertain and finite length of life, where $T_x < \infty$ is a random variable following a known mortality law.

For a random lifetime, the probability of ruin

$$Z_{T_x} := \int_0^{T_x} e^{-\left(\frac{\mu - 1/2\sigma^2}{2}\right)t - \sigma B_t} dt$$  (12)

$$\phi (w) = 1 - \Pr[Z_{T_x} < w].$$  (13)

---

9 $\phi (w) = \Gamma \left(\frac{a}{w}, b\right)$, where $a = \frac{2\mu}{\sigma^2} - 1$ and $b = \frac{2}{\sigma^2}$.
However, since the density function of $Z_{T_x}$ is unknown, we need an approximation method to compute the probability of ruin when the length of life is uncertain. Huang et al. outline an approximation based on a moment matching approach. Using the law of iterated expectations, the first moment of the random variable $Z_{T_x}$ is

$$
E(Z_{T_x}) = E[E[Z_t|T_x = t]]
$$

$$
= E \left[ E \left[ e^{-(\mu - 1/2\sigma^2)s - \sigma B_s} ds \right] | T_x = t \right]
$$

$$
= E \left[ \int_0^t e^{-(\mu - 1/2\sigma^2)s} E(e^{-\sigma B_s}) ds | T_x = t \right]
$$

$$
= E \left[ \int_0^t e^{-(\mu - 1/2\sigma^2)s + s\sigma^2} ds | T_x = t \right]
$$

$$
= \int_0^\infty e^{-(\mu - \sigma^2)t} \, t \, p_x \, dt
$$

(14)

where $t \, p_x$ is the conditional probability of an individual surviving $t$ more years, having reached age $x$. Given an instantaneous force of mortality (hazard rate) $\lambda(t)$, the conditional probability of survival, $t \, p_x$, can be expressed as

$$
t \, p_x = \exp \left[ - \int_x^{x+t} \lambda(s) \, ds \right].
$$

(15)

To evaluating the integral (14) we need to specify a density function for the lifetime random variable. The simplest approach is to assume that survival is exponentially distributed over the later years of life so that the hazard rate is constant and

$$
t \, p_x = \exp \left[ - \int_x^{x+t} \lambda ds \right] = \exp [-\lambda t].
$$

For an exponential mortality law we can evaluate (14):

$$
M_e^{(1)} = E[Z_{T_x}] = \int_0^\infty \exp \left[ - (\mu - \sigma^2) s \right] s \, p_x \, ds
$$

$$
= \int_0^\infty \exp \left[ - (\mu - \sigma^2 + \lambda) s \right] ds
$$

(16)

$$
= \frac{1}{\mu - \sigma^2 + \lambda}.
$$
To derive the second moment takes more algebra, but Huang et al. show that

\[ M^{(2)}_e = E \left[ Z_{T_x}^2 \right] = \frac{2}{(\mu - 2\sigma^2)} \int_0^\infty \left\{ \exp \left[ - \left( \mu - \sigma^2 \right) s \right] - \exp \left[ - \left( 2\mu - 3\sigma^2 \right) s \right] \right\} \cdot dp_x ds \]

\[ = \frac{2}{(\mu - 2\sigma^2)} \left( \frac{1}{\mu - \sigma^2 + \lambda} - \frac{1}{2\mu - 3\sigma^2 + \lambda} \right) \]

\[ = \frac{2}{(\mu - \sigma^2 + \lambda) (2\mu - 3\sigma^2 + \lambda)} . \quad (17) \]

Despite making that analysis simple, the constant hazard rate is not the best representation of mortality later in life and the Gompertz law offers a more accurate model of survival by allowing the hazard rate to vary with time. The Gompertz function sets

\[ \lambda(x) = \frac{1}{b} \exp \left( \frac{x - m}{b} \right) . \quad (18) \]

where \( b \) and \( m \) are the scale (dispersion) and mode parameters of the distribution. Using (18) and the definition of \( \cdot p_x \) in (15) we can express the conditional survival probability as:

\[ \cdot p_x = \exp \left[ - \int_x^{x+t} \frac{1}{b} \exp \left( \frac{u - m}{b} \right) du \right] \]

\[ = \exp \left[ b\lambda_x \left( 1 - e^{\frac{t}{b}} \right) \right] . \quad (19) \]

For Gompertz mortality the first moment integral in (14) evaluates to

\[ M^{(1)}_G = E \left[ Z_{T_x} \right] = A \left( \xi | m, b, x \right) , \xi = \left( \mu - \sigma^2 \right) \quad (20) \]

\[ A \left( \xi | m, b, x \right) = b \exp \left\{ \exp \left[ \frac{x - m}{b} \right] + (x - m) \xi \right\} \Gamma \left( -b\xi , \exp \left[ \frac{x - m}{b} \right] \right) \]

where \( \Gamma (u, v) = \int_v^\infty e^{-t} t^{u-1} dt \) is the incomplete Gamma function. Similarly, the second moment is

\[ M^{(2)}_G = E \left[ Z_{T_x}^2 \right] = \left( \frac{2}{\mu - 2\sigma^2} \right) \left[ A \left( \mu - \sigma^2 | m, b, x \right) - A \left( 2\mu - 3\sigma^2 | m, b, x \right) \right] . \quad (21) \]

