A Constrained Coalitional Approach to Price Formation*

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Abstract

Since Edgeworth (1925) it has been understood that the Bertrand price setting game does not produce a pure strategy equilibrium in a number of simple settings. Two prominent examples are price competition with convex costs and spatial competition with finite buyers.

Building on work by Hamilton MacLeod and Thisse (1991), this paper develops an alternative model of price formation. We examine as a non-transferable utility coalitional game the set of outcomes that are feasible in the Bertrand price setting game. We prove that the core of this NTU coalitional game is equivalent to the set of outcomes that can be produced by undercut-proof prices. Moreover we show that the market clearing price is always in the core, and that where competition exists on both sides of the market there is a sense in which the core collapses to only admit market clearing outcomes.

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Within industrial organisation price formation tends to be modelled as a Bertrand price setting game. Implicit in the Bertrand price setting game is the assumption that each seller is constrained such that she must offer the same unit price to every potential buyer. This assumption effectively constrains the range of payoffs that may arise as outcomes of the game.

The Bertrand price setting game has proved to be problematic in a number of settings. In numerous simple models — such as the switching costs problem — the Bertrand price setting game fails to produce a pure strategy Nash equilibrium. Similar difficulties also arise where sellers face non-trivial capacity constraints, in models of location choice and where sellers face upward sloping marginal cost curves. Edgeworth (1925) was amongst the first to identify this problem stating that in the presence of capacity constraints,

“…there will be an indeterminate tract through which the index of value oscillate, or rather will vibrate irregularly for an indefinite length of time. There will never be reached that determinate position of equilibrium…”

Edgeworth, writing before the concept of a mixed strategy had been developed, proposed that capacity constraints would instead give rise to what we now know as the Edgeworth cycle.

As the structure of game theory solidified, it was determined that the Edgeworth cycle itself does not satisfy the requirements of an equilibrium. Authors tackling this problem turned either to a mixed strategy solution, or sought a pure strategy outcome by modelling price formation as a dynamic game. Neither approach has proven to be particularly satisfactory. Mixed strategy equilibria are difficult to interpret when the prices being modelled will persist for more than an instant, while dynamic games frequently lack predictive power due to the large range of prices that may be supported as an equilibrium.

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1Prices can also be backed out of the Cournot quantity setting game. However, as Kreps and Scheinkman (1983) have demonstrated, Cournot competition can be interpreted as a Bertrand price setting game in which sellers must unilaterally commit to a capacity in advance of the market.
The principal alternative to the Bertrand price setting game is to model transactions as a transferable utility (TU) coalitional game. TU games are relatively easy to work with and tend to produce precise and robust results. However, TU coalitional games admit as possible outcomes arbitrary allocations of value, most of which cannot be implemented through linear prices. Even where the core of a TU coalitional game can be implemented through a vector of linear prices, the form of the core may still be a consequence of the existence of non-core allocations that cannot.

If we accept that for a given market linear prices are not simply the outcome of the price setting game, but rather the consequence of a constraint exogenously imposed upon transactions, the TU approach is not acceptable. However, there does exist a third alternative, a synthesis of Bertrand price setting and a coalitional game, that obeys the linear price constraint while still delivering the precise and robust results that we seek.

This game, which we term the constrained coalition price setting game, examines as a non-transferable utility (NTU) coalitional game the set of outcomes that are feasible in the Bertrand price setting game. Thus far, games with this structure have only been applied to a very narrow range of problems. Spulber (1989) employed this structure to determine where a natural monopoly charging linear prices will be robust against an entrant targeting a specific subset of its customers. Probably the most complete existing treatment of this game comes from Hamilton, MacLeod & Thisse (1991) who employ the method to solve the location choice problem on the Hotelling line.

In this paper we extend and formalise the constrained coalitional price setting game to a general setting. Armed with this technique we proceed to tackle a series of previously intractable and anomalous problems in industrial organisation. The game is shown to be very flexible, making no assumptions concerning the distribution of bargaining power between buyers and sellers. Consequently, the constrained coalitional price setting game

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2 The terms buyer and seller are used in place of the more traditional consumer and firm in recognition of the fact that firms may be either a buyer or a seller in the market, and in order to avoid the common characterisation of consumers as passive participants in the game. Throughout the paper we will refer to buyers with masculine pronouns and
may be employed in the analysis of intermediate markets where it is possible for firms to exercise both monopoly and monopsony power.

