Optimal Consumption and Portfolio Choice for Pooled Annuity Funds

Michael Z. Stamos*

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Abstract

This paper presents the optimal continuous time dynamic consumption and portfolio choice for pooled annuity funds. A pooled annuity fund constitutes an alternative way to protect against mortality risk compared to purchasing a life annuity. The crucial difference between the pooled annuity fund and purchase of a life annuity offered by an insurance company is that participants of a pooled annuity fund still have to bear some mortality risk while insured annuitants bear no mortality risk at all. The population of the pool is modeled by employing a Poisson process with time-dependent hazard-rate. It follows that the pool member's optimization problem has to account for the stochastic investment horizon and for jumps in wealth which occur if another pool member dies. In case the number of pool members goes to infinity analytical solutions are provided. For finite pool sizes the solution of the optimization problem is reduced to the numerical solution of a set of ODEs. A simulation and welfare analysis show that pooled annuity funds insure very effectively against longevity risk even if their pool size is rather small. Only very risk averse investors or those without access to small pools are more inclined to pay a risk premium to access private life annuity markets in order to lay off mortality risk completely. As even families constitute such small pools the model provides theoretical justification for the low empirical annuity demand.

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*Department of Finance, Goethe University (Frankfurt, Germany), e-mail: stamos@finance.uni-frankfurt.de, phone +49 (69) 798 - 22898. This research was conducted with support from the German Research Foundation (DFG). Opinions and errors are solely those of the author and not of the institutions with whom the author is affiliated.

1 Introduction

The most prominent financial risk in the late stage of the investor's life-cycle is to run out of savings and to have nothing left to cover the very basic needs - called longevity risk.[1] On the flipside, the investor also faces brevity risk as she could decease without consuming all her estate but leave an unintended bequest. Both outcomes are results of the unknown time of death also referred to as mortality risk.

The typical means to insure against both outcomes of mortality risk is to purchase a life-annuity offered by an insurance company during the late stage of the life-cycle. In return for an initially paid premium the investor retains a life long stream of income. This conversion of life-time savings into a life-annuity is called annuitization. As the insurance company redistributes wealth of deceased former members to surviving members the implicit rate of return is higher than the capital market return. The excess return finances longevity and is often referred to as mortality credit. The mortality credit comes with some opportunity costs since the annuity purchase is irrevocable and the bequest potential is lost. Insurance companies completely take over the mortality risk of the pool as they guarantee (with a certain probability) that the mortality credit is paid out in a deterministic manner commensurate to an ex ante specified mortality table. However, the risk transfer has to reduce the mortality credit by a certain risk premium required by the insurance to ensure risk adequate solvency capital and to control the default risk.[2] Impressive effort has been undertaken in the literature to discuss the consumption and portfolio choice problem with life annuities since the introduction of life annuities in a portfolio context is not straight forward as one has to deal with the irreversibility of the purchase of life annuities, their age dependent risk/return

[1]There is a vast literature studying the personal bankruptcy risk of investors who do not insure against longevity risk and follow the so called self-annuitization strategies: see, for instance, Albrecht and Maurer (2002); Ameriks, Veres, and Warshawsky (2001); Bengen (1994, 1997); Ho, Milevsky, and Robinson (1994); Hughen, Laatsch, and Klein (2002); Milevsky (1998, 2001); Milevsky and Robinson (2000); Milevsky, Ho, and Robinson (1997); and Pye (2000, 2001).

[2]Depending on the organization of the insurance there may arise incentive problems regarding the risk adequate pricing of annuities and/or the adequate default risk. Regulation and supervision is needed to resolve potential conflict of interests between annuitants (policy owners) and insurance owners.

profile, and the stochastic investment horizon of individuals.[3]

An alternative way to insure against mortality risk is to construct a pooled annuity fund (also sometimes called tontine) and follow a group self annuitization strategy. Contrary to the individual self annuitization strategy alluded above a group of investors pool their wealth into one annuity fund. Pooled annuity funds combine the features of an insurance product and a mutual fund. Just like in the case of mutual funds, one advantage of pooling is that this fund can usually better diversify investments due to a bigger size and can also be managed by an investment professional. The crucial difference to a mutual fund is that in case one investor perishes, her fund's assets are redistributed among the surviving investors just like in the case of a life annuity.[4] In contrast to life-annuities, the redistribution of the mortality credit is uncertain due to the remaining mortality risk. The participation in a pooled annuity fund constitutes an endogenous bequest motive. The investor is willing to leave estate to the other pool members to be in turn rewarded with the estate of prematurely died pool members. In fact, pooled annuity funds are the most common type of longevity insurance as each family can be regarded as a small annuity pool (Kotlikoff and Spivak (1981)).

The most recent literature studying pooled annuity funds is the work of Piggott, Valdez, and Detzel (2005) and Valdez, Piggott, and Wang (2006). Piggott, Valdez, and Detzel (2005) provided an excellent analysis of the organization and mechanics of pooled annuity funds. They derive the recursive evolution of payments over time given that investment returns and mortality deviate from expectation. However, they do not model the underlying risk factors explicitly. Valdez, Piggott, and Wang (2006) find within a two period model theoretical evidence that the adverse selection problem inherent in life annuities markets is alleviated in

[3]Beginning with Yaari (1965) the list of literature discussing the optimal consumption and portfolio choice problem with life annuities has become quite respectable. For an overview of the field of literature I refer the interested reader to Horneff et al. (2006). Most recent contributions include Babbel (2006), Cairns, Blake, and Dowd (2006), Davidoff, Brown, and Diamond (2005), Horneff, Maurer, and Stamos (2006a,2006b), Horneff et al. (2006,2007), Koijen, Nijman, and Werker (2006), Milevsky and Young (2006), and Milevsky, Moore, and Young (2006).

[4]Pooled annuity funds potentially bear moral hazard problems as pool members can profit from the decease of others. This holds true for life annuity contracts offered by insurance companies. The problem could be alleviated if the pool is organized anonymously. After all, this moral hazard problem would require the offender to have a high degree of criminal energy.

pooled annuity funds. The reason is that investors of pooled annuity funds cannot exploit the gains of adverse selection as well as when purchasing a life annuity because investors cannot perfectly predict the distribution of mortality credits anymore.

Although there is work done on the dynamic optimal consumption and portfolio choice problem for both cases individual self annuitization and life annuities there is to the author's best knowledge no study done in the context of group self annuitization strategies. The present paper closes this gap as it derives in the tradition of Merton (1969,1971) the optimal consumption and portfolio choice for a finite number l (potentially thousands) of homogenous investors collecting their wealth in a pooled annuity fund. Exploring group self annuitization strategies in a consumption and portfolio choice context is not trivial as one has to model the finite population of the pooled annuity fund in order to determine when the mortality credits are released. We employ for each investor a separate Poisson process with time-dependent jump intensity that determines the time at which a member perishes and committed funds are redistributed among the survivors.[5] The evolution of the pool's population is a sum of Poisson processes which is itself a Poisson process with (globally) stochastic hazard rate and jump size as both depend on the number of current pool members.

Investors are assumed to be expected lifetime-utility maximizer who derive no utility from bequest.[6] Within the pool surviving investors can continuously choose the consumption fraction withdrawn from the fund as well as the fund's asset allocation in stocks and money market. The assumed framework contains the special case $l = 1$, reflecting the individual

[5] Very similar to modeling defaults in reduced form credit risk models the time of death is the first jump time of a Poisson process.

[6] Naturally, the investor would put only that part of wealth into a pooled annuity fund which is not intended to serve as bequest. Therefore, throughout the paper I consider only that part of wealth which is reserved for consumption needs. Empirical studies such as Kotlikoff and Summers (1981) find that almost 80 percent of the total accumulated wealth in the United States is due to intergenerational transfers. This fact raises the question as to whether bequests are accidental or intentional. The literature on intentional bequests distinguishes between altruistic and strategic bequest motives as opposite ends of the spectrum. For instance, Abel and Warshawsky (1988) study the altruistic bequest motive in a reduced form and find a joy of giving parameter that is of a substantial magnitude. Bernheim, Shleifer, and Summers (1985) analyze the strategic bequest motive and discover empirical evidence. By contrast, Hurd (1987) does not find any evidence of bequest motives because the pattern of asset decumulation is similar among different household sizes. In addition, Hurd (1989) can support his prior findings by showing that the nature of most bequests is accidental because the date of death is uncertain to an individual.

self annuitization strategy and $l = \infty$, reflecting the perfect pool which is often assumed to model a life-annuity.