Having identified the first two moments of the true but unknown density function
of $Z_{T_s}$ under two alternative mortality distributions, the issue is to what known density function can they be approximated? The limiting distribution for $Z_\infty (T \to \infty)$ is a reciprocal Gamma distribution,

$$\Pr [Z < z] := \frac{\beta^{-\alpha}}{\Gamma(\alpha)} \int_0^t y^{-(\alpha+1)} e^{-y/\beta} dy$$

which has first and second moments of

$$M^{(1)} = \frac{1}{\beta (\alpha - 1)}, \quad M^{(2)} = \frac{1}{\beta^2 (\alpha - 1)(\alpha - 2)}$$

so that

$$\alpha = \frac{2M^{(2)} - M^{(1)}M^{(1)}}{M^{(2)} - M^{(1)}M^{(1)}}, \quad \beta = \frac{M^{(2)} - M^{(1)}M^{(1)}}{M^{(2)}M^{(1)}}. \quad (23)$$

Given this limiting result, Huang et al. propose approximating the distribution of $Z_{T_s}$ using the moments derived above in equations (16) - (17) or (20) - (21) substituted into (23) and evaluated as a reciprocal Gamma random variable. For exponential utility, the expressions for the reciprocal Gamma parameters are simple:

$$\hat{\alpha}_e = \frac{2\mu + 4\lambda}{\sigma^2 + \lambda} - 1, \quad \hat{\beta}_e = \frac{\sigma^2 + \lambda}{2}, \quad (24)$$

but we use a numerical evaluation for the Gompertz case.

The value we are primarily interested in is the probability of ruin, or the probability that the stochastic present value of the constant consumption path, given uncertainty over returns and longevity, is less than initial wealth $W_0 = w$. Since the probability that

$$\phi(w) = 1 - \Pr[Z_{T_s} < w] = \Pr[Z_{T_s} > w],$$

we can use the parallel between the Gamma and reciprocal Gamma distributions to expedite calculation: the probability that a reciprocal Gamma random variable is greater than a particular value is equal to the probability that a Gamma random variable is less than the inverse of that value, hence

$$\phi(w) = \Pr[\inf_{0 \leq s \leq T} W_s \leq 0 | W_0 = w] \approx G\left(\frac{1}{w} | \alpha, \beta\right). \quad (25)$$

where the right hand side is the probability that a random variable with a Gamma distribution defined by $\alpha$ and $\beta$ is less than $\frac{1}{w}$. The Gamma distribution function is embedded
in standard software packages. (For the Gompertz moments, we need to evaluate the Gamma distribution over a negative parameter which requires rescaling by a method described in Appendix A.)

Equation (25) could also be used to infer required wealth for a given age, consumption plan and probability of ruin, to infer a required rate of return, or the safest investment portfolio.

4 Parameter selection

Replicating the Age Pension using the stochastic present value method requires three parameters - the drift and diffusion terms for the portfolio process $\mu$ and $\sigma$, that is the return and volatility of the portfolio selected by the retiree, and the instantaneous force of mortality, $\lambda$, (or its Gompertz equivalent using the scale and mode parameters $b$ and $m$).

4.1 Portfolio return and volatility

Consistent with our aim of establishing how much wealth a self-funding retiree would need to generate a consumption stream equal to the Age Pension in value and certainty, we confine ourselves to the portfolios typically offered to Australian superannuants.

We label our five artificial portfolios as High Growth, Growth, Balanced, Conservative and Capital Stable where each is a combination of two or more asset classes from Australian shares, international shares, Australian property securities, Australian fixed interest and cash.\textsuperscript{10} Portfolio weights are set out in Figure 3. The portfolios decline in exposure to growth assets, from 90% allocation to shares and property in the High Growth fund, 70% in Growth, 50% in Balanced, 30% in the Conservative fund and Capital Stable entirely invested in cash and fixed-interest securities.\textsuperscript{11}

\textsuperscript{10}We make no claim that these constructed investments are optimal since a wider variety of assets and possibly more efficient weighting schemes are available. However a quick survey of providers (say, AMP, Vanguard or BT) will convince the reader that our choices are typical.

\textsuperscript{11}The average Australian retiree chooses to hold a relatively high proportion of growth assets in their portfolio - more than 50% in property and equities - according to survey data (see Bateman et al. 2007).
To estimate real returns and volatility for each of these portfolios we collect monthly time series of returns indices for each asset class over the 16-year period, 30 December 1989 – 30 December 2005. For each asset class, we compute a monthly periodic return and apply portfolio weights (Figure 3) to get a portfolio return, so that \( (1 + i_{P,t}) = \sum_{j=1}^{n} \omega_j (1 + i_{j,t}) \) where \( (1 + i_{P,t}) \) is the gross nominal monthly portfolio return over month \( t \), \( \omega_j \) is the proportion allocated to asset class \( j \) and \( (1 + i_{j,t}) \) is the nominal monthly gross return to asset index \( j \). To translate this to a real return, we derive a monthly percentage change in the quarterly Consumer Price Index by linear interpolation \( h_t \) (or for MTAWE, \( h_t \))

\(^{12}\) Returns indices approximate an aggregate growth in capital value and re-investment of dividends.
$n_t$) and compute the monthly log-change in the real portfolio return as

$$r_{P,t} = \ln S_t - \ln S_{t-1} = \ln (1 + i_{P,t}) - \ln (1 + h_t)$$  

(26)

or if we deflate by the greater of inflation and earnings growth

$$r_{P,t} = \ln S_t - \ln S_{t-1} = \ln (1 + i_{P,t}) - \ln (\max [(1 + h_t), (1 + n_t)]) .$$  

(27)

The annualised expected value and volatility of this process are:

$$\mu = 12 \frac{1}{T} \sum_{t=1}^{T} (r_{P,t}) + \frac{1}{2} \sigma^2$$

$$\sigma = s \sqrt{12},$$

$$s = \sqrt{\frac{1}{T-1} \sum_{t=1}^{T} (r_{P,t})^2 - \frac{1}{T(T-1)} \left[ \sum_{t=1}^{T} (r_{P,t}) \right]^2}$$

where $T$ is the number of observations.

