Abstract
We observe that returns are driven by asymmetric and fat-tail distribution while the mean-variance criterion only provides a good approximation of expected utility maximisation under the assumption of normality. It turns out that the incorporation of skewness and kurtosis into investor’s decision will change the asset allocation of an optimal portfolio. This paper investigates the importance of higher moments in explaining the variation of small stock returns in ASX 300 and in S&P 500 respectively. We implement the polynomial goal programming model to obtain a portfolio with higher moments under the constraints on the first two moments. The results of this paper may help investors, especially Australian investors, in allocating their funds efficiently.
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1. Introduction

The foundation of capital asset pricing model (CAPM, hereafter) by Markowitz (1952) Sharpe (1964) and Lintner (1965) has led to numerous studies on asset allocation based upon on mean-variance (the first two moments) criterion of return distributions. Under the two moments, asset returns are assumed to be normally distributed. However it has been known that stock returns are driven by asymmetric or fat-tail distributions and extreme returns occur too often to be consistent with normality. This creates a controversy in capital asset pricing literature on whether higher moments (skewness and kurtosis) should be accounted for explaining a gap between theory and reality.

The third moment, skewness, is a measure of the asymmetry of the probability distribution of returns. This risk refers to any quantitative model that is based upon the normal distribution while its independent variables are skewed more than what the normal distribution can tolerate. Ignoring skewness risk in predicting stock returns causes the capital asset pricing model to underestimate the risk of returns with high skewness. There are two types of skewness: conditional and unconditional skewness. The conditional skewness (co-skewness, hereafter) is more widely used in empirical testing since it takes into consideration the correlation with the market performance. The fourth moment, kurtosis, is a measure of the "peakedness" of the probability distribution of returns. A distribution with high kurtosis has a sharper peak and fatter tails as compared to normal distribution. Kurtosis risk or "fat tail" risk means that we have more extreme observations than we do at the tails of the normal distribution. Like co-skewness, conditional kurtosis (co-kurtosis, hereafter) is more preferred as compared to unconditional skewness if investors want to compare with the benchmark.

Empirical evidences show that returns are highly correlated when below the mean (i.e in bear market) than when above the mean (in bull market). Investors treat downside losses and upside gains asymmetrically, giving the former much heavier weight in their decisions than the latter (Kahneman and Tversky’s loss aversion preferences). Therefore, they demand a higher premium for holding stocks with high downside risk. However, most of capital asset pricing models fail to capture this variation in risk factors loadings as they suggest that expected excess return or return premium is linearly proportional to the market premium in both bull and bear markets. Inter-temporal CAPM by Merton (1973), Arbitrage Pricing Theory (APT) by Roll (1977) or Fama and French three factor model (1992) try to increase the power of explanation of CAPM by including more explanatory variables rather than incorporating the asymmetry in the factor loadings across up and down markets. Therefore, the variation of return due to market condition is still left unexplained.

Investors desire a model which can incorporate the asymmetry of stock return distributions and the correlation between returns and market condition. And this study seeks to address this shortcoming.

Consistent with the current literature, the goal of this thesis is to provide a further understanding about the roles of higher moments in capital asset pricing models. This study firstly examines the importance of skewness and kurtosis in asset allocation for ASX 300 and S&P 500 stocks. Thereafter; the study presents an asset pricing model when skewness and kurtosis are factored on. A program is then created to optimize a portfolio’s asset allocation given the historical assets’ returns. This program applies the polynomial goal programming model\(^{1}\) to solve the conflict between maximizing expected return and skewness and minimizing variance and kurtosis simultaneously. Finally, portfolios under the higher moment model and under two moment model are constructed. The study expects that the four-moment based portfolio will outperform the mean-variance based portfolio. If successful, this finding would provide important implications for investors, especially Australian investors, in allocating their funds efficiently.