In the case $l = 1$, representing the consumption and portfolio choice problem of one investor with uncertain investment horizon, the Hamilton-Jacobi-Bellman-equation can be solved analytically. Merton (1971) already explored the implications of uncertain investment horizons on consumption and portfolio choice, but restricted the analysis to constant jump-intensities. Since exponentially distributed time of death clearly does not relfect empirical mortality rates we expand the analysis to time-dependent jump intensities. This case serves as a benchmark case against which group self annuitization strategies can be compared.

In the case $l = \infty$, representing a perfectly diversified pool, analytical solutions are available, too. Since mortality risk is completely eliminated, this case resembles the purchase of a variable life-annuity. Consequently, the derived optimal consumption strategy is equivalent to the optimal payout structure of life-annuities. Thus, the present paper contributes to the prior life-annuity literature by deriving also the optimal payout pattern of life annuities while the previously mentioned life-annuity related literature has fixed the payout pattern exogenously.[7] The derivation of the optimal annuity payout structure is a result of our assumption that all pool members are homogenous so that all would follow the equal consumption strategy. It is shown that having access to a perfect pool substantially increases the optimal consumption fraction compared to the benchmark case ($l = 1$) since the additional income from the deterministic mortality credit reduces the need to hold capital stock.

For finite pool sizes $1 < l < \infty$ the problem is solved up to the solution of a set of $l - 1$ ODEs (as for $l = 1$ the analytic solution is available).[8] The finite pool size implies that the earned mortality credit realizes at uncertain times. Anytime a pool member persishes the wealth of the surivors jumps by a certain amount which depends on the number of current

[7]The previous literature on asset allocation and consumption problems with life-annuities, e.g. Milevsky and Young (2006a), Milevsky, Moore, and Young (2006b), Koijen, Nijman, and Werker (2006), have assumed that the life-annuity payment pattern is exogenously given by imposing a certain assumed interest rate (AIR).

[8]As the numerical solution of the set of ODEs is rather straightforward, the calculation with up to thousands of pool members is finished within the matter of minutes.

survivors. Hence, the optimal policy depends on both age and the number of current pool members l. The higher the pool size, the higher is the optimal withdrawal rate, since the anticipated mortality credit increases with l. For small pool sizes the optimal consumption policy converges to the individual self annuitization case and for large pool sizes to the life-annuity case. In all considered cases the optimal portfolio follows exactly the constant mix Merton rule since the investment opportunity set is constant and the correlation between asset returns and mortality risk is zero. Further the size of the mortality credit does not depend on in which asset classes the annuity fund invests.

A final welfare analysis suggests that pooling generates remarkable utility gains equivalent to a wealth increase of up to 45% at age 60 and above 100% from age 80 on as the mortality credit surges. The utility gain of pooled annuity funds with size above $l = 100$ is around 90% of the utility gain generated by life annuities. Even pooled annuity funds with a small number of members $5 < l < 10$ (such as families) hedge mortality risk very effectively.

The remainder of this paper is organized as follows. Section II presents the mechanics of the pooled annuity fund comprising the population model, financial markets, and the fund's wealth dynamics. In section III the analytical and numerical solution of the dynamic optimization problem of pool members is described. Section IV continues with a numerical calibration of the model including a Monte Carlo and welfare analysis based on computations of certainty equivalents. Section V concludes.

2 The Pooled Annuity Fund

2.1 Population Model

When modeling pooled annuity funds it is essential to employ an appropriate population model to determine the dates at which the funds from perished members are reallocated among the survivors. As the pool size could be low, it is especially important that the population model accounts for the finiteness of the pool size. Furthermore, the model should also be able to capture empirical mortality rates. Finally, it is required that the model is tractable in a dynamic consumption and portfolio choice framework. To this end, I set up a model in which the time of death of each pool member is determined by the first jump time of a Poisson process with time-dependent pre-jump intensity. Let $L_0 \in \{1, 2, ...\}$ be the initial number of living pool members. Each pool member's time of death τ_i, $i \in \{1, ..., L_0\}$ is determined by the first jump time of an inhomogeneous Poisson process $N_i = \{N_{i,t}\}_{t\geq 0}$ with time-dependent pre-jump intensity $\lambda_{t,i}$:

$$\tau_i = \{\min t : \; N_{t,i} = 1\}. \tag{1}$$

To model homogeneous investors I assume that the jump intensity does not depend on i. This assumption could well correspond to the case in which all individuals are of same age and gender. The pre-jump intensity λ_t is assumed to be deterministic over time. It follows that the probability of survival between two dates t and $s \geq t$ is given by

$$p(t, s) = \exp\left\{ -\int_t^s \lambda_u \, du \right\}. \tag{2}$$

The number of living fund members in t follows from the above assumptions and is given by $L_t = L_0 - \sum_{i=1}^{L_0} N_{i,t}$. Accordingly, the number of surviving investors evolves according to

$$dL_t = -\sum_{i=1}^{L_t} dN_{t,i}. \tag{3}$$

Equation (3) can be rewritten as follows

$$dL_t = -dN_t, \tag{4}$$

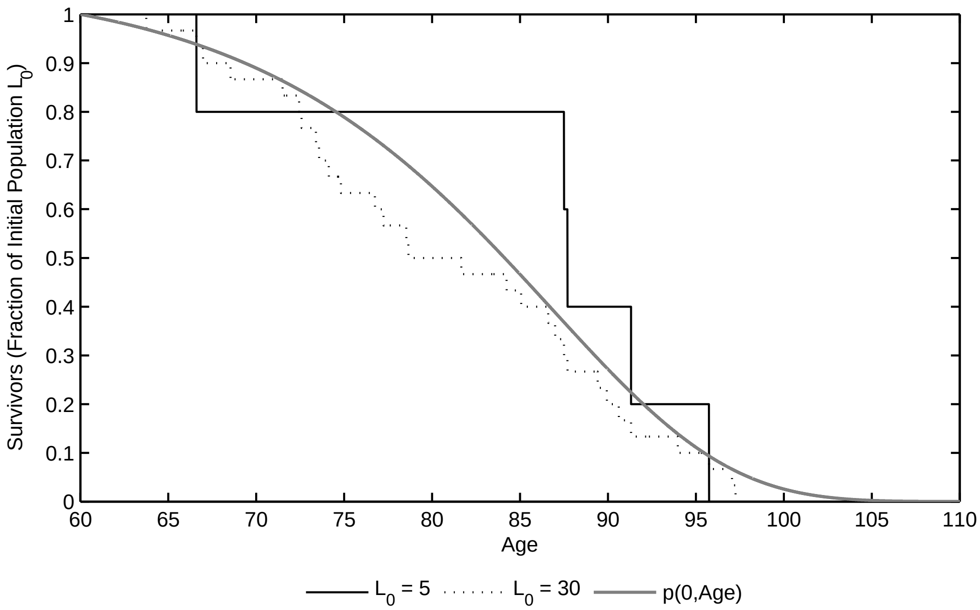

Figure 1: Illustration of Simulated Populations. This figure presents simulated paths of the development of populations for initial pool sizes $L_0 = 5$ and 30 and the survival probabilities p(0,Age-60). At $t = 0$ investors are assumed to be aged 60. Survival probabilities are calibrated according to the 1996 population 2000 basic mortality table for US females ($m = 86.85$, $b = 9.98$).

since the sum of Poisson processes $\sum_{i=1}^{L_t} N_i$ is itself a Poisson process $N = \{N_t\}_{t \geq 0}$ with $E[dL_t] = \sum_{i=1}^{L_t} \lambda_t dt = L_t \lambda_t dt$.[9] As a consequence, the population can be modelled with only one risk factor which will below facilitate the solution of the consumption and portfolio choice problem. It is convenient to assume that λ_t follows the Gompertz law of mortality:

$$\lambda_t = \frac{1}{b} e^{(t-m)/b}, \tag{5}$$

where $m > 0$ denotes the modal time of death and $b > 0$ is a dispersion parameter. This assumption implies that the dynamics of λ_t is given by $d\lambda_t = \frac{\lambda_t}{b}\, dt > 0$. The Gompertz law of mortality constitutes the standard mortality law in population models since it is parsimonious to handle as only two parameters are to be estimated. Further, it captures

[9] It is just required that the Poisson processes N_i, $i \in \{1, ..., L_0\}$ do not jump at the same time which is fulfilled almost surely.

empirical mortality rates remarkably well.[10] The probability of survival between two dates t and $s \geq t$ (2) can be rewritten as $e^{b\lambda_t(1-e^{(s-t)/b})}$.