Table 2 shows the nominal and real returns and volatilities for each of the portfolios. Returns and volatilities, not surprisingly, increase with exposure to growth assets, and the average inflation rate is 2.8% and the average earnings-augmented deflation is 4.4%. These real returns are tempered by poor international equity results in the later part of the sample, but, coming as they do out of 16 years of economic expansion in Australia, and coinciding with strong domestic equity and property performance, may be overly optimistic as a proxy for future returns. However to match up with net returns to retirement investors, we also deduct an indicative management fee from the real returns, such as are charged by allocated pension providers offering similar investments (see, for example, AMP Allocated Pension Product Disclosure Statement for accounts of value 100-499K).

<Insert Table 2 here>

### 4.2 Force of mortality

Over the past hundred years, mortality rates in Australia have been declining rapidly. By the publication of the most recent (2000-2002) Life Tables (Commonwealth of Australia
2004), mortality rates were 40-45% lower than in the mid-1960s. Improved life expectancy implies longer retirements and raises the value of a guaranteed income stream such as the Age Pension. Even so, uncertainty over the length of life is a crucial factor in lifecycle planning and one of the advantages of the stochastic present value model is that it incorporates this risk via approximations to the survival density.

### 4.2.1 Exponential force of mortality

Assuming that the instantaneous force of mortality, $\lambda$, for a person aged $x$ is constant, (i.e. exponentially distributed), we can compute an estimate of $\lambda$ directly from the improved expectation of life from the Life Tables, since $E(T) = \frac{1}{\lambda}$. To allow for improvements to mortality over the remaining lifetimes of current retirees, we adjust expected remaining lifetime by the specified 25-year improvement factors (see Life Tables documentation for the method). Table 3 displays the remaining lifetimes and mortality rates for males and females from ages 60 to 80.

<Insert Table 3 here>

### 4.2.2 Gompertz force of mortality

We estimate parameters $b$ and $m$ using non-linear least squares as $\log(p_x) = \exp\left(\frac{x-m}{b}\right) \left(1 - \exp\frac{1}{b}\right)$, taking discrete mortality data, $p_x$, from the Australian Life Tables 2000-2002. The conditional survival probability $p_x$ we use in estimation is adjusted by the 25 year improvement factors as described in the Life Tables. Model fit worsens if the sample includes the thin mortality data at extreme old age, so the sample runs from ages 50 to 90. Table 4 reports estimation results for males and females.

<Insert Table 4 here>

Having estimated a range of parameter values to reflect investment choices and current mortality for Australians eligible for the Age Pension we input these to equation (25) and infer a probability of retirement ruin. Alternatively we can fix the likelihood of ruin, the drift and diffusion, and infer the initial wealth needed to fund a payment scheme.
5 Replicating the Age Pension

The stochastic present value model can be used to predict the sustainability of a self-funded retiree’s investment and spending plan without using simulation experiments. As an example, Table 5 shows the probability that an individual will run out of money before the end of life. In this example our investor reaches age 65 and retires with a net $1,000,000 and decides on a fixed real spending plan of between $20,000 and $100,000 each year. The retirement ruin probabilities in Table 5 are based on Gompertz mortality measures with the lowest probability of ruin for each draw-down rate marked with an asterisk. The probability of ruin is generally higher for women than for men, because of longer life expectancy. Which investment strategy creates the most sustainable spending path depends on the required level of drawdown: for rates of drawdown between 4-6% of initial retirement wealth, the growth portfolio is least likely to be exhausted, whereas at higher expenditure (8% of initial wealth) the high growth portfolio is least-risk for women.

If preferences are measurable in terms of ruin probability, then a retiree could decide on an investment and spending plan by trading off an increase in ruin probability against an increase in spending. However we note that in some respects the Stochastic Present Value method is at odds with conventional utility theory. A constant real level of consumption is not an optimal strategy for a power utility maximizer - the best plan is a constant rate of draw-down (leaving aside survival uncertainty). As Brown (2000) points out, a constant level of consumption implies infinite risk aversion for an individual with power utility preferences over consumption, manifesting in complete unwillingness to transfer consumption across time. And further, the utility maximising consumer with conventional preferences will never allow wealth to fall to zero because at zero wealth the marginal utility of consumption is infinite. While theoretically sub-optimal for conventional preferences, a constant real consumption stream is exactly what is offered under

---

13 Probabilities based on exponential mortality are reported in Appendix B.
14 The Association of Superannuation Funds of Australia (ASFA) estimate that a single, home-owning retiree in 2006 needed around $18,200 p.a. for a modest lifestyle and $35,400 p.a. to maintain a comfortable lifestyle.
the Age Pension and similar basic pension schemes around the world, so the stochastic present value method seems a reasonable valuation approach.

While it could be argued that, given the Australian Government’s historical support for the program, the likelihood of running out of money when receiving the Age Pension approaches zero, in the analysis below we accept some regulation risk. We assume that the Age Pensioner faces a very small but non-zero probability of ruin, hence we entertain a small possibility that pension support could be removed or modified in such a way as to be out of the reach of the retiree.