\(^{1}\) Polynomial goal programming model is an extension or generalisation of linear programming to handle multiple, normally conflicting objective measures. Each of these measures is given a goal or target value to be achieved. Unwanted deviations from this set of target values are then minimised in an achievement function.
The structure of this proposal is as follows. Section 2 provides a brief overview of basic literature on capital asset pricing. Section 3 describes the data to be used and how they are employed in our empirical testing for higher moments’ statistic importance. The preliminary results are also briefly discussed in this section. In section 4, a higher moment model is presented and methodology for optimizing asset allocation is included. Further issues and limitation are discussed in section 5. Section 6 concludes.

2. Literature Review

Capital asset pricing is modelled based on an intuition that a rational investor is only willing to bear more risk if he is compensated by higher return. The risk is referred as a return volatility measured by the standard deviation of its returns (the second moment). The theory relates the expected rate of return of an asset to its systematic risk which can not diversified by increasing the portfolio size.

According to Markowitz, the efficiency of a market portfolio is solely driven by mean-variance criterion. The traditional asset allocation problem of a portfolio is solved based on the mean-variance framework with the assumption of normality for the distribution of assets returns (Elton, Gruber, Brown and Goetzmann; 2003). However CAPM fails to explain the returns of the smallest market capitalized deciles (small-cap) and returns from specific strategies such as ones based on momentum. This strategy focuses on short term stock prices’ movements in bull and bear markets rather than the fundamental risk of companies. Thus the lack of consistent risk-based explanation in momentum effect is the most challenging asset pricing anomaly (Rachev, Jasic, Biglova and Fabozzi ;2005). Contributing to the puzzles of asset pricing theory, empirical evidences done by Jondeau and Rockinger (2006) show that if the investor’s utility is not quadratic, the maximisation of investors’ expected utility function with respect to mean and variance is different from their maximising asset allocation. Jondeau and Rockinger explain that expected utility maximisation is ineffective due to the fact that asset return distribution violates the assumption of normality. Thus under downside risk criteria, an asset allocation optimization for a portfolio constructed only by bonds and stocks results in a corner solution in the sense of a bond only portfolio (Hyung and Vries; 2007). Samuelson (1970) shows that when the investment decision is restricted to a finite time interval, the mean-variance efficiency becomes inadequate and the higher moments become more relevant to portfolio selection.

Fama and French (1992) correct the shortcomings of CAPM by including additional factors such as size (SMB)\(^2\) and book to market equity (HML)\(^3\) into the CAPM to increase the explanation power of the model. However, the model also fails to capture the asymmetry in the factors loading across up and down markets.

Different from Fama and French, Kraus and Litzenberger (1976) extend the Sharpe-Lintner CAPM model to incorporate the effect of skewness in return distributions with the assumption that investors prefer positive skewness in their portfolios. Their basic approach is to expand a utility function beyond the second moment in a Taylor series to examine skewness effects. They assume that investors are averse to variance in their portfolios, and therefore beta in individual assets. Thus, they prefer positive skewness in their portfolios. Friend and Westerfield (1980) find the Kraus-Litzenberger model unsuccessful in modifying the Sharpe-Lintner CAPM although they find some but not conclusive evidence that investors may pay premium for positive skewness in their portfolios.

Harvey and Siddique (2000a) test the three moment CAPM proposed by Kraus and Litzenberger but focus more on conditional skewness (systematic skewness). They found that stock with large negative co-skewness with the market will earn higher risk premia. They construct the conditional skewness factor following the methodology

\(^2\) SMB (Small minus Big) is the difference between returns on diversified portfolios of small and big stocks.

\(^3\) HML (High minus Low) is the difference between the returns on diversified portfolios of high and low book to market values.
used by Fama and French. They also find that the model incorporating co-skewness is helpful in explaining some of the non-systematic components in cross sectional variation of equity returns for portfolios where previous studies have been unsuccessful.

Smith (2007) fits the conditional three model factor proposed by Harvey and Siddique by allowing the prices of risk factor to vary over time. He finds that adding SMB and HML, the size and book to market factors of Fama and French (1993) has little impact on the price of market beta risk when conditional skewness risk is included in the model. It implies that part of the ability of SMB and HML to explain the variation in returns is related to systematic skewness.