2.2 Financial Markets

The fund assets can be invested into one risky asset (the stock market) and one riskless asset (the money market). The price dynamics of the risky asset is given by the standard geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dZ_t, \tag{6}$$

where dZ_t is the increment of the one-dimensional Brownian motion $Z = \{Z_t\}_{t\geq 0}$ which is not correlated with the jump process. The riskless asset evolves according to

$$\frac{dB_t}{B_t} = r\, dt, \tag{7}$$

where r is the locally and globally riskless interest rate. The simple conception of the asset model maybe deserves some justification. As the focus of this paper does not lie on the effects of grouping mortality risks a more complex asset model will not improve the intuition of group self annuitization strategies. The investment horizon of this paper is potentially a couple of decades. Hence, short term active trading on changes in the investment opportunity set lies not at the heart of this study. However, stock market price risk clearly is the dominant source of risk in a portfolio and has the strongest impact on welfare.

2.3 The Wealth Dynamics of the Pooled Annuity Fund

At $t = 0$ investors $i \in \{1, ..., L_0\}$ pool their wealth $W_{i,0}$ inside the annuity fund. The total initial value is therefore given by

$$W_{AF,0} = \sum_{i=1}^{L_0} W_{i,0}. \tag{8}$$

[10] For an illustration of the goodness of fit of the Gompertz mortality law see for example Milevsky (2006b).

The value of the annuity fund evolves according to

$$\frac{dW_{AF,t}}{W_{AF,t}} = [r + \pi_t(\mu - r) - c_t]\,dt + \pi_t\sigma\,dZ_t, \tag{9}$$

where π_t is the portfolio weight of stocks and c_t is the rate withdrawn from the fund at time t. The fraction of fund wealth owned by investor i is given by $h_{i,t} = W_{i,t}/W_{AF,t}$. Anytime a pool member $j \neq i$ dies prematurely his remaining wealth $W_{j,t-} = h_{j,t-}W_{AF,t-}$ becomes reallocated among survivors i so that their wealth $W_{i,t-}$ jumps according to

$$\begin{aligned} W_{i,t} &= W_{i,t-} + \frac{h_{i,t-}}{1 - h_{j,t-}} W_{j,t-} \\ &= W_{i,t-}\left(1 + \frac{h_{j,t-}}{1 - h_{j,t-}}\right). \end{aligned}$$

From Itô's lemma for jump-diffusion processes follows the evolution of $W_{i,t}$:

$$\frac{dW_{i,t}}{W_{i,t-}} = [r + \pi_t(\mu - r) - c_t]\,dt + \pi_t\sigma\,dZ_t + \sum_{j=1,j\neq i}^{L_0} \frac{h_{j,t-}}{1 - h_{j,t-}}\,dN_{j,t}, \qquad t < \tau_i. \tag{10}$$

So, the fraction of fund wealth owned by the surviving members $h_{i,t} = W_{i,t}/W_{AF,t}$ evolves according to

$$dh_{i,t} = h_{i,t-} \sum_{j=1,j\neq i}^{L_0} \frac{h_{j,t-}}{1 - h_{j,t-}}\,dN_{j,t}, \quad t < \tau_i, \tag{11}$$

until $h_{i,t} = 1$ in case i is the only remaining survivor. In turn, if the considered pool member i herself dies accumulated funds are spread among the other surviving members, hence $dW_{i,t=\tau_i} = -W_{i,t-}$,$dh_{i,t} = -h_{i,t-}$.

Since pooling has only a positive impact on the wealth evolution over life time it is obviously a dominant strategy to conduct the group self annuitization strategy if no bequest motive is present. Much more, the presence of a pool produces the motive to bequeath remaining estate to other pool members in order to share their released estate. Further notice, that the chosen asset allocation has no influence on the size mortality credit as it is just an additional source of income.

As the size of the mortality credit depends on how much wealth was owned by the deceased pool member, it is necessary to record the wealth fraction of each living pool member. For the sake of simplicity and to diminish the curse of dimensionality it is assumed that each investor has the same initial endowment W_0 since otherwise the number of state variables would be equal to the number of living investors. Thus, the fraction of pool wealth owned by each investor in t is $h_t = 1/L_t$. Hence, the dynamics (10) can be simplified to

$$\frac{dW_{i,t}}{W_{i,t-}} = [r + \pi_t(\mu - r) - c_t]\, dt + \pi_t \sigma \, dZ_t + \frac{1}{L_{t-} - 1} dN_t, \qquad t < \tau_i, L_{t-} > 1, \tag{12}$$

where the instantaneous probability that one of the other pool members dies is $E[dN_t] = (L_{t-} - 1)\lambda_t \, dt$. For instance, if one other pool member dies and the pool size is $L = 2$, the additional income is $1/(L_{t-} - 1) = 100\%$, if the pool size is $L = 3$ the additional income reduces to 50% and so on. The smaller the pool size the higher is the additional return but the smaller is the probability that one of the other investors dies. Of course, if $L = 1$, the last survivor earns no mortality credit anymore and has also no loss of bequest potential:

$$\frac{dW_{i,t}}{W_{i,t}} = [r + \pi_t(\mu - r) - c_t]\, dt + \pi_t \sigma \, dZ_t, \qquad L_t = 1. \tag{13}$$

Calculating the instantaneously expected mortality credit gives:

$$E\left[\frac{1}{L_{t-} - 1}\, dN_t\right] = \lambda_t \, dt, \qquad L_{t-} > 1. \tag{14}$$

It can be seen that the expected instantaneous mortality credit is independent from the pool size as the term $(L_{t-} - 1)$ cancels out. The intuition is that the higher the pool size L is the lower is the mortality credit but the higher is the probability to earn the mortality credit. The only relevant state variable is age because the older the pool and thus the higher λ_t is, the more released funds will be shared among fewer survivors.

The look at the instantaneous variance of the mortality credit gives insight about the associated (local) risks:

$$Var\left[\frac{1}{L_{t-}-1}\,dN_t\right] = \frac{\lambda_t}{L_{t-}-1}\,dt, \qquad L_{t-} > 1. \tag{15}$$

While the expected mortality credit does only depend on the current hazard rate λ_t , the local variance depends also on the number of living investors L_{t-}. The higher the number of investors the more predictable becomes the mortality credit. In fact, the (idiosyncratic) jump risk can be completely eliminated for infinitely large pools:

$$\lim_{L_{t-}\to\infty} Var\left[\frac{1}{L_{t-}-1}\,dN_t\right] = 0. \tag{16}$$

Hence, this would constitute a perfect insurance pool as mortality risk is eliminated. $1/(L_{t-}-1)\,dN_t$ in (12) can be replaced with the expected mortality credit $\lambda_t\,dt$. This result is a consequence of the standard assumptions that the mortality of all pool members is equal and that the hazard rate does not vary stochastically over time.