How much retirement wealth would an individual self-funded retiree need to replicate a constant real income of $14,000 p.a. and what would be the least-risk approach to generating that income stream? The stochastic present value model shows that the investment required to fund the Age Pension payment depends on age, gender and investment strategy. Table 6 sets out the amount of initial wealth needed to support the pension payment for men and women of average mortality who invest in standard managed funds. We compute this wealth amount for ages 65 and 70 years in order to evaluate the impact of delaying retirement (or changing the eligibility date) and report it as a multiple of average annual earnings. In February 2007, full-time adult ordinary time earnings were $1071.70 per week or $55728.40 p.a. seasonally adjusted (ABS release 6302.0).

The required wealth level at retirement for a 65 year-old female ranges from as much as 8.24 times average annual earnings for very high or low risk investment portfolios at 0.5% probability of failure, to 5.31 times earnings for the growth portfolio with a 5% probability of failure. At our benchmark 1% failure probability the most efficient investment allocation is to a balanced fund which requires 6.79 times average earnings or accumulated wealth of $378,581. This amount is 8.6 times more than current average superannuation balances for females 60-65 years (approximately $44,000, ASFA 2007). More conservative and more risky investment strategies need more savings to generate the real income stream with the desired level of security. By contrast, commercial insurance firms currently offer CPI-indexed single life annuities paying $14,000 p.a. at a premium of $337,455, or 6.06 times
average annual earnings.\textsuperscript{15}

For a female retiring at age 70, growth portfolios are slightly better than balanced portfolios, but the required wealth is not much reduced by a five year delay. At 1% risk, 6.44 times average annual earnings is required, or $358,945, about $20,000 less than the level for 65 years.

For males, wealth requirements are similar to females at the 1% probability of failure. The slightly riskier growth portfolios, with 70% exposure to equity and property assets, are most efficient for generating the real income stream. A 65 year old male needs 6.8 times average earnings ($379,194), and a 70 year old male needs 6.34 times average earnings ($353,399), again a reduction of around $20,000 by delaying retirement. By age 70, lower life expectancies for men mean that required wealth is less than for females. These amounts are 3 times as large as estimates of the current average male accumulation (approximately $130,000, ASFA 2007). The cost of a single-life CPI indexed annuity for a 65 year old male is $328,844 or 5.9 times average earnings, which, as for females, is less costly than self-insurance.

As discussed in Section 2, Age Pension payments are adjusted to be no less than 25% of MTAWE. In Table 7 we value this connection with earnings growth by computing the wealth needed at retirement to generate an income stream that maintains real value and parity with earnings. We do this by ‘deflating’ nominal returns by the maximum of monthly changes in prices and earnings over the sample. Since earnings have outpaced inflation historically, larger accumulations are needed to match the growth in wages.

\textless Insert Table 7 here\textgreater

At our benchmark 1% probability of failure, a 65 year old female needs 8.55 times average annual earnings ($476,694) in retirement savings to generate the earnings and inflation adjusted pension payment. The amount drops to 7.88 times average earnings ($439,296) if retirement is delayed to age 70. These amounts are 26% and 22% more than the required wealth if the pension tracks inflation only and we conclude that the link to earnings is very valuable to pensioners. Males require 8.44 times average earnings at age

\textsuperscript{15}Average purchase price to generate $14000 p.a. using CPI indexed single life annuities without guarantee for 65 year old male and female, Table F, DeXX&R Retirement Incomes League Tables, Quarterly Statistics ending December 2006.
65, and 7.64 at age 70, a 24% and 20% increase over the amount needed to match inflation increases only. The closest single life annuity to the earnings-linked Age Pension payment is a single life indexed to rise at 5% p.a. A 65 year old female would pay $421,559 or 7.56 times average earnings for a 5% indexed annuity paying $14,000 in the first year, whereas the premium for a male is currently $368,227, or 6.6 times average annual earnings, again below the cost of self-insurance.

We conclude that the public pension creates a large notional transfer of wealth at retirement that would be difficult for a typical worker to replicate under current rates of earnings and retirement savings. The most efficient way to replicate the pension is via a single life annuity which incorporates mortality credits, but purchases of life annuities in Australia is very low compared with other retirement income streams: most retirees appear to prefer to maintain control over their capital when given a choice.\textsuperscript{16} For retirees wishing to replicate an income stream to the value of the Age Pension using individual investment accounts, balanced or growth portfolios are most efficient, but very large accumulations are needed, upwards of 6 times average earnings to generate a real income stream equal to the current payment and close to 8.5 times average earnings to keep up with increases in wages and prices.

6 Conclusion

The majority of elderly Australians rely on a targeted public pension, the Age Pension, as their main source of retirement income. Despite the maturing of the mandatory, earnings-linked superannuation scheme begun in 1992, individual retirement accumulations are modest and likely to remain small in coming decades. General dependence by the elderly on transfers from public revenue is forecast to continue. However, Australian Government policy aims to alleviate increasing demands on public funds by encouraging more reliance on private retirement savings rather than the basic pension. Here we ask the question, how much private savings would an individual have to accumulate to replicate the payments and insurance features of the basic public pension? We compute the retirement wealth

\textsuperscript{16} Purchases of life annuities are less than 0.2% of the retirement incomes market (Plan for Life 2006).
that would allow a retiree to enjoy the same benefits as the basic pension using the standard draw-down products of the Australian retirement incomes market. This amount represents the stochastic present value at retirement of the Age Pension payment stream.

Since the exact density function of the stochastic present value of any retirement spending plan is not known when lifetimes are uncertain, we use a moment-matching approximation (Milevsky and Robinson 2000, 2005 and Huang et al. 2004) to value a spending plan equivalent to the pension. Allowing for a very low probability of reaching ‘ruin’, we back out the initial nest egg needed to replicate a $14,000 pension while accounting for investment and longevity risk. We interpret this initial wealth as the value that a self-insured retiree would attach to full pension eligibility.