Most of researchers concentrate on the first three moments, i.e within the mean-variance –skewness framework while neglecting kurtosis. The studies of Mandelbrot (1963) and Mandelbrot and Taylor (1967) show that stock returns on financial market are not normal but exhibit excess kurtosis (fat tails). Clark (1973) explains that different evolution of price series on different days is due to the coming of new information. Campbell and Hentschel (1992) argue that news generate volatility and cluster. Trading is slow when no information is available. As soon as information is released to the market, stock prices start varying significantly. Therefore, the larger the kurtosis is, the higher volatility is, the higher the probabilities of extreme events happen. Thus the higher volatility is, the larger the kurtosis is, the higher the probabilities of extreme events occurring. However, they also argue that good news increases stock prices, yet some of this increase is diminished by the increase in risk premium requested for the higher volatility. On the other hand, bad news lowers stock prices and this drop is amplified further by the increase in the risk premium. Because of the clustering of news, the left tail of return distribution is thicker than the right tail. Kirchler and Huber (2007) find a significant positive relationship between the degree of heterogeneity of fundamental information and the absolute returns. Heterogeneity of fundamental information is the main driving force for trading activity, volatility and the emergence of fat tails. Kirchler and Huber also discover that with respect to volatility clustering, the decrease of absolute returns after new information is released follows an intra-periodical pattern, which yields a long lasting positive autocorrelation of absolute returns. When information is rejected into the market, prices fluctuate largely. This volatility reduces quickly as traders learn from past prices and react quickly to that information. This leads to the relatively stable prices until new information is released again. It is therefore important to include both skewness and kurtosis in the capital asset pricing to allocate assets in a portfolio efficiently. And this is also the key point of the paper.

With the presence of skewness and kurtosis, portfolio selection is the trade off between the objectives such as maximizing expected return and skewness and minimizing risk and kurtosis simultaneously. For example, Andritt (1967) and Ingersoll (1975) discuss that investors with decreasing absolute risk aversion have to forgo the expected portfolio return if they want to gain more benefit from increasing its skewness, vice versa; while risk averse investors use its expected return as an expense to achieve portfolio risk reduction. Lai, et al (1991) proposes that the polynomial goal programming model (PGP,; hereafter) is significantly efficient in solving the conflict between competing portfolio objectives. The important feature of PGP is that the feasible solutions of PGP always exist. Therefore, PGP is helpful to provide the guidance on optimal asset allocation decision, such as, which investment strategies should be included or how much capital should be allocated to each asset.

Brinson, et al (1986) proposes the importance of precision in asset allocation by empirical analysis. According to their analysis, more than 90 percent of the performance of a portfolio is determined by asset allocation. Therefore, good methodology about asset allocation leads to an excellent performance for a portfolio.

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4 Information relates to the economic well-being of a company such as revenue, earnings, assets, liabilities and growth. These factors are used to determine the worth of an investment in fundamental analysis.
3. The importance of skewness and kurtosis in capital asset pricing

3.1 Hypothesis

In order to propose a CAPM with higher moments for asset allocation, we first need to show that higher moments are important to capital asset pricing. Current literature advocates the importance of higher moments for emerging markets (Satchell and Hwang; 1999). This study believes that higher moments are also important to small capitalization (small cap) stocks. Therefore, this research investigates the behaviour of small stock returns in ASX 300 and in S&P500 respectively.

The hypothesis for this part is as follows:

Ho: Higher moments are not statistically significant for small cap in ASX 300 and in S&P 500.
Ha: Higher moments are statistically significant for small cap in ASX 300 and in S&P 500.

3.2 Methodology and expected results

3.2.1 Preliminary test

The goal of this paper is to examine how asset allocation changes when returns are driven by a non-normal distribution. Therefore the test of normality for the distributions of small cap in ASX 300 and S&P500 is crucial in our analysis. The hypothesis of the normality test is as follows:

Ho: small cap returns in ASX 300 and S&P 500 are normally distributed.
Hₘ: small cap returns in ASX 300 and S&P 500 are not normally distributed.

Jarque-Bera (JB) test is employed in this hypothesis. The JB statistic measures the difference of the skewness and kurtosis of the series with those from the normal distributions. Under the null hypothesis of the normal distribution, the JB statistic is distributed with 2 degrees of freedom. Null hypothesis is rejected if the probability (P-value) associated with JB statistic is less than 5 percent level of significance. JB statistic and P-value can be calculated using E-View package.

