Optimal Dynamic Consumption and Portfolio Choice for Pooled Annuity Funds

by

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15th Colloquium of Superannuation Researchers
Introduction (I)

- **Individual Self-Annuitization (ISA)**: Ruin risk and utility implications well understood

- **Optimal annuitization strategies**: Optimal asset allocation and timing of (variable/fixed) life-annuity purchases

- **Current consensus: mortality risk should be hedged**
  e.g. Mitchell et al. (1999) and Brown et al. (2001) report utility gains of around 40%!


  **e.g. Yaari (1965), Richard (1975), Babbel and Merrill (2006), Cairns et al. (2006), Kojien et al. (2006), Milevsky and Young (2007), Milevsky et al. (2006), and Horneff et al. (2006a, 2006b, 2006c, 2007)**
Alternative mortality hedge: Group Self-Annuitization (GSA)

Construction of a Pooled Annuity Fund:
- Individuals pool their retirement wealth into an annuity fund
- in case one participant dies, survivors share the released funds (mortality credit)

In fact: GSA very common since families self-insure (Kotlikoff and Spivak, 1981)
Trade Offs between mutual fund, pooled annuity fund and life-annuity:

- Common characteristics:
  - assets underlying the different wrappers can be broadly diversified

- GSA / life-annuity versus mutual fund
  - earn the mortality credit but lose bequest potential
  - lost flexibility since purchase is irrevocable (due to severe adverse selection)

- GSA versus life-annuity:
  - group bears some mortality risk, but has to pay no risk premium to owners of an insurance provider
Prior Literature on Pooled Annuity Funds

  - Mechanics of pooled annuity funds: recursive evolution of payments over time given that investment returns and mortality deviate from expectation
  - But, no explicit modeling of risks

  - Two period model
  - Result: adverse selection problem inherent in life annuities markets is alleviated in pooled annuity funds
  - Rational: investors of pooled annuity funds cannot exploit perfectly the gains of adverse selection since mortality credit is stochastic
Contributions

Derivation of the optimal consumption and portfolio choice for pooled annuity funds

- If $l$ is the number of homogenous participants:
  - $l = 1$: Individual Self Annuitization
  - $1 < l < \infty$: Group Self Annuitization
  - $l \to \infty$: Ideal Life-Annuity

Integration of Individual / Group-Self-Annuitization and Life Annuity in a Merton continuous time framework with stochastic investment horizon

Prior literature on life-annuities sets the payout pattern exogenously (via so-called “assumed interest rate”, $AIR$)

Evaluation of self-insurance effectiveness for various pool sizes
Population Model I

- Stochastic time of death of investor $i$ determined by inhomogeneous Poisson Process $N_i$ with jump intensity $\lambda(t)$

$$\tau_i = \{ \min t : N_{t,i} = 1 \}$$

- Probability of no-jump (surviving) between $t$ and $s > t$:

$$p(t, s) = \exp \left\{ - \int_t^s \lambda_u \, du \right\}$$

with $\lambda(t)$ according to Gompertz Law (fits empirical data, easy to estimate).
Figure 1: Illustration of Simulated Populations. This figure presents simulated paths of the development of populations for initial pool sizes $L_0 = 5$ and $30$ and the survival probabilities $p(0,\text{Age-60})$. At $t = 0$ investors are assumed to be aged 60. Survival probabilities are calibrated according to the 1996 population 2000 basic mortality table for US females ($m = 86.85$, $b = 9.98$).
Risky asset e.g. globally diversified stock portfolio:

\[ \frac{dS_t}{S_t} = \mu \, dt + \sigma \, dZ_t \]

Riskless asset e.g. local money market:

\[ \frac{dB_t}{B_t} = r \, dt \]

Parsimonious asset model to focus on effects of mortality risk
Wealth Dynamics: Total Annuity Fund Wealth

- \( L_0 \) homogenous participants pool their wealth \( W_{i,0} \) in annuity fund (AF)

\[
W_{AF,0} = \sum_{i=1}^{L_0} W_{i,0}
\]

- Dynamics of the total fund value

\[
\frac{dW_{AF,t}}{W_{AF,t}} = \left[ r + \pi_t (\mu - r) - c_t \right] dt + \pi_t \sigma \, dZ_t
\]

\( c_t \): continuous withdrawal-rate from pooled annuity fund

\( \pi_t \): portfolio weights
Wealth Dynamics: Individual Wealth (I)

- Fraction of AF assets belonging to individual $i$
  $$h_{i,t} = \frac{W_{i,t}}{W_{AF,t}}$$

- Reallocation of individual wealth in case $j$ dies:
  $$W_{i,t} = W_{i,t-} + \frac{h_{i,t-}}{1 - h_{j,t-}} W_{j,t-}$$

- Dynamics of individual’s wealth (Ito’s Lemma):
  $$\frac{dW_{i,t}}{W_{i,t-}} = [r + \pi_t (\mu - r) - c_t] dt + \pi_t \sigma dZ_t + \sum_{j=1, j \neq i}^{L_0} \frac{h_{j,t-}}{1 - h_{j,t-}} dN_{j,t}$$
  $$dW_{i,t=\tau_i} = -W_{i,t-}$$
If all investors have equal share $h_i$:

$$
\frac{dW_{i,t}}{W_{i,t^-}} = \left[r + \pi_t(\mu - r) - c_t\right] dt + \pi_t \sigma dZ_t + \frac{1}{L_{t^-} - 1} dN_t \quad t < \tau_i, L_{t^-} > 1
$$