Consequently, having access to a perfect pool yields a deterministic income $\lambda_t dt$ from earned mortality credits so that the dynamics (12) can be restated as follows

$$\frac{dW_t}{W_t} = [r + \pi_t(\mu - r) + \lambda_t - c_t]\,dt + \pi_t\sigma\,dZ_t. \tag{17}$$

It is also informative to derive the long term wealth dynamics in order to get insight into the long term wealth effects of the earned mortality credit. According to Itô's lemma for jump diffusions the growth rate of wealth between t and $s \geq t$ is:

$$\begin{aligned}\frac{W_{i,s}}{W_{i,t}} = \exp\left\{\int_t^s (r + \pi_u(\mu - r) - c_u - \frac{1}{2}\,\pi_u^2\sigma^2)\,du + \int_t^s \pi_u\sigma\,dZ_u\right\}\\ \times \prod_{j=1}^{\Delta N(t,s)}\left(1 + \frac{1}{L_t - j}\right), \quad N_{i,s} = 0\end{aligned} \tag{18}$$

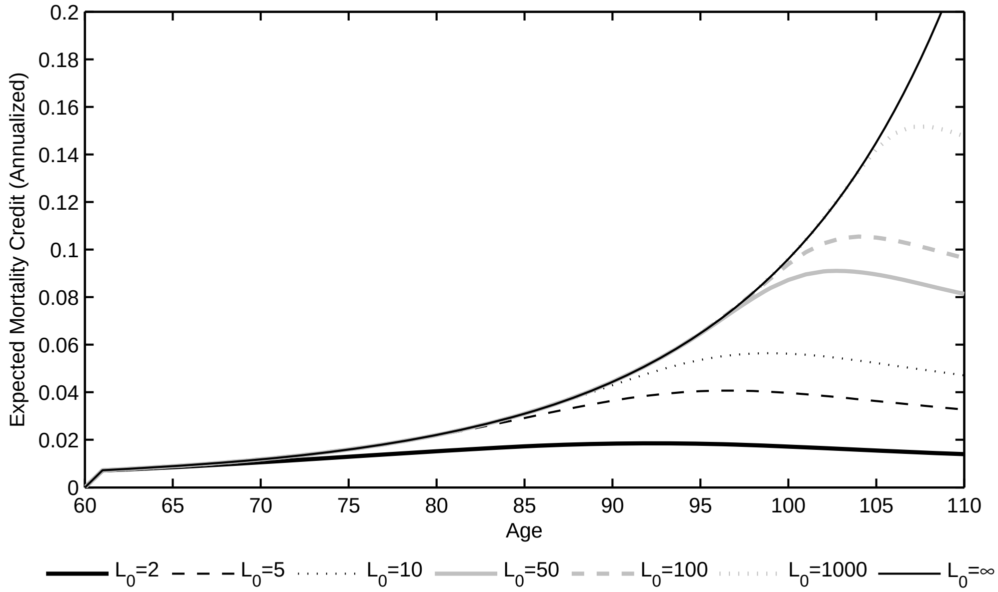

Figure 2: Expected Annualized Mortality Credit for Various Ages and Initial Pool Sizes. The annualized expected mortality credit is defined as $(EMC(l, t = 0, s))^{1/s} - 1$. The calibration of survival probabilities corresponds to that of Figure (1).

where $\Delta N(t, s) = N_s - N_t$ is the number of deaths within the pool excluding the considered investor. The lower product term expresses the extra return component which is backed by the sum of earned mortality credits. Given that the current pool size is $L_t = l$, the at t expected mortality credit which is received between t and s conditional on the survival of the considered pool member i is given by

$$\begin{aligned} EMC_i(l, t, s) &= E\left[\prod_{i=1}^{\Delta N(t,s)} \left(1 + \frac{1}{L_t - i}\right) |L_t = l, N_{i,s} = 0\right] \\ &= P(L_s \geq 1 | L_t = l)\, MC(t, s), \end{aligned} \tag{19}$$

where $P(L_s \geq 1|L_t = l) = \sum_{k=0}^{l-1} \binom{l}{k} (1 - p(t, s))^k\, p(t, s)^{l-k}$ is the probability that at least one pool member survives until s and $MC(t, s)$ is the (deterministic) mortality credit $\exp \int_t^s \lambda_u \, du$ in case of a perfect pool.[11] Contrary to the expected instantaneous mortality credit $\lambda_t \, dt$

[11]Please see the Appendix A for a detailed derivation of (19).

derived above, the expected cumulative mortality credit increases with the size of the pool l because $P(L_s \geq 1|L_t = l)$ increases with l monotonically to $\lim_{l \to \infty} P(L_s \geq 1|L_t = l) = 1$.

Figure (2) illustrates how the annualized expected mortality credit evolves over time for $t = 0$ (age=60) and varying pool sizes $L_0 = l$. For example if the initial pool size is $L_0 = 5$ and the considered pool member survives until age 90 the expected annualized addition return from pooling is 4%. As the considered investor would on average be able to increase the consumption rate by 4% the effect from pooling should have substantial impact on utility which is analyzed below. A further implication of the figure is that high pool sizes only contribute to the expected income from mortality credits in the very late stage of the life-cycle.

3 The Dynamic Optimization Problem

3.1 The Value Function

Having derived the dynamics of the pool members' wealth, the structure of the mortality credit, and the connection between indvidual self insurance ($L_0 = 1$), group self insurance ($1 < L_0 < \infty$) and a perfect pool $L_0 = \infty$, the arising questions are manifold: how does the mortality credit affect the optimal consumption and portfolio choice, how crucial is it to account for the finiteness of the pool size, how high is the utility gain from pooling funds, on the other hand how high is the potential utility loss if the pool size is only finite compared to a perfect pool.

To answer these questions I assumed that all investors have homogenous preferences described by

$$U = \int_0^\tau e^{-\delta t} u(C_t)\, dt, \tag{20}$$

where $\delta > 0$ denotes the time preference rate, τ denotes the stochastic time of death and $C_t = c_t W_t$ is the level of consumption at time t. The utility function $u(C)$ is assumed to be of the standard CRRA type given below:

$$u(C) = \frac{C^{1-\gamma}}{1-\gamma} \qquad \gamma \neq 1, \gamma > 0, \tag{21}$$

where γ is the level of relative risk aversion. The value function of any pool member at time t is given by

$$V(w,l,t) = \sup_{[\pi_s, c_s]_t^\infty} E\left[\int_t^\tau e^{-\delta(s-t)} u(C_s)\, ds \,|\, W_t = w, L_t = l\right]. \tag{22}$$

The value function depends on the three state variable wealth W_t, number of living investors L_t, and time t. First, the value function (22) can be rewritten by integrating over the stochastic time of death:

$$\begin{aligned} V(w,l,t) &= \sup_{[\pi_s, c_s]_t^\infty} E\left[\int_t^\infty e^{-\int_t^\tau \lambda_u du} \int_t^\tau e^{-\delta(s-t)} u(C_s)\, ds\, d\tau \,|\, W_t = w, L_t = l\right] \\ &= \sup_{[\pi_s, c_s]_t^\infty} E\left[\int_t^\infty e^{-\int_t^s (\lambda_u + \delta)\, du} u(C_s)\, ds \,|\, W_t = w, L_t = l\right]. \end{aligned}$$

Thus, utility at a certain point of time $s > t$ is discounted by the factor $\exp(-\int_t^s (\lambda_u + \delta)\, du)$ to account for both time preference and mortality risk. Finally, the indirect value function is given by

$$V(w,l,t) = \sup_{[\pi_t, c_t]} \lim_{\Delta t \to 0} \left\{ \frac{(W_t c_t)^{1-\gamma}}{1-\gamma} \Delta t + e^{-(\lambda_t + \delta)\, \Delta t} E\left[V(w,l,t+\Delta t) \,|\, W_t = w, L_t = l\right] \right\}. \tag{23}$$