The hypothesis of normality is expected to be rejected in both cases. Thus, the assumption of normality is questionable under mean-variance criterion.

3.2.2 Testing for the significance of skewness and kurtosis

After rejecting the normality hypothesis, in order to verify the importance of skewness and kurtosis in asset pricing model we then run a regression for a CAPM when skewness (S) and kurtosis (K) are added:

\[ R_p(t) - R_f(t) = \alpha + \beta(R_m(t) - R_f(t)) + \gamma_1 S + \gamma_2 K + e(t) \]  

(1)

Where \( R_p(t) \) is the return of a portfolio at time \( t \), \( R_f(t) \) is the risk free rate at time \( t \), \( R_m(t) - R_f(t) \) is the market premium per unit of risk, \( S \) and \( K \) are the sensitivity of a portfolio ‘s excess returns to the skewness and kurtosis of its returns over time.

Hypothesis test of skewness and kurtosis significance:

Ho: \( \gamma_1 = 0 \) and Ho: \( \gamma_2 = 0 \)
Ha: \( \gamma_1 \neq 0 \) and Ha: \( \gamma_2 \neq 0 \)

We expect that skewness and kurtosis are statistically significant 5% level of significance.
3.2.3 Testing for the power of explanation of four moment factor model:

In order to support the advantage of higher moment model in asset allocation compared to other previous models, we compare the power of explanation (R-squared) of higher moment model versus Fama and French model. We first run regression on Fama and French three factor model to obtain the adjusted R-squared:

\[
R_p(t) - R_f = \alpha + \beta(R_m(t) - R_f(t)) + s \times SMB + h \times HML + e(t) \tag{2}
\]

We expect that adjusted R-squared obtained in equation (1) \( R^2_1 \) is larger than in equation (2) \( R^2_2 \). This means the higher moment model explains the variation in returns better than Fama and French model. In other words, the four moment model provides a better fit to the data than the Fama and French model.

3.2.4 Testing whether SMB and HML are proxies for skewness and kurtosis

The motivation of this test is that there is a possibility that SMB and HML are related to skewness and kurtosis and that the Fama and French three factor model can be attributed to the factors proxying for omitted skewness and kurtosis. If this is the case, equation (3) below and equation (2) are indifferent as skewness and kurtosis are no longer statistically significant in equation (3). Therefore, the hypothesis is to test the joint significance of skewness and kurtosis in the equation below:

\[
R(t) - R_f = \alpha + \beta(R_m(t) - R_f(t)) + s \times SMB + h \times HML + \gamma_3 S + \gamma_4 K + e(t) \tag{3}
\]

**Ho:** \( \gamma_3 = 0, \gamma_4 = 0 \)  
**Ha:** \( \gamma_3 \neq 0, \gamma_4 \neq 0 \)

Smith (2007) argues that adding SMB and HML into higher moment model has little impact on the price of market beta risk. Therefore, we expect that the coefficients of skewness and kurtosis in this test are still significant at 5% level of significance. In addition, we also expect that \( \gamma_3 \) and \( \gamma_4 \) in (3) are smaller than \( \gamma_1 \) and \( \gamma_2 \) in (1), which means skewness and kurtosis are correlated to SMB and HML and part of the power of explanation of skewness and kurtosis are from SMB and HML. However, SMB and HML may not be good enough to substitute for skewness and kurtosis as \( \gamma_3 \) and \( \gamma_4 \) are still significant and therefore contribute to the explanation of the stock variation.

3.2.5 Robustness checks

To avoid spurious results from OLS regression in the case of heteroskedastic and serial correlation errors, the heteroskedastic auto-correlation regressions are used for Fama and French equation and the four moment model equation. The same conclusion like above is expected to be reached.