Expected instantaneous mortality credit:

$$
E \left[ \frac{1}{L_{t^-} - 1} dN_t \right] = \lambda_t dt
$$

Instantaneous variance of mortality credit:

$$
Var \left[ \frac{1}{L_{t^-} - 1} dN_t \right] = \frac{\lambda_t}{L_{t^-} - 1} dt \text{ for } L_{t^-} > 1
$$
Wealth Dynamics: Mortality Credit (I)

- Long-run expected mortality credit for finite pools with size \( l \): expected growth-rate between \( t \) and \( s \) due to reallocation of wealth

\[
EMC_i(l, t, s) = P(L_s \geq 1|L_t = l)\ MC(t, s)
\]

\( MC(t, s) \): deterministic mortality credit \( \exp \int_t^s \lambda_u \ du \) if \( l \to \infty \)
Figure 2: Expected Annualized Mortality Credit for Various Ages and Initial Pool Sizes. The annualized expected mortality credit is defined as $(EMC(l, t = 0, s))^{1/s} - 1$. 
Optimization Problem

Investors have CRRA preferences

\[ U = \int_0^\infty e^{-\delta t} u(C_t) \, dt \]
\[ u(C) = \frac{C^{1-\gamma}}{1-\gamma} \quad \gamma \neq 1, \gamma > 0, \]

Optimize expected utility by choosing \( c_t \) and \( \pi_t \) subject to:

\( 1 < L_t < \infty \)

\[ \frac{dW_{i,t}}{W_{i,t-}} = [r + \pi_t(\mu - r) - c_t] \, dt + \pi_t \sigma \, dZ_t + \frac{1}{L_{t-} - 1} \, dN_t \]

\( L_t = 1 \)

\[ \frac{dW_{i,t}}{W_{i,t}} = [r + \pi_t(\mu - r) - c_t] \, dt + \pi_t \sigma \, dZ_t \]

\( L_t \to \infty \)

\[ \frac{dW_t}{W_t} = [r + \pi_t(\mu - r) + \lambda_t - c_t] \, dt + \pi_t \sigma \, dZ_t \]
Analytical Results

- Optimal stock fraction for all cases:
  \[ \pi^* = \frac{\mu - r}{\gamma \sigma^2} \]

- Optimal Consumption fraction

  - \( L_t = 1 \):
    \[ c(1, t) = \left\{ \int_t^\infty e^{\gamma \int_t^s (A - \lambda_u) \, du} \, ds \right\}^{-1} \]

  - \( L_t \to \infty \):
    \[ c(\infty, t) = \left\{ \int_t^\infty e^{\gamma \int_t^s (A - \gamma \lambda_u) \, du} \, ds \right\}^{-1} \]

  - \( 1 < L_t < \infty \) : Solve ODEs numerically for \( f(l, t) \) and plug into \( c(l, t) \)
    \[ \frac{f_t(l, t)}{f(l, t)} + \gamma f(l, t)^{1/\gamma} + (A - \lambda_t l) + \lambda_t (l - 1) \left( \frac{l}{l - 1} \right)^{1-\gamma} \frac{f(l - 1, t)}{f(l, t)} = 0 \]
    \[ c(l, t) = f(l, t)^{-\frac{1}{\gamma}} \]
Figure 3: Optimal Withdrawal Policies for Varying Age and Population Size. The upper graph presents the optimal discrete withdrawal fraction \(1 - \exp(-c(l, t))\) of the base case for different ages and population sizes \(l\). For example, at age 80 the optimal withdrawal rate is 0.066 if the current pool size is \(l = 1\) and 0.095 if \(l = 5\).
Numerical Results: Optimal Withdrawal Rate/Payout Profile

![Graph showing optimal consumption fraction vs. age for different lifespans and gender.](image-url)
Numerical Results: Expected Consumption Path

Figure 4: Expected Optimal Consumption Path for Varying Age and Initial Pool Size. The calculation of the expectations is done based on the optimal consumption policy $c(l, t)$ via 100,000 Monte Carlo simulations in which the time step is set to $\Delta t = 1/240$. 
Numerical Results: Welfare Increase Relative to $l=1$

Figure 5: Equivalent Wealth Increase for Various Pool Sizes and Ages. This figure presents the equivalent wealth increase $R(l, t)$ which is calculated according to equation (38). $R(l, t)$ is the additional fraction of wealth needed in the case without pooling ($l = 1$) to be as well off as in the case with pooling ($l > 1$).
### Table I

Equivalent Increase in Initial Wealth: Impact of Gender and Risk Aversion

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$L_0 = 5$</th>
<th>$L_0 = 10$</th>
<th>$L_0 = 100$</th>
<th>$L_0 = 1000$</th>
<th>$L_0 = \infty$</th>
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<tr>
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<td>29.64</td>
<td>62.48</td>
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</table>

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<td>38.23</td>
<td>82.39</td>
<td>95.15</td>
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</tr>
</tbody>
</table>

**Note:** Table I reports the equivalent increase in initial wealth (in percent) for varying gender and risk aversion if $L_0 > 1$ investors pool their wealth compared to the case without pooling $L_0 = 1$. The equivalent increase in initial wealth can be interpreted as the additional fraction of initial wealth needed in the case without pooling to have at age 60 the same expected utility as in the cases with pooling. All parameters but risk aversion and mortality laws are set according to the base case. The calculations are done according to equation (38).
Conclusion

- Integration of Individual / Group-Self-Annuitization and Life Annuity in a Merton continuous time framework with stochastic investment horizon

- Derivation of the optimal consumption and portfolio choice for
  - $l = 1$: Individual Self Annuitization
  - $1 < l < \infty$: Group-Self-Annuitization
  - $l \rightarrow \infty$: Life-Annuity with optimal payout profile

- GSA is an effective mortality hedge

- Utility gains almost as high as those of optimal and actuarially fair life-annuities