The approach taken in this paper was inspired by Telser’s (1987) study of efficient co-operation and competition. Telser characterises games in which each player acts unilaterally — such as Bertrand price setting — as being competitive, while games in which decisions are made via coalitional interaction are termed co-operative. Telser conjectures that in order for a market to operate efficiently it requires a balance of co-operative and competitive forces. For example, fierce competition can be socially harmful when it leads to wasteful duplication, whereas a degree of co-operation in activities such as research and development has the capacity to enhance efficiency.

An important distinction between Telser’s approach and that adopted here is that Telser views core outcomes as being the product of a co-operative interaction between players. This characterisation fails to account for the fierce competition that may arise in a coalitional game. MacDonald and Ryall (2002) have shown that the ability of a player to appropriate value in a TU coalitional game is dependent on the ease with which they can be replaced in both the grand coalition, and all possible deviating coalitions. A player is not guaranteed to appropriate value in excess of their reservation payoff unless they add value to two or more coalitions in a way that cannot be replicated by a rival player. Similar factors are at work in an NTU coalitional game, although here the inability of players to arbitrarily assign value between the members of a coalition complicates the analysis.

In this regard Brandenberger and Stuart’s (2006) discussion of bi-form games is closer to the approach that we adopt here. Brandenberger and Stuart consider a game in which each player takes a unilateral action prior to participating in a coalitional game. Both environments are regarded as being competitive, however Brandenberger and Stuart recognise that some decisions — such as choice of capacity — are taken in highly structured environments best modelled as unilateral actions in a game, whilst other decisions are made in a less structured environment for which a coalitional sellers with feminine pronouns. This convention is adopted in the interests expositional simplicity.
approach is more appropriate.

The model developed in this paper should be viewed as a complement to, rather than as an extension of, these works. The interaction between players that generates prices in the constrained coalitional game is richer, and less structured, than is assumed by the Bertrand price setting game. At the same time we retain all the constraints on the range of feasible payoffs that are implicit in the Bertrand game.

Section 1 sets out the basic structure of the constrained coalitional price setting game. This description is more complete and substantially more general than that presented by Hamilton et. al. (1991). Included is a method for endogenising rationing in problems where capacity at a given price is less than demand. We also develop an important result that relates the cores of a TU and constrained coalitional analysis of a market.

The special case examined in section 2 generalises the central result developed by Hamilton et. al. (1991) to a generic spatial setting. Specifically, it is shown that an allocation lies in the core of a market with spatially differentiated products if and only if it can be generated by prices that are undercut-proof. As a simple example this result is applied to the switching costs problem.

Section 3 analyses a model of price competition in which sellers face convex cost functions. As Kreps and Scheinkman (1983) have demonstrated, when faced with convex costs the market may fail to possess a pure strategy Bertrand Nash equilibrium. It is shown that the market clearing price is always in the core of the constrained coalitional price setting game, and that where competition exists on both sides of this market there is a sense in which the core collapses to only admit allocations corresponding to market clearing prices. This model is also useful for illustrating the way in which the constraints imposed upon transactions can lead to inefficient outcomes lying within the core.

The final example considered in this paper concerns vertical contracting in which an input sold in an upstream market is transformed into a product sold in a downstream market. Once again it is shown that it is a core outcome for both markets to operate at the market clearing price. Moreover, the double
marginalisation that may arise in a Bertrand analysis of this market is shown to be an non-core outcome in the constrained coalitional price setting game.

The paper concludes with a discussion of the model and results, including the implications for anti-trust policy.

1 The Price Setting Game

In this section we formalise and extend the structure of the constrained coalitional price setting game pioneered by Hamilton et. al. (1991). As is the case with all NTU coalitional games, this game can be completely described by the set of players and a correspondence that indicates the set of payoffs that are feasible for each possible coalition of players. This correspondence is commonly referred to as the feasible set. It follows that in order to formalise the game we need only identify the range of payoffs that can be achieved by each coalition.

The first step is to set out the form of each player’s payoff. At this time we make no assumptions concerning the ways in which each buyer’s payoff depends on the prevailing prices and quantities consumed, however we do set out a specific form for seller payoffs that will hold throughout the paper. Next, we determine the ways in which output may be rationed between buyers. As is the case in the Bertrand price setting game rationing is not trivial, this can best be seen in the presence of capacity constraints and where sellers “tie” in the quality of the offer that they make to a given buyer.\footnote{Thus rationing also functions to endogenise the “tie breaking rule”.} The only demand that we place on the endogenous rationing rule is that the distribution of output be consistent with the structure of the game. The section concludes with a result linking the cores of TU and NTU coalitional games, and a guide to interpreting the core.