3.3 Data

For Australian market, the data used in this study consists of about 300 stocks listed in the constituent list of ASX 300. The advantage of this data is that it is easily used to build blocks for portfolio construction. The proxy for market index is the ASX300 index which is the value weighted average of 300 largest Australian companies based on market capitalization. A 90-day Bank-Accepted bill rate is used as proxy for risk free rate. The stock and index returns are collected daily from Datastream spanning the period 01/01/2001 (when the data is available) to 10/07/2007, yielding 1700 observations (holidays and weekends excluded).

For U.S market, we extract daily returns of 500 stocks listed in the constituent list of S&P 500. The advantage of the data is that it is most recognized as the best single estimate of U.S equity market which includes 500 leading
companies in leading American industries. Like ASX300, it is a core component of U.S indices and can be easily used as building blocks for portfolio construction. S&P 500 index is used as proxy for market index and 30 day Treasury bills is used as proxy for risk free rate. The data are extracted from Datastream starting from 01/01/2002 (when the constituent list is available in Datastream) to 10/7/2007. There are a total of 1440 observations, excluding holidays and weekends.

The data gathering of this study differs from the previous one in the following aspects: (1) only monthly returns were used to do the study of Fama and French and higher moments models, while we use daily data. It is documented by Kirchler and Huber (2007) that skewness and kurtosis become prominent when the higher frequency data is used (stylized fact). (2) There is little evidence has been published using Australian data to demonstrate the importance of higher moments to the asset allocation. Lastly, there is no literature compared the impact of higher moments between American stocks and Australian stocks while the current study seeks to address this relationship.

3.3.1 Sorting data

SMB and HML estimation:
This study adopts the methodology similar to that employed by Fama and French (1993). The sample data of each year is ranked by the market capitalization of that year and then broken into five (quintile) size groups with an equal number of stocks for each group. Quintile 1 contains the biggest stocks and quintile 5 comprises the smallest. The portfolio SMB, meant to mimic the risk factor in returns related to the size, is the difference between returns of stocks in quintile 5 and quintile 1. Independently, the sample is also ranked by market value to book value (or price to book value PTBV; hereafter). The sample is broken into five groups with an equal number of stocks in each group. Quintile 1 comprises the highest PTBV stocks and quintile 5 the lowest. The portfolio HML meant to mimic the risk factors in returns related to PTBV is the difference between stock returns in quintile 1 and quintile 5. The 25 portfolios are then created by the intersection of the five size and the five PTBV groups. These portfolios are able to capture the effect of size and market to book value (PTBV)

Co-skewness and co-kurtosis estimation:
The co-skewness and co-kurtosis risks of portfolios are calculated as follows:

\[
\gamma_{\Phi M} = \frac{E[\{r_p - E(r_p)\} \{r_M - E(r_M)\}^2]}{\sqrt{E[\{r_p - E(r_p)\}^2]E[\{r_M - E(r_M)\}^2]}} = \frac{\text{Cov}(r_p, r_M^2)}{\sigma_p \sigma_M^2}
\]

\[
\theta_{\Phi M} = \frac{E[\{r_p - E(r_p)\}^2 \{r_M - E(r_M)\}^2]}{E[\{r_p - E(r_p)\}^2]E[\{r_M - E(r_M)\}^2]} = \frac{\text{Cov}(r_p^2, r_M^2)}{\sigma_p^2 \sigma_M^2}
\]

Where \(r_p\) and \(r_M\) are the daily returns for the portfolio and the index. \(E()\) is the mean function and \(\sigma^2\) is the variance.

Harvey and Siddique(2000) shows that the co–skewness is related to the coefficient obtained from regressing the portfolio return on the square of market return. In other words, the indirect way to estimate co-skewness is to multiply the slope of the regression by a constant which is made up by three factors: standard deviation of the portfolio returns, standard deviation of index return, and standard deviation of square of index return. By estimating this way, t-statistics of the co-skewness is the same to that of the slope. Similarly, co-kurtosis can be estimated by the coefficient from regressing the square of portfolio return on the square of market return. In the preliminary results below, we will apply this method.
To measure the sensitivity of returns to co-skewness and co-kurtosis the study uses the mimicking portfolios constructed based on the methodology of Harvey and Siddique (2000). The sample data is classified by the magnitude of co-skewness (co-kurtosis) by descending and again broken into five groups with an equal number of stocks. Quintile 1 contains the highest co-skewness (co-kurtosis). The difference in two portfolios’ returns of low co-skewness (co-kurtosis) and high co-skewness are used as explanatory variables in equation 1. Each of the 25 portfolios constructed by the intersection of five co-skewness and five kurtosis groups is used as the dependent variable.