We estimate that the implicit public transfer to Age Pensioners is substantial, in the order of $450,000 at age 65 or 8.5 times current average annual earnings. This amount is many times larger than current average superannuation accumulations. The implicit transfer is generally larger for women because of longer life expectancy, and harder for women to attain by private savings because earnings-related accumulations are commonly smaller than for men. Delaying retirement by five years reduces required wealth by only 5% or less. On the other hand, 25% more wealth is needed to maintain the relative level of the pension with wages, as compared with indexing to consumer prices. Finally, despite their marked unpopularity with the retired, commercial life annuity products replicate public pension payment paths much more cheaply than drawn-down plans invested in managed funds.

Dramatic increases in retirement savings are needed if lower levels of Age Pension reliance are to be realised. The likelihood of an average worker accumulating sufficient in second pillar savings to replicate the basic pension seems remote. On the other hand, the high implicit value of pension eligibility creates incentives for retirees to draw down private savings faster in order to access pension benefits.
References


Appendix A: Rescaling the incomplete gamma function

A numerical complication arises from the fact that the incomplete gamma function which appears in the moments (20) and (21) is difficult to compute in most software packages because $-1 < -\xi b < 0$ and the packages will not return gamma values defined over negative parameters. A rescaling derived from Milevsky (2001) allows the incomplete gamma function to be rewritten over $(-\xi b + 1) > 0$.

The standard probability density function of the gamma distribution is

$$g_a(x) = \frac{e^{-x}x^{a-1}}{\Gamma(a)},$$

where $x, a > 0$, and the cumulative density function is given by:

$$G_a(c) = \Pr [X \leq c] = \int_0^c \frac{e^{-x}x^{a-1}}{\Gamma(a)} dx.$$  \hfill (29)

To evaluate the moments we need values of the incomplete gamma function:

$$\Gamma(a, c) = \int_c^\infty e^{-x}x^{a-1}dx \quad (30)$$

$$= \Gamma(a)(1 - G_a(c)). \quad (31)$$

where $a = -\xi b$ and $c = \exp \left[ \frac{x-m}{b} \right]$. The negative value $-\xi b$ rules out using standard software to retrieve these values. Milevsky (2001) suggests redefining the incomplete gamma function over $-\xi b + 1$, which will be non-negative, and then rescaling to get back to the original problem.

Integrating (30) by parts gives

$$\int e^{-x}x^{a-1}dx = e^{-x} \frac{1}{a} x^a + \frac{1}{a} \int e^{-x}x^a dx,$$

and so
\[ \Gamma(a, c) = -\frac{ce^{-c}}{a} + \frac{1}{a} \Gamma(a + 1, c). \] (33)

Using (33) we can rewrite \( \Gamma(a, c) \) as:

\[ \Gamma(a, c) = \frac{1}{a} \Gamma(a + 1)(1 - G_{a+1}(c)) - \frac{ce^{-c}}{a}. \] (34)

Equation (34) is easily programmed into standard spreadsheet packages. For the cases where \(-\xi b < -1\), we rescale to \(-\xi b + 2\) using the recursion:

\[ \Gamma(a, c) = \frac{1}{a} \left[ \frac{1}{a+1} \Gamma(a + 2)(1 - G_{a+2}(c)) - \frac{ce^{a+1}e^{-c}}{a+1} \right] - \frac{ce^{-c}}{a}. \]
Appendix B: Probability of retirement ruin with exponential mortality

Probability of retirement ruin for male (female) age 65, initial wealth $1 million dollars, exponential mortality

Table shows the area under the Gamma density \( \phi(w) = \Pr[\inf_{0 \leq s \leq T} W_s \leq 0|W_0 = w] \cong G\left(\frac{1}{w}|\alpha, \beta\right) \) where the percentage real spending rate is \( 1/w \) and \( \alpha \) and \( \beta \) are the parameters of the Gamma density determined by moment matching (see section 2 above). The probabilities reported are that a 65 year old female (male) Australian retiree with initial wealth of $1 million will reach zero wealth before the end of life, when the probability of survival is approximated using the exponential distribution.

<table>
<thead>
<tr>
<th>Probability of retirement ruin (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Female 65 years</strong></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>High Growth</td>
</tr>
<tr>
<td>Growth</td>
</tr>
<tr>
<td>Balanced</td>
</tr>
<tr>
<td>Conservative</td>
</tr>
<tr>
<td>Capital Stable</td>
</tr>
</tbody>
</table>