3.2 The Hamilton-Jacobi-Bellman Equation

The Hamilton-Jacobi-Bellman (HJB) equation follows immediately when calculating the expected drift of the value function using Itô's lemma:

$$
\begin{aligned}
(\lambda_t+\delta)V = \sup_{[\pi,c]} \Bigg\{ & \frac{(cw)^{1-\gamma}}{1-\gamma} + V_t + V_w\, w(r+\pi(\mu-r)-c) + \frac{1}{2} V_{ww}\, \pi^2\sigma^2 w^2 \\
& + \lambda_t(l-1)\left[V\left(w\left(1+\frac{1}{l-1}\right), l-1, t\right) - V\left(w,l,t\right)\right] \Bigg\},
\end{aligned}
\tag{24}
$$

where V_t and V_w denote the first partial derivatives of V with respect to t and w, respectively and $V_{w,w}$ the second partial derivative with respect to w. While the upper term of the right hand side reflects the standard HJB without pooling ($l = 1$) the lower term reflects the positive impact on utility if the investor receives the mortality credit. If one pool member dies wealth jumps to $w\left(1+\frac{1}{l-1}\right)$ and the number of the other pool members reduces to $l-1$. After solving the first order condition for π the optimal portfolio policy is given by:

$$
\pi^* = \frac{\mu - r}{\gamma\,\sigma^2}. \tag{25}
$$

It becomes apparent that the optimal portfolio follows the constant mix Merton rule and is independent of w, l, and t. The independence from w is the consequence of assuming CRRA preferences. The independence from l stems from the fact that the relative size of the mortality credit does not depend on the asset allocation of the pooled annuity fund. Further, the investment opportunity set does not depend on the calendar date t. In order to derive the optimal consumption policy, $V(w,l,t)$ is conjectured to have the form

$$
V(w,l,t) \;\; = \;\; f(l,t)\frac{w^{1-\gamma}}{1-\gamma}, \tag{26}
$$

with terminal conditions $\lim_{t\to\infty} f(l,t) = 0$, and $f(0,t) = 0$. The optimal consumption policy is then obtained by

$$c(l,t) = f(l,t)^{-\frac{1}{\gamma}}. \tag{27}$$

The dependence from t is induced by the time-dependent hazard rate. The increasing hazard rate implies two conflicting effects: the older investors become the higher is their time preference rate $\delta + \lambda_t$, but the higher is also the expected instantaneous mortality credit $\lambda_t dt$. The dependence from l is induced as the potential mortality credit increases with l as shown above. Due to CRRA preferences the optimal consumption rate is independent from w.[12]

The guess can be verified by plugging (26) and (25) in the HJB (24) so that $f(l,t)$ must obey

$$\frac{f_t(l,t)}{f(l,t)} + \gamma f(l,t)^{-1/\gamma} + (A - \lambda_t l) + \lambda_t (l-1) \left(\frac{l}{l-1}\right)^{1-\gamma} \frac{f(l-1,t)}{f(l,t)} = 0, \tag{28}$$

where $f_t(l,t)$ denotes the partial derivative with respect to t and with A being a constant

$$A = (1-\gamma)\left[r + \frac{1}{2\gamma}\left(\frac{\mu - r}{\sigma}\right)^2\right] - \delta. \tag{29}$$

Analytic solutions to the set of ODEs (28) can only be derived for the two extreme cases $l = 1$ and $l = \infty$.

3.3 Special Case: $l = 1$

This case reflects the situation where only one investor lives either because no other investor is there or because all other investors already have perished. In this case equation (28) reduces to the following ODE with time-dependent coefficients:

$$\frac{f_t(1,t)}{f(1,t)} + \gamma f(1,t)^{-1/\gamma} + (A - \lambda_t) = 0. \tag{30}$$

[12]This is also a consequence of the assumption that the hazard rate does not depend on the level of wealth.

It can be easily verified that

$$f(1,t)=\left\{\int_t^\infty e^{\frac{1}{\gamma}\int_t^s (A-\lambda_u)\,du}\,ds\right\}^\gamma \tag{31}$$

satisfies equation (30) for all t when it is plugged back in (30). Thus the optimal consumption policy (27) becomes

$$c(1,t)=\left\{\int_t^\infty e^{\frac{1}{\gamma}\int_t^s (A-\lambda_u)\,du}\,ds\right\}^{-1}. \tag{32}$$

It can be seen that the optimal consumption fraction increases over time as present consumption becomes more valuable the older the individual is and the more the hazard rate increase.[13]

3.4 Special Case: $l=\infty$

Using the wealth dynamics for a perfect pool (17) to derive the HJB and applying guess (26) yields

$$f(\infty,t)=\left\{\int_t^\infty e^{\frac{1}{\gamma}\int_t^s (A-\gamma\lambda_u)\,du}\,ds\right\}^\gamma \tag{33}$$

and

$$c(\infty,t)=\left\{\int_t^\infty e^{\frac{1}{\gamma}\int_t^s (A-\gamma\lambda_u)\,du}\,ds\right\}^{-1}. \tag{34}$$

Comparing the integrand $A-\gamma\lambda_u$ with the integrand $A-\lambda_u$ of the previous case $l=1$ shows that the optimal consumption rate increases if $\gamma>1$ and decreases if $\gamma<1$. The intuition is that in case $\gamma>1$ lower capital stock is needed since the mortality credit finances future consumption. Contrary, in case $\gamma<1$, the investor bets on a long life and postpones consumption into the late stage of the life-cycle in order to attain the mortality credit later on.

[13]The optimal consumption rate would only be constant if we impose the same restriction as in Merton (1971) where $\lambda_t=\lambda$. However, this assumption would imply that the individual never ages as the probability of death is constant.

3.5 Case: $1 < l < \infty$

As no analytical results are available for this case equation (28) has to be solved numerically by using finite difference methods.[14] The function $f(l,t)$ is approximated by values $\hat{f}_{l,i}$ where $l \in \{1, ..., L_0\}$, $i \in \{0, 1, ..., M\}$ with $t = i\,\Delta t$. The boundary conditions for this problem are $\hat{f}_{l,M} = \lim_{t\to\infty} f(l,t) = 0$, and $\hat{f}_{1,i} = f(1,t)$, and $\hat{f}_{\infty,i} = f(\infty,t)$. The partial derivative $f_t(l,t)$ is approximated by the forward differential quotient $(\hat{f}_{l,i+1} - \hat{f}_{l,i})/\Delta t$. Plugging this approximation back into equation (28) yields

$$\frac{\hat{f}_{l,i+1} - \hat{f}_{l,i}}{\hat{f}_{l,i}\,\Delta t} + \gamma \hat{f}_{l,i}^{-1/\gamma} + (A - \lambda_t l) + \lambda_t (l-1)\left(\frac{l}{l-1}\right)^{1-\gamma} \frac{\hat{f}_{l-1,i}}{\hat{f}_{l,i}} = 0, \tag{35}$$

If $\hat{f}_{l-1,i}$ and $\hat{f}_{l,i+1}$ are known the only variable to make the equation hold is $\hat{f}_{l,i}$. After calculating the boundaries $\hat{f}_{l,M} = \lim_{t\to\infty} f(l,t) = 0$ and $\hat{f}_{1,i} = f(1,t)$ equation (35) is solved recursively. Holding $l = 2$ constant, $\hat{f}_{2,i}$ is computed backward from $i = M$ to $i = 0$. Then this procedure is repeated for $l = 3$ and so on. Thus the solution of the equation has the complexity of solving $l - 1$ ODEs. Then $\hat{f}_{l,i}$ is used to compute the approximate solution of the optimal consumption strategy

$$\hat{c}_{l,i} = \hat{f}_{l,i}^{-1/\gamma}. \tag{36}$$

4 Numerical Example

4.1 Optimal Consumption Rules for Different Pool Sizes

In order to show concrete results and assess the economical relevance of the analytical derivations above I set up a base case with stylized parameters. I assume that the level of relative risk aversion is $\gamma = 5$ representing a moderate level and that the time preference parameter is equal to $\delta = 0.04$. The parameter γ determines the investor's willingness to substitute

[14] Even if one treats l as constant in order to solve the l resulting ODEs iteratively, there is no analytical solution available.

consumption among both states and time as the intertemporal elasticity of consumption is given by $1/\gamma$. In the later welfare analysis results are reported for a range of risk aversion coefficients from $\gamma = 2$ to $\gamma = 30$.

The expected instantaneous real stock return is $\mu = 0.06$ and the instantaneous volatility is $\sigma = 0.18$. The real interest rate is set to $r = 0.02$. The implicit discrete equity premium of the above settings is $\exp(\mu) - \exp(r) = 0.042$. Especially the choice of the equity risk premium has been subject to numerous theoretical and empirical articles in the past (see, for instance, Siegel (2005) for a survey). While the equity premium in the US has been around 6 percent since 1872 most forward looking economists doubt whether this is also true for the future. A current consensus in the long term portfolio choice literature lies around 3-4 percent (see for example Cocco, Gomes, and Maenhout (2005) or Gomes and Michaelides (2005)). Under this parameterization follows immediately that the optimal stock fraction is $\pi^* = 24.69\%$ according to equation (25).