3.4 Preliminary results: Do co-skewness and co-kurtosis exist in the examined returns data?

3.4.1 The U.S market

<table>
<thead>
<tr>
<th>Size</th>
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<th>3</th>
<th>4</th>
<th>Small</th>
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<tr>
<td>Co-skewness (t-statistics)</td>
<td>0.31968</td>
<td>0.46028</td>
<td>0.43432</td>
<td>0.44175</td>
<td>0.31968</td>
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<td></td>
<td>(1.806115)*</td>
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<td>(4.014509)*</td>
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<td></td>
<td>(62.78511)</td>
<td>(75.74747)*</td>
<td>(104.2022)*</td>
<td>(109.7402)*</td>
<td>(322.2238)*</td>
</tr>
</tbody>
</table>

(*the t-statistics is significant at 5% level)

Table 1: The estimation of co-skewness and co-kurtosis for portofolios sorted by size, January 2002 – July 2007

The table shows that as size of the portfolios decreases, co-skewness and co-kurtosis both increase.

Table 1 and figure 1 present the summary statistics that compare the different measures of co-skewness and co-kurtosis across five portfolios groups ranked by size. As can be seen from the table, all of the statistics are significant at 5% level. This means that co-skewness and co-kurtosis do exist in the portfolios classified by size. Interestingly, the co-higher moments are significantly increases when size decreases. These preliminary results strengthen the belief of this study about the importance of higher moments for small cap.

Figure 1: The plots of co-skewness and kurtosis along the size of the portfolios.

These plots confirm the conclusion from table 1. In addition, it shows that co-higher moments of the portfolios ranked by size are significantly higher than the index.
The table shows that co-higher moments both increase from highest to lowest PTBV but the trend of co-kurtosis is not clear.

The plot of co-skewness confirms the conclusion from table 2. However, the second plot does not show clearly the increasing trend of co-kurtosis. The co-higher moments of the portfolios ranked by PTBV are significantly higher than those of the index.

Table 2 and figure 2 present the statistics of co-higher moments for portfolios ranked by PTBV. They show that co-skewness does increase as PTBV decreases. Although the trend for co-kurtosis is not clear, the co-higher moments of the portfolios are significantly higher than those of the index. If the index is assumed to be normally distributed, the results in figure 1 and figure 2 show that the return distributions of the portfolios, especially the smallest size or lowest PTBV portfolios, display a large deviation from normality.

### 3.4.2 The Australian market

<table>
<thead>
<tr>
<th>Size</th>
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<td>Co-skewness</td>
<td>-0.59277</td>
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<td>-0.40291</td>
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<td>-0.31279</td>
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<tr>
<td>(t-statistics)</td>
<td>-11.4868*</td>
<td>-10.2392*</td>
<td>-7.64892*</td>
<td>-6.265343*</td>
<td>-5.89775*</td>
</tr>
<tr>
<td>Co—kurtosis</td>
<td>5.604421</td>
<td>5.083594</td>
<td>4.294878</td>
<td>3.361199</td>
<td>4.848194</td>
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<tr>
<td>(t-statistics)</td>
<td>-11.486</td>
<td>-10.23921</td>
<td>-7.648924</td>
<td>-6.265343</td>
<td>-5.897752</td>
</tr>
</tbody>
</table>

(* the t-statistics is significant at 5% level)

Table 3: The estimation of co-skewness and co-kurtosis for portfolios sorted by size, January 2001 – July 2007

Although it is not clear compared to U.S market, co-skewness does increase as the size of the portfolio is smaller. The trend of co-kurtosis along the size is inconclusive.
Figure 3: The plots of co-skewness and co-kurtosis according to the size of the portfolios. The plots confirm the conclusion from Table 3. The absolute magnitude of co-higher moments of the portfolios is significantly higher than those of the index. It suggests a large deviation of the return behaviour from the index.