<p>| <strong>Male 65 years</strong> | <strong>Real Spending rate, $000 p.a.</strong> |
|-------------------|
|                    | 20 | 40 | 60 | 80 | 100 |
| High Growth        | 0.3 | 3.3 | 11.0 | 22.9 | 36.7 |
| Growth             | 0.3 | 3.0 | 10.6 | 22.5 | 36.8 |
| Balanced           | 0.3 | 3.4 | 11.9 | 24.9 | 40.0 |
| Conservative       | 0.4 | 4.1 | 13.9 | 28.2 | 44.1 |
| Capital Stable     | 0.6 | 5.8 | 17.8 | 33.8 | 50.4 |</p>
<table>
<thead>
<tr>
<th>Age Pension and Related Benefits</th>
<th>Regular Payment</th>
<th>Annual value</th>
<th>Adjustment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age pension payment</td>
<td>$525 fortnightly</td>
<td>$13653</td>
<td>(\max(CPI, 0.25MTAVE))</td>
</tr>
<tr>
<td>Pharmaceuticals allowance</td>
<td>$6 fortnightly</td>
<td>$151</td>
<td>CPI</td>
</tr>
<tr>
<td>Rent assistance</td>
<td>$104 fortnightly</td>
<td>$2704</td>
<td>CPI</td>
</tr>
<tr>
<td>Telephone allowance</td>
<td>$21 quarterly</td>
<td>$86</td>
<td>CPI</td>
</tr>
<tr>
<td>Utilities allowance</td>
<td>$53 semi-annually</td>
<td>$106</td>
<td>CPI</td>
</tr>
<tr>
<td>Remote area allowance</td>
<td>$18 fortnightly</td>
<td>$473</td>
<td>by legislation</td>
</tr>
<tr>
<td>Pensioner Concession Card</td>
<td>Access to PBS plus lower state and local government charges e.g. water, property and vehicle taxes, energy bills, public transport fares</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Source: Centrelink, Department of Human Services, Australian Government
Table 2. Portfolio summary statistics.

This table presents estimates of nominal returns, real returns and standard deviation values corresponding to each of the portfolios. Portfolio returns are the annualised log change in the weighted sum of monthly periodic returns to the component asset classes, less the monthly log change in the CPI for real returns or less the monthly log change in the greater of the CPI and MTAWE for earning-adjusted returns. (Weights for each portfolio are given in Figure 3.)

We compute monthly gross returns to each asset class index where Australian equities are the Australia-DS Market index, International equities are the AC WORLD INDEX ex AUSTRALIA translated into Australian dollars at the end-month AUD/USD exchange rate, fixed income is the UBS Composite All Maturities index for Australia, property is the S&P/ASX 300 Property index and cash is the UBS AU Bank Bills All Maturities index, all from Datastream. The total return price index (RI) of the relevant asset class index was used for calculations of the periodic monthly returns. As a measure of inflation (earnings), we use a linear interpolation of quarterly annualised growth in the CPI (MTAWE), translated into a monthly log change. The CPI data are from the Reserve Bank of Australia database and MTAWE is from the Australian Bureau of Statistics Publication 6302.0. Indicative management fees are taken from the AMP Allocated Pension Product Disclosure Statement (AMP 2007) for managed fund investments of similar risk exposure and account size $100-$499k.

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Nominal Return</th>
<th>CPI-adj Return</th>
<th>CPI/MTAWE adj Return</th>
<th>less fees</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>High Growth</td>
<td>10.4%</td>
<td>7.6%</td>
<td>5.7%</td>
<td>6.0%</td>
<td>9.9%</td>
</tr>
<tr>
<td>Growth</td>
<td>10.0%</td>
<td>7.2%</td>
<td>5.4%</td>
<td>5.6%</td>
<td>7.9%</td>
</tr>
<tr>
<td>Balanced</td>
<td>9.2%</td>
<td>6.4%</td>
<td>4.7%</td>
<td>4.8%</td>
<td>5.8%</td>
</tr>
<tr>
<td>Conservative</td>
<td>8.5%</td>
<td>5.7%</td>
<td>4.0%</td>
<td>4.1%</td>
<td>3.8%</td>
</tr>
<tr>
<td>Capital Stable</td>
<td>7.5%</td>
<td>4.7%</td>
<td>3.1%</td>
<td>3.1%</td>
<td>1.5%</td>
</tr>
</tbody>
</table>
Table 3: Average remaining lifetimes and constant mortality rates

This table shows estimated constant instantaneous force of mortality $\lambda$, where $E(T) = \frac{1}{\lambda}$ using expected remaining lifetime $e0x$ from the Australian Life Tables 2000-2002, with and without 25-year improvements in mortality (see Life Tables documentation for improvement method).

<table>
<thead>
<tr>
<th>Age</th>
<th>Exptd Life</th>
<th>Mortality Rate $\lambda$</th>
<th>Exptd Life</th>
<th>Mortality Rate $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>without</td>
<td></td>
<td>with</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Improvements</td>
<td></td>
<td>Improvements</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>21.7</td>
<td>0.0462</td>
<td>23.9</td>
<td>0.0419</td>
</tr>
<tr>
<td>65</td>
<td>17.7</td>
<td>0.0565</td>
<td>19.5</td>
<td>0.0514</td>
</tr>
<tr>
<td>70</td>
<td>14.1</td>
<td>0.0710</td>
<td>15.4</td>
<td>0.0649</td>
</tr>
<tr>
<td>75</td>
<td>10.9</td>
<td>0.0918</td>
<td>11.8</td>
<td>0.0846</td>
</tr>
<tr>
<td>80</td>
<td>8.2</td>
<td>0.1214</td>
<td>8.8</td>
<td>0.1132</td>
</tr>
</tbody>
</table>

Panel B: Females

<table>
<thead>
<tr>
<th>Age</th>
<th>Exptd Life</th>
<th>Mortality Rate $\lambda$</th>
<th>Exptd Life</th>
<th>Mortality Rate $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>without</td>
<td></td>
<td>with</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Improvements</td>
<td></td>
<td>Improvements</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>25.4</td>
<td>0.0393</td>
<td>27.3</td>
<td>0.0366</td>
</tr>
<tr>
<td>65</td>
<td>21.1</td>
<td>0.0473</td>
<td>22.8</td>
<td>0.0439</td>
</tr>
<tr>
<td>70</td>
<td>17.1</td>
<td>0.0586</td>
<td>18.4</td>
<td>0.0544</td>
</tr>
<tr>
<td>75</td>
<td>13.3</td>
<td>0.0750</td>
<td>14.3</td>
<td>0.0699</td>
</tr>
<tr>
<td>80</td>
<td>10.0</td>
<td>0.1002</td>
<td>10.6</td>
<td>0.0939</td>
</tr>
</tbody>
</table>
Table 4: Estimated Gompertz parameters.