The survival probabilities are fitted by least squares estimation to the survival probabilities between age 60 and 115 given by the 1996 Population 2000 Basic Mortality Table. I will do the numerical calculation for men and women as mortality risk depends crucially on the gender of pool members. For US Females the resulting parameters are $m = 86.85$, $b = 9.98$ and for males $m = 81.90$, $b = 11.05$. These parameter estimators for m reflect the higher mortality of men having a modal time of death lying at 81.9 years versus the modal of women at age 86.85. The estimated value for b shows that the dispersion of the time of death (mortality risk) is higher in the case of men than in the case of women. In the numerical calculations the starting age is set to 60 (at $t = 0$) and the maximum age to 110 ($T = 110 - 60$) as it is very unlikely that a 60 year old individual will survive that long (see Figure (1)).

The upper graph of Figure (3) reveals the impact of pooling on the optimal consumption/withdrawal decision. From age 60 on investors will start with higher withdrawals if they participate in a pool. The higher the number of pool participants is the higher are

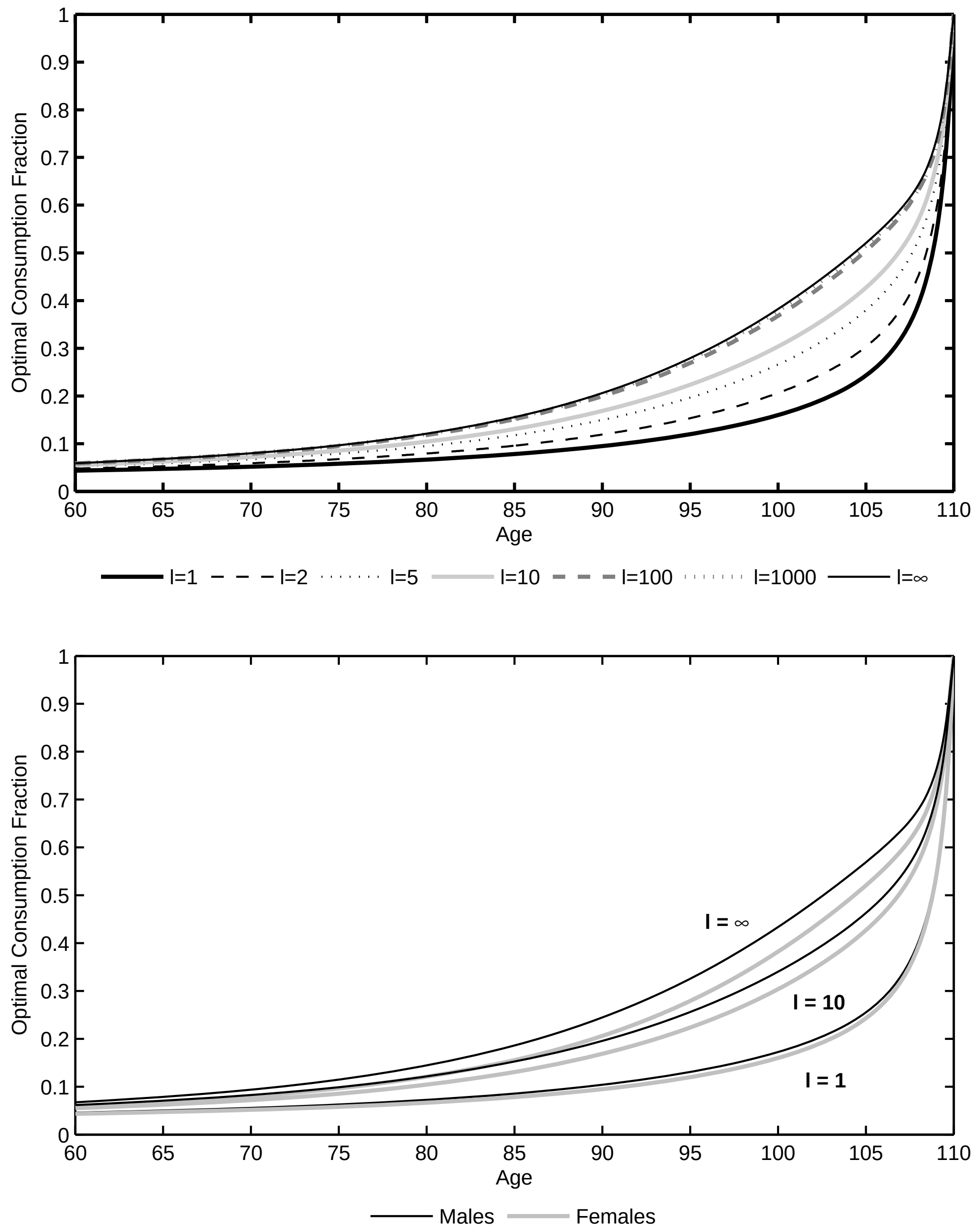

Figure 3: Optimal Withdrawal Policies for Varying Age and Population Size. The *upper graph* presents the optimal discrete withdrawal fraction $(1 - \exp(-c(l, t)))$ of the base case for different ages and population sizes l. For example, at age 80 the optimal withdrawal rate is 0.066 if the current pool size is $l = 1$ and 0.095 if $l = 5$. The *lower graph* presents the optimal discrete withdrawal fraction of the base case for different ages and gender. The optimal policies are depicted for $l = 1$, $l = 10$, and $l = \infty$. For the cases $l = 1$ and $l = \infty$ analytical solutions are given in (32) and (34), respectively. For $1 < l < \infty$ the set of ODEs (35) is solved numerically. The minimum age is set to 60 ($t = 0$), the maximum age to 110 ($T = 110 - 60$) and the time step is one month, $\Delta t = 1/12$. Thus, the grid size in which the ODEs are solved is $\{L_0 = 1000\} \times \{T/\Delta t = 600\}$.

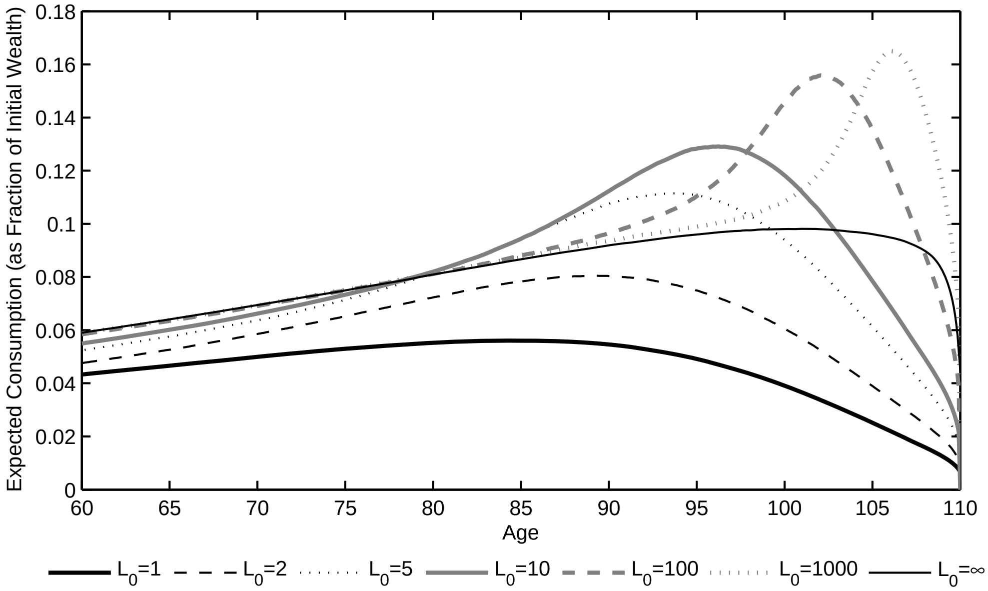

Figure 4: Expected Optimal Consumption Path for Varying Age and Initial Pool Size. The calculation of the expectations is done based on the optimal consumption policy $c(l, t)$ via $100,000$ Monte Carlo simulations in which the time step is set to $\Delta t = 1/240$.

the withdrawal rates because the more do pool members benefit from the mortality credit. Investors anticipate the extra income from deceasing pool members so that less capital stock is saved. The the smaller the pool size the larger is the increase of optimal consumption in l. Even if the pool size is only $l = 5$ or $l = 10$, the optimal consumption rate almost doubles from age 80 on. In cases $l \geq 50$ the anticipated mortality credit is so high that the optimal consumption rate more than doubles from age 70 on. Focusing the cases $l \geq 50$ also makes clear that the optimal consumption rate is only slightly below the one of the case $l = \infty$. This indicates that the individual anticipates a stream of mortality credits that is similar to the one of the perfectly diversified pool. The lower graph of Figure (3) demonstrates the optimal withdrawal policy of women vis-à-vis that of men. Although the expected age of men is much lower then that of women, men can afford only a slightly higher withdrawal rate in the case $l = 1$. But if male investors pool their wealth, they can afford much higher consumption rates (up to 5 percent points more in the case $l = \infty$) than women since they can expect higher mortality credits due to their shorter live.