Table 3 and figure 3 show the co-higher moments’ statistics for Australian portfolios ranked by size. Different from U.S market, a negative co-skewness for Australian portfolios means the return of the portfolio and its variance are negatively related. As mentioned early in the literature, investors prefer a positive skewness to the negative one. Therefore, Australian investors should demand higher premium to compensate for the negative skewness risk they have to tolerate. Although the trend of co-higher moments are not clear as compared to U.S market, the fact that the co-moments of small cap are relatively higher than those of the index is one of the evidence confirming the non-normal distribution of small cap portfolios.

<table>
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<tr>
<th>PTBV</th>
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<th>3</th>
<th>4</th>
<th>Lowest</th>
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<td>-0.45997</td>
<td>-0.30561</td>
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<td>-0.2943</td>
</tr>
<tr>
<td>(t-statistics)</td>
<td>(-4.59934)*</td>
<td>(-8.77825)*</td>
<td>(-5.75969)*</td>
<td>(-4.64098)*</td>
<td>(-5.54245)*</td>
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<tr>
<td>Co-kurtosis</td>
<td>2.29867</td>
<td>5.820545</td>
<td>3.983114</td>
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<tr>
<td>(t-statistics)</td>
<td>(-4.599341)*</td>
<td>(-8.778251)*</td>
<td>(-5.759689)*</td>
<td>(-4.640984)*</td>
<td>(-5.542450)*</td>
</tr>
</tbody>
</table>

(* the t-statistics is significant at 5% level)

Table 4: The estimation of co-skewness and co-kurtosis for the portfolios sorted by PTBV, January 2001 – July 2007. The movement of co-higher moments are inclusive although it does show that both co-skewness and co-kurtosis increase in term of absolute magnitude.

Figure 4: The plots of co-skewness and co-kurtosis according to the PTBV of the portfolios. They again confirm the conclusion from Table 4. Co-higher moments are relatively higher than those of the index.
In summary, both markets show that co-higher moments are important to the distribution of stock returns, especially the small cap. In general, both skewness and kurtosis increase when size and PTBV decrease. However, the trend in the Australian market is not clear compared to the U.S market. It can be explained by the number of stocks included in each sample and the representativeness of each market. S&P 500 contains more stocks than ASX300 and well represents the global equity market while ASX 300 only contributes about 2% equity to the global market. Therefore, the statistics produced for S&P 500 portfolios are more unbiased and accurate.

Due to the time constraint and the scope of the present paper, we cannot provide a full result of all empirical tests listed above. Final results and conclusion for this part will be reached in the next state of this thesis. However, the preliminary results provide strong evidence about the importance of higher moments in explaining the variation in stock returns, particularly in small-cap returns. This supports us in designing the methodology for the next part.

4. Asset Allocation under mean-variance-skewness-kurtosis model

The goal of this part is to develop a program that can optimize the asset allocation under higher moment CAPM model. Following Lai (1991), Konno (1993) and Watanabe (2006), we apply the polynomial goal programming model to solve the conflict between maximizing expected return and skewness, and minimizing variance and kurtosis simultaneously. Optimal asset allocation is obtained after this problem is solved. The hypothesis for this part is as follows:

**Ho:** Asset allocation under higher moments does not beat asset allocation under mean-variance criterion.

**Ha:** Asset allocation under higher moments does beat asset allocation under mean-variance criterion.

The methodology for this part is formulated based on the following assumptions:

- **-** Investors are risk-averse and have a power utility function.
- **-** Investors have a non-decreasing concave utility. It means expected utility is positively related to expected return and skewness while negatively related to variance and kurtosis.

4.1 Methodology

Let n and T be the number of assets in the portfolio and number of observations during the period examined.