Table displays estimated coefficients and fit statistics for the non-linear least-squares estimation of the Gompertz equation, \( p_x \), is the improved probability of surviving one more year having reached age \( x \), and \( b \) and \( m \) are the scale and mode parameters of the distribution. Data are the discrete survival probabilities for males (females) aged 50 to 90 years from the 2000-2002 Australian Life Tables, improved by the 25-year improvement factors.

Estimated equation: \( \log(p_x) = \exp \left( \frac{x-m}{b} \right) \left( 1 - \exp \frac{1}{b} \right) \)

Sample: 50-90 years

<table>
<thead>
<tr>
<th></th>
<th>Males</th>
<th>Females</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coefficient</td>
<td>(\hat{m})</td>
<td>(\hat{m})</td>
</tr>
<tr>
<td>p-value</td>
<td>(0.000)</td>
<td>(0.000)</td>
</tr>
<tr>
<td>(R^2)</td>
<td>0.998</td>
<td>0.999</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>(\hat{b})</th>
<th>(\hat{b})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coefficient</td>
<td>8.95</td>
<td>7.60</td>
</tr>
<tr>
<td>p-value</td>
<td>(0.000)</td>
<td>(0.000)</td>
</tr>
</tbody>
</table>


Table 5: Probability of retirement ruin for male (female) age 65, initial wealth $1 million dollars.

Table shows the area under the Gamma density \( \phi(w) = \Pr\left[\inf_{0\leq s\leq T} W_s \leq 0|W_0 = w\right] \cong G\left(\frac{1}{w^2}\right)\) where the percentage real spending rate is \(1/w\) and \(\alpha\) and \(\beta\) are the parameters of the Gamma density determined by moment matching (see Section 3 above). The probabilities reported are that a 65 year old female (male) Australian retiree with initial wealth of $1 million will reach zero wealth before the end of life, when the probability of survival is approximated using an estimated Gompertz distribution. Probabilities marked with an asterisk are the least for each spending rate, indicating the most efficient portfolio allocation.

<table>
<thead>
<tr>
<th>Probability of retirement ruin (%)</th>
<th>Portfolio</th>
<th>Real Spending rate, $000 p.a.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Female 65 years</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Portfolio</strong></td>
<td><strong>Real Spending rate, $000 p.a.</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>40</td>
</tr>
<tr>
<td>High Growth</td>
<td>0.03</td>
<td>2.6</td>
</tr>
<tr>
<td>Growth</td>
<td>0.008</td>
<td>1.8*</td>
</tr>
<tr>
<td>Balanced</td>
<td>0.004*</td>
<td>1.8</td>
</tr>
<tr>
<td>Conservative</td>
<td>0.004</td>
<td>2.2</td>
</tr>
<tr>
<td>Capital Stable</td>
<td>0.01</td>
<td>3.9</td>
</tr>
<tr>
<td>Male 65 years</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Portfolio</strong></td>
<td><strong>Real Spending rate, $000 p.a.</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>40</td>
</tr>
<tr>
<td>High Growth</td>
<td>0.04</td>
<td>2.1</td>
</tr>
<tr>
<td>Growth</td>
<td>0.02</td>
<td>1.6*</td>
</tr>
<tr>
<td>Balanced</td>
<td>0.01*</td>
<td>1.7</td>
</tr>
<tr>
<td>Conservative</td>
<td>0.02</td>
<td>2.1</td>
</tr>
<tr>
<td>Capital Stable</td>
<td>0.03</td>
<td>3.3</td>
</tr>
</tbody>
</table>
Table 6: Wealth required at the beginning of retirement to produce inflation adjusted $14000 p.a. real income, Ages 65 and 70.

Table shows the wealth needed at retirement to generate the inflation-adjusted Age Pension income stream of $14000 p.a. as a multiple of average earnings, for females and males with Gompertz mortality (see Section 3). Initial wealth is computed as the inverse gamma spending rate, $G^{-1}(\phi(w)|\alpha, \beta)$, where $\phi(w) = 0.005, 0.01, 0.03, \text{ and } 0.05$, and where $\alpha$ and $\beta$ are the parameters of the Gamma density determined by moment matching (see Section 2 above).

Annual average earnings ($55728.40) is based on February 2007 seasonally adjusted estimate of full-time adult ordinary time earnings, Australian Bureau of Statistics release 6302.0.