It is also instructive to report the expected consumption levels for the different pool sizes in order to get a grasp of the expected consumption path and the magnitude of the mortality credit. To this end a Monte Carlo simulation is conducted in which the wealth dynamics (12) is discretized. The time step is set to $\Delta t = 1/240$ and the number of iterations to $100,000$. For dates t not lying on the grid of the numerical solution of the set of ODEs (35) the optimal consumption rate is approximated by linear interpolation. The results of the calculation are shown in Figure (4). The consideration of the expected consumption paths shows that the realized level of consumption can be raised even if the initial pool size is rather small. Interesting is also the hump shaped consumption profile at very high ages. This shape can be explained by the high skewness in consumption which is generated by the jumps in wealth due to the stochastic mortality credit at high ages. The jumps are larger the older the individual becomes as the pool size has shrunk in the meanwhile so that released funds are allocated among fewer survivors.

4.2 Welfare Analysis: Self Annuitization versus Life Annuity versus Group Self Annuitization

The substantial increase in consumption rates suggests that utility gains from pooling are economically significant. In order to estimate the economic importance of pooling, I calculate the equivalent wealth increase of cases $l > 1$ relative to the case $l = 1$. Especially it is of interest how high the pool size l has to be in order to reach utility levels close to the ideal case $l = \infty$. The equivalent wealth increase $R(l,t)$ is defined by the following equation:

$$\frac{(W_0(1 + R(l,t)))^{1-\gamma}}{1-\gamma} f(1,t) = \frac{W_0^{1-\gamma}}{1-\gamma} f(l,t), \tag{37}$$

which imposes that the lifetime utility of the case $l = 1$ and the general case $l > 1$ has to be equated with the equivalent wealth increase $R(l,t)$. It follows that

$$R(l,t) = \left(\frac{f(l,t)}{f(1,t)}\right)^{1/(1-\gamma)} - 1. \tag{38}$$

Consequently, $R(l,t)$ can be interpreted as the additional fraction of initial wealth that is needed in the case without pooling ($l = 1$) to be as well off as in the case with pooling ($l > 1$). Figure (5) shows the results under our base case parameterization. The figure presents $R(l,t)$ for each combination of $l \in \{1, 2, 3, ..., 100, \infty\}$ and $0 \leq t \leq 60$ (i.e. $60 \leq \text{age} \leq 110$) . The positive effect of pooling on lifetime utility is quite strong. For instance, in the case $l = 100$, age $= 60$ the equivalent wealth increase $R(l,t)$ is around 45%.[15] For very old investors $R(l,t)$ can easily exceed 100% and more. $R(l,t)$ increases with t because of the increasing mortality intensity λ_t. As expected, $R(l,t)$ rises also with l since the potential mortality credit rises with the number of pool members as mentioned above. Surprisingly, the increase of $R(l,t)$ in l is initially so steep that $10 - 20$ pool members are already sufficient to generate utility gains of more than 90% of that of perfect pools $l = \infty$. The increase in utility gains becomes rather small from around $l = 50$ upwards.

To understand how the various pool sizes would be assessed by men and women with different levels of risk aversion, the same welfare analysis is done for $\gamma = 2$ in order to reflect an investor with low risk aversion, $\gamma = 5$ for moderate risk aversion, and $\gamma = 30$ for high risk aversion . The impact of different gender and risk aversions on welfare gains are reported in Table I. Irrespective of gender and risk aversion, investors enhance their wellbeing the larger the initial pool size L_0 is. This is reasonable in light of the previously shown result that the mortality credit increases with the pool size. By contrast, utility gains do not increase uniformly for a given L_0 if the risk aversion parameter is raised. Specifically, in case the investor is a female and the pool size is $L_0 = 5$, they drop from about 28% to 17% if risk

[15] The approach to assess the utility gains be certainty equivalent comparisons is similar to the welfare analysis undertaken by Mitchell et al. (1999) and Brown, Mitchell, and Poterba (2001) for the case of life annuities. This study reported comparable utility gains in the range of 40%.

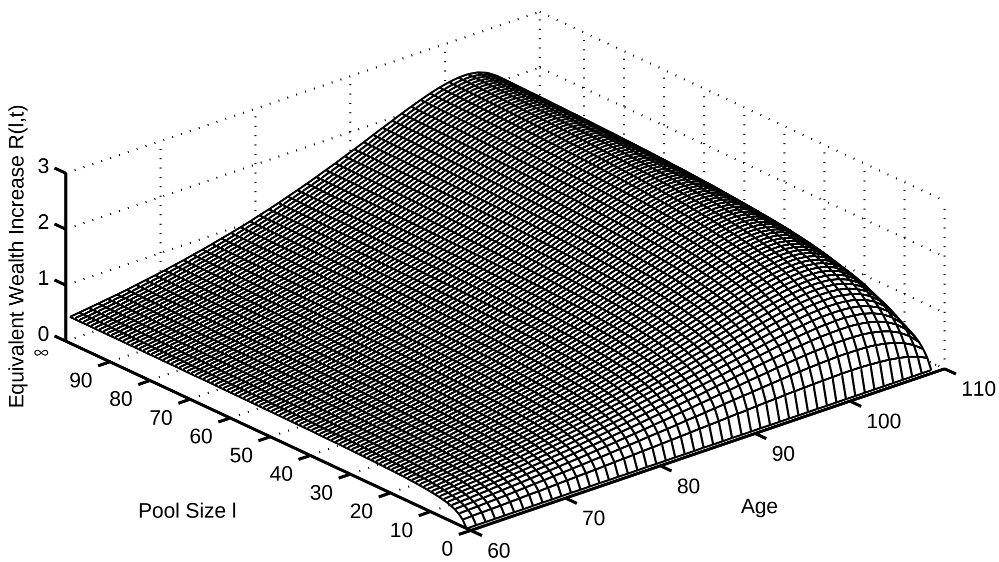

Figure 5: Equivalent Wealth Increase for Various Pool Sizes and Ages. This figure presents the equivalent wealth increase $R(l,t)$ which is calculated according to equation (38). $R(l,t)$ is the additional fraction of wealth needed in the case without pooling ($l=1$) to be as well off as in the case with pooling ($l>1$).

aversion is increased from $\gamma = 5$ to $\gamma = 30$. This utility decrease also holds true for males and pool size $L_0 = 10$. The reason is that benefits from self-insuring in a small sized pool are reduced because high risk averse investors evaluate the riskiness of the mortality credit more critically than low risk averse investors. In turn, very risk averse individuals would be more inclined to pay a premium in order to access a pool with higher pool size or a life annuity.

Comparing utility gains of men and women shows that men benefit more from pooling. This seems at first glance surprising as the problem to finance longevity is more eminent in case of longer living women. But the calibration of the Gompertz law parameter b indicates that mortality risk (dispersion of the time of death) is higher in case of men than in case of women whereas the fact that women life longer on average is deterministic. Accordingly men are more exposed to the risk to life longer or less than expected and hence can benefit more from insurance.