Let $X^{TR}$ be the transpose weight vector of assets in the portfolio; $R$ and $R(x)$ be return vector and portfolio return function at time t, $\bar{R}$ and $\bar{R}(x)$ be the expected return vector and the expected portfolio return function; $\sigma_i^2$ and $V(x)$ be the variance of asset i and the variance of portfolio function; $s_i^3$ and $S(x)$ and $K(x)$ be the skewness of asset i and skewness of portfolio; $V(R)$ and $S(R)$ be the variance and skewness of return vector R.
Step 1: (Two moments or mean-variance CAPM)
The target of this test is to minimize the portfolio variance function:
\[ V(x) = E \left( X^{TR} (R - \bar{R}) \right)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij} \]
Under given constraints:
1. \( \bar{R} (x) = RI = \sum_{i=1}^{n} r_i x \)
2. \( X^{T} I = \sum_{i=1}^{n} x_i = 1 \)
3. \( X \geq 0 \)
\( c \) is the return investors expect to have for their portfolio. If we let \( c \) vary from minimum to maximum possible returns, we can generate the Efficient Frontier.

Step 2: (Now consider skewness into the mean-variance model)
The target of this test is to maximize the portfolio skewness function:
\[ S(x) = E \left( X^{T} (R - \bar{R}) \right)^3 \]
\[ = E \left( \sum_{i=1}^{n} (r_i - \bar{r}_i) x_i \right)^3 = \frac{1}{T} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} (r_i - \bar{r}_i) x_i \right)^3 \]
Under given constraints:
1. \( V(X) \leq v \) (\( v \) is the minimum variance generated from step 1)
2. \( \bar{R} (x) = RI = \sum_{i=1}^{n} r_i x \)
3. \( X^{T} I = \sum_{i=1}^{n} x_i = 1 \quad X \geq 0 \)

Step 3: (Now we consider kurtosis into the mean-variance-skewness model)
We target to minimize the portfolio kurtosis function:
\[ K(x) = E \left( X^{T} (R - \bar{R}) \right)^4 \]
$$=E \left( \sum_{i=1}^{n} (R_i - \bar{R})x_i \right)^4 = \frac{1}{T} \left( \sum_{i=1}^{T} \sum_{j=1}^{n} (R_j - \bar{R})x_j \right)^4$$

Under given constraints:
1. $S(X) \geq s$ (the value generated from step 2)
2. $V(X) \leq v$ (the value generated from step 1)
3. $\bar{R} \ (x ) = \bar{R} \ I = \sum_{i=1}^{n} r_i x_i = c$
4. $X^T I = \sum x_i = 1 \quad X \geq 0$

Benchmark: Without any preference of returns (without $c$), given the level of risk free rate, we also can generate the finest (the most efficient) portfolio under mean-variance criterion. This will give investors a benchmark to make decisions.

4.2 How good is the proposed model?

In order to test the effectiveness of the four moment model, this study constructs a portfolio based on the higher moment model and a portfolio based on mean-variance model. The process of optimizing asset allocation of the higher moment portfolio follows the proposed methodology. Then the performance of these two portfolios is compared with the expectation that the asset allocation of higher moment portfolios will outperform mean-variance asset allocation.

5. Limitation

Like any other quantitative research, this study uses historical data to predict the future stock returns. The mismatch between the actual future returns and what is predicted by the proposed model is unavoidable. However, this difference is expected to be minor and will not crucially affect investor’s decisions. In addition, due to the time constraint, this research assumes that investors’ preferences toward the four moments are symmetric. However, different investors’ preferences affect the four moment statistics of returns and therefore affect the asset allocation of a portfolio and investment strategies. This issue raises an interesting question for further research and we expect to expand this study in future time.

6. Conclusion

It is apparent from part I that mean-variance-skewness-kurtosis model works well in explaining the variation in stock returns as compared to mean-variance model. In addition, part II propose that the portfolio constructed under four moment criterion will outperform the portfolio under two moment criterion. If this proposition turns out to be correct, this would amount support the investors in their asset allocating decisions. Moreover, it would greatly support the expansion of this study to incorporate the asymmetric investment’s preferences toward the mean, variance, skewness and kurtosis of returns. If our contention is incorrect, this paper will nonetheless reiterate an important message to researchers working in the same interest that higher moments are not statistically significant for American and Australian stock indices. In either case, the findings of this study are worth to contribute to the literature of capital asset pricing.
References