<table>
<thead>
<tr>
<th>Initial wealth level for $14000 income</th>
<th>0.5%</th>
<th>1%</th>
<th>3%</th>
<th>5%</th>
<th>0.5%</th>
<th>1%</th>
<th>3%</th>
<th>5%</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Female 65 yrs</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ruin probability</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>70 yrs</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>High Growth</td>
<td>8.24</td>
<td>7.39</td>
<td>6.13</td>
<td>5.41</td>
<td>7.72</td>
<td>6.85</td>
<td>5.58</td>
<td>5.03</td>
</tr>
<tr>
<td>Growth</td>
<td>7.54</td>
<td>6.84</td>
<td>5.79*</td>
<td>5.31*</td>
<td>7.19</td>
<td>6.44*</td>
<td>5.34*</td>
<td>4.85*</td>
</tr>
<tr>
<td>Balanced</td>
<td>7.43*</td>
<td>6.79*</td>
<td>5.82</td>
<td>5.38</td>
<td>7.16*</td>
<td>6.45</td>
<td>5.39</td>
<td>4.92</td>
</tr>
<tr>
<td>Conservative</td>
<td>7.59</td>
<td>6.97</td>
<td>6.01</td>
<td>5.41</td>
<td>7.35</td>
<td>6.64</td>
<td>5.57</td>
<td>5.04</td>
</tr>
<tr>
<td>Capital Stable</td>
<td>8.24</td>
<td>7.56</td>
<td>6.52</td>
<td>6.04</td>
<td>7.94</td>
<td>7.17</td>
<td>6.00</td>
<td>5.49</td>
</tr>
<tr>
<td>Life annuity</td>
<td>6.06</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Male 65 yrs</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ruin probability</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>70 yrs</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>High Growth</td>
<td>8.19</td>
<td>7.24</td>
<td>5.86</td>
<td>5.27</td>
<td>7.63</td>
<td>6.65</td>
<td>5.28</td>
<td>4.70</td>
</tr>
<tr>
<td>Growth</td>
<td>7.63</td>
<td>6.80*</td>
<td>5.60*</td>
<td>5.07</td>
<td>7.21*</td>
<td>6.34*</td>
<td>5.10*</td>
<td>4.56*</td>
</tr>
<tr>
<td>Balanced</td>
<td>7.62*</td>
<td>6.84</td>
<td>5.68</td>
<td>5.16</td>
<td>7.25</td>
<td>6.40</td>
<td>5.18</td>
<td>4.65</td>
</tr>
<tr>
<td>Conservative</td>
<td>7.85</td>
<td>7.06</td>
<td>5.88</td>
<td>5.36</td>
<td>7.48</td>
<td>6.61</td>
<td>5.36</td>
<td>4.82</td>
</tr>
<tr>
<td>Capital Stable</td>
<td>8.54</td>
<td>7.67</td>
<td>6.38</td>
<td>5.80</td>
<td>8.08</td>
<td>7.13</td>
<td>5.75</td>
<td>5.16</td>
</tr>
<tr>
<td>Life annuity</td>
<td>5.90</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 7: Wealth required at the beginning of retirement to produce the earnings and inflation adjusted equivalent to $14000 p.a., Ages 65 and 70.

Table shows the wealth needed at retirement to generate the earnings and inflation-adjusted Age Pension income stream of $14000 p.a. as a multiple of average earnings, for females and males with Gompertz mortality (see Section 3). Initial wealth is computed as the inverse gamma spending rate, $G^{-1} (\phi (w) | \alpha, \beta)$, where $\phi (w) = 0.005, 0.01, 0.03,$ and 0.05, and where $\alpha$ and $\beta$ are the parameters of the Gamma density determined by moment matching (see Section 2 above). Annual average earnings ($55728.40) is based on February 2007 seasonally adjusted estimate of full-time adult ordinary time earnings, Australian Bureau of Statistics release 6302.0.

<table>
<thead>
<tr>
<th>Initial wealth level for $14000$ income</th>
<th>Ruin probability</th>
<th>70 yrs</th>
<th>Ruin probability</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.5%  1%  3%   5%</td>
<td></td>
<td>0.5%  1%  3%   5%</td>
</tr>
<tr>
<td><strong>Female 65 yrs</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>High Growth</td>
<td>10.48  9.30  7.57  6.82</td>
<td></td>
<td>9.52  8.36  6.70  5.99</td>
</tr>
<tr>
<td>Growth</td>
<td>9.60   8.62  7.16*  6.52*</td>
<td></td>
<td>8.89  7.88*  6.42*  5.79*</td>
</tr>
<tr>
<td>Balanced</td>
<td>9.46*  8.55*  7.20   6.59</td>
<td></td>
<td>8.85*  7.89   6.48   5.87</td>
</tr>
<tr>
<td>Conservative</td>
<td>9.74   8.84   7.49   6.88</td>
<td></td>
<td>9.14   8.17   6.74   6.11</td>
</tr>
<tr>
<td>Capital Stable</td>
<td>10.83  9.82   8.30   7.62</td>
<td></td>
<td>10.06  8.97   7.38   6.68</td>
</tr>
<tr>
<td><strong>Life annuity</strong></td>
<td></td>
<td>7.56</td>
<td></td>
</tr>
<tr>
<td><strong>Male 65 yrs</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>High Growth</td>
<td>10.26  8.96   6.64   6.34</td>
<td></td>
<td>9.27   8.18   6.24   5.51</td>
</tr>
<tr>
<td>Growth</td>
<td>9.57   8.44*  6.82*  6.12*</td>
<td></td>
<td>8.78*  7.64*  6.03*  5.36*</td>
</tr>
<tr>
<td>Balanced</td>
<td>9.55*  8.47   6.90   6.22</td>
<td></td>
<td>8.81   7.70   6.12   5.46</td>
</tr>
<tr>
<td>Conservative</td>
<td>9.91   8.81   7.21   6.51</td>
<td></td>
<td>9.14   8.00   6.37   5.68</td>
</tr>
<tr>
<td>Capital Stable</td>
<td>11.02  9.77   7.95   7.17</td>
<td></td>
<td>9.97   8.75   6.94   6.17</td>
</tr>
<tr>
<td><strong>Life annuity</strong></td>
<td></td>
<td>6.61</td>
<td></td>
</tr>
</tbody>
</table>

35