Table I indicates that if the pool size is $l = 100$, the difference in utility gains to the perfect

Table I
Equivalent Increase in Initial Wealth: Impact of Gender and Risk Aversion

			Females		
γ	$L_0 = 5$	$L_0 = 10$	$L_0 = 100$	$L_0 = 1000$	$L_0 = \infty$
2	23.78	28.21	32.97	33.30	33.59
5	27.66	35.75	46.82	48.12	48.64
30	17.13	29.64	62.48	71.39	72.92

			Males		
γ	$L_0 = 5$	$L_0 = 10$	$L_0 = 100$	$L_0 = 1000$	$L_0 = \infty$
2	30.52	36.53	43.25	43.75	44.12
5	35.21	46.01	61.30	63.17	63.91
30	22.11	38.23	82.39	95.15	97.44

Note: Table I reports the equivalent increase in initial wealth (in percent) for varying gender and risk aversion if $L_0 > 1$ investors pool their wealth compared to the case without pooling $L_0 = 1$. The equivalent increase in initial wealth can be interpreted as the additional fraction of initial wealth needed in the case without pooling to have at age 60 the same expected utility as in the cases with pooling. All parameters but risk aversion and mortality laws are set according to the base case. The calculations are done according to equation (38).

pool $l = \infty$ becomes rather small, irrespective of the investor's risk aversion and gender. This suggests that the remaining risks of non perfect pools are negligible if investors are homogenous and have CRRA preferences if the pool size is around $l = 100$. The results imply that people with access to a large pooled annuity fund would only pay low risk premiums in order to access the private life annuity market. Thus, if external capital for insurance companies is costly, grouped annuity funds may be preferred. Moreover, even very small pools ($L_0 = 5$,$L_0 = 10$) such as families can replicate more than 50% of the utility gains of perfect pools which could provide an explanation for the empirical low demand for life annuities.[16] Only investors with restricted opportunity to create a small pool (e.g. due to small family size) or investors with high risk aversions seem to be more inclined to afford the risk premium for a life annuity.

[16]This result indicates that even a family could manage longevity risk effectively. See also Kotlikoff and Spivak (1981) for a theoretical discussion of family self-insurance.

5 Conclusion

Pooled annuity funds are alleged to constitute an alternative to the classic longevity insurance via life annuities offered by insurance companies (see Piggott, Valdez, and Detzel (2005) and Valdez, Piggott, and Wang (2006)). To answer the question how pooled annuity funds perform compared to life annuities this paper analyzes the optimal dynamic consumption and portfolio choice problem of pooled annuity funds in a continuous time framework with CRRA preferences. It is shown that the research question translates into an optimal control problem with stochastic investment horizon and jumps in wealth. Wealth of surviving members of the fund jumps if one other fund member dies and forfeit wealth is reallocated among the survivors.

The framework embraces also the case $l = 1$ reflecting the self annuitization strategy and the case $l = \infty$ reflecting the purchase of a life-annuity. In these cases analytical solutions for the optimal consumption rate are derived. In the case $l = 1$ we extent the analysis of Merton (1971) by allowing time-dependent hazard rates to capture empirical mortality patterns. In the case $l = \infty$, we contribute to the prior literature on life-annuities which exogenously imposed a certain payout structure of life-annuities. The present study endogenously derives the optimal payout pattern by assuming that all investors of the annuity are homogenous.

The present paper also contributes to the literature by solving the optimal dynamic consumption and portfolio choice problem for a finite number of homogenous pool members $1 < l < \infty$. The optimization problem can be reduced to a set of ODEs whose number is equal to the number of annuity fund members l. It is shown that the optimal consumption fraction increases monotonically with the number of fund members and age. The reason for the first result is that the more fund members are in the pool the more mortality credits can be paid out in the future. The increase in age is explained by the increasing time preference due to the increasing mortality when becoming old. The optimal portfolio follows the standard Merton rule as mortality risk is assumed to be not correlated with the given investment opportunity set and because the size of the mortality credit does not depend on the fund's asset allocation. The participation in a pooled annuity fund constitutes an endogenous bequest motive because the investor is willing to leave estate to the other pool members to be in turn potentially rewarded with their estate.

Welfare calculations suggest that the utility increase from pooling is substantial, e.g. 45% in the case of hundred pool members $l = 100$. The magnitude of this number is comparable to those attained by Mitchell et al. (1999) and Brown, Mitchell, and Poterba (2001) for the case of life-annuities. At first glance maybe surprisingly it turns that pooling generates higher utility gains for men than for women since women live longer on average and are thus thought to be more exposed to longevity risk. However the opposite is true if longevity risk is seen as the risk to live longer than expected. Then, men face higher risk since the mortality risk i.e. the dispersion of the men's time of death is higher than that of women.

It is also demonstrated that even rather small sized pools with about $l = 100$ members can produce around 90% of those utility gains generated by a perfect pool or equivalently life annuities. Thus, the remaining mortality risks which have to be borne by pooled annuity fund members seem to be economically negligible under CRRA preferences indicating that pooled annuity funds may be preferred if external capital for insurance companies is costly. Even pools being not bigger than a family can self-insure against mortality effectively and reach more than 50% of the utility gains of a perfect pool. This result is in line with the low empirical take up for private life annuities. Only investors with high risk aversion may be inclined to pay a risk premium in order to transfer mortality risk completely to a insurance company.

It remains to be explored whether stochastic changes in the mortality probabilities over time would change the above given picture structurally. However, empirically improvement in mortality was rather smooth and predictable. Thus, considering the risk of changing hazard rates in a pure diffusion model calibrated to historical data should also have a negligible impact on utility. However, it would be interesting to introduce extreme mortality changes via jumps in mortality hazard rates in order to compare the utility outcome of group self annuitization strategies versus life annuities. This analysis should then also take into account that also the insurance company could default in such extreme events.

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APPENDIX

A. Derivation of the Expected Cumulative Mortality Credit

Given that the current pool size is $L_t = l$, the at t expected mortality credit which is received between t and s conditional on the survival of the considered pool member i is given by

$$EMC_i(l,t,s) = E\left[\prod_{j=1}^{\Delta N(t,s)} \left(1 + \frac{1}{L_t - i}\right) |L_t = l, N_{i,s} = 0\right]. \tag{39}$$

The result of the product $\prod_{i=1}^{\Delta N(t,s)} \left(1 + \frac{1}{L_t - i}\right)$ is $L_t/(L_t - \Delta N(t,s))$. Thus the expected mortality crdit can be rewritten as

$$EMC_i(l,t,s) = \sum_{k=0}^{l-1} P(\Delta N(t,s) = k|L_t = l, N_{i,s} = 0)\frac{l}{l-k}, \tag{40}$$

where $P(\Delta N(t,s) = k|L_t = l, N_{i,s} = 0)$ denotes the probability that N jumps k times between t and s given that $N_{i,s} = 0$ or in other words that k of the other $L_t - 1$ pool members $j \neq i$ die between t and s. According to equation (2) the individual survival probability for each pool member is given by $p(t,s)$. Thus, the total number of perished members $\Delta N(t,s)$ is binomially distributed according to:

$$P(\Delta N(t,s) = k|L_t = l, \{N_{i,u} = 0\}_{u=0}^{s}) = \binom{l-1}{k} (1 - p(t,s))^k \, p(t,s)^{l-1-k}. \tag{41}$$

Plugging (41) in (19) yields

$$\begin{aligned} EMC_i(l,t,s) &= \frac{\sum_{k=0}^{l-1} \binom{l}{k} (1 - p(t,s))^k \, p(t,s)^{l-k}}{p(t,s)} && (42) \\ &= \frac{P(\Delta N(t,s) \leq l-1)}{p(t,s)}. && (43) \end{aligned}$$

$P(\Delta N(t,s) \leq l-1)$ can be rewritten as the probability that at least one pool member survives until s and $1/(p(t,s)) = \exp \int_t^s \lambda_u \, du$ is exactly the (deterministic) mortality credit of a perfect pool $MC(t, s)$. So, the expected mortality credit of a finite pool $EMC(l, t, s)$ is a fraction of $MC(t, s)$:

$$EMC_i(l, t, s) = P(L_s \geq 1 | L_t = l)\, MC(t, s). \tag{44}$$