The constrained coalitional price setting game differs from Bertrand price setting only insofar as the strategic context in which prices are formed is coalitional rather than unilateral. Bertrand price setting is correctly described as a two-stage game. In the first stage each seller unilaterally, and in ignorance of the actions other sellers, sets a unit price for her product. In the second
stage the menu of prices is revealed to buyers, and buyers unilaterally optimise their consumption accordingly. Described this way the Bertrand Nash equilibrium is in fact a sub-game perfect Nash equilibrium of the two-stage game.

The constrained coalitional price setting game retains the structure of the Bertrand price setting game. However, instead of each seller making a unilateral pricing decision in the first stage, the prices of every seller are determined via unstructured interaction between all market participants. We make no assumptions concerning the nature of this interaction, and we specifically avoid imposing timing or any other structure. As is the case in Bertrand price setting, the outcome of the first-stage of the constrained coalitional price setting game is a menu of prices. In the second stage buyers unilaterally optimise their consumption.

In both games sellers are prevented from discriminating between buyers. Instead, each seller is constrained to offer the same terms of trade to all buyers. While this paper considers only linear prices, more complex terms of trade such as two part tariffs could also be considered.

Formally, given a possibly uncountable set $B$ of buyers, and a finite set $S$ of sellers, let $p_j \in \mathbb{R}$ denote the price set by the seller $s_j \in S$, and let $q_{ij} \in \mathbb{R}_+$ denote the quantity purchased by the buyer $b_i \in B$ from the seller $s_j$. Given prevailing prices $p = \{p_j\}_{j \in S}$ and quantities traded $q = \{q_{ij}\}_{i \in B, j \in S}$, the payoff to a player $k$ is denoted $x_k(p, q)$.

For the coalitional analysis it is also important to understand the payoffs that players receive when a coalition $Q \subseteq N = B \cup S$ of players deviate and trades amongst themselves.\footnote{In the language of coalitional game theory $N$ is the grand coalition consisting of all the players — buyers and sellers — active in the market.} We adopt the assumption that trades are only possible within a coalition that is able to agree upon terms of trade; formally $q_{ij} = 0$ where $\{i, j\} \not\in Q$.

Throughout this paper it is assumed that the payoff to a seller $s_j \in S$ takes the form,

$$x_{s_j}(p, q) = p_j \sum_{i \in B} q_{ij} - C_j \left(\sum_{i \in B} q_{ij}\right), \quad (1.1)$$
where the function $C_j(\cdot)$ denotes the cost to $s_j$ of producing the quantity $\sum_{i \in B} q_{ij}$. On the other hand the form of each buyer’s payoff is arbitrary and may depend upon the quantities consumed by other buyers.

### 1.1 Rationing Rules

Both constrained coalitional and Bertrand price setting games are solved recursively. In the second-stage of both games buyers observe the vector of prices and optimise their purchases accordingly. Even so, quantifying $q$ for a given $p$ (and $Q \subseteq N$) is not a trivial matter. Consider, for example, the case of Bertrand competition with homogeneous goods and capacity constraints examined by Kreps and Scheinkman (1983). In this model one seller may designate a price that lies below that of its rival, without being able to satisfy market demand at that price. In cases such as this some mechanism must be specified to ration the low price product to buyers.

Kreps and Scheinkman employ an efficient rationing rule. Formally, $q$ satisfies efficient rationing if and only if it maximises the aggregate surplus that accrues to buyers given the prevailing prices and capacity constraints. Efficient rationing is not uncontroversial as it precludes sellers employing rules such as first-come first-served, and may require a seller to have detailed knowledge of each buyer’s individual preferences. Moreover, even where sellers are able to institute efficient rationing it will not generally be in the best interests of sellers as efficient rationing tends to minimise the total quantity traded. These features are illustrated in example 1.1 below.

Hamilton et. al. (1991) do not consider how to incorporate rationing into the constrained coalitional price setting game as ration in the Hotelling model is straightforward. However, for the more general formulation presented here the issue of rationing must be addressed. This rationing rule rule is designed with two competing goals in mind. We wish to maximise the range of options available to players, while at the same time requiring the rationing to be consistent with the structure of the game. Specifically, rationing must account for the fact that once the menu of prices is posted each players is acting to

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5For games with uncountable buyers the sums in (1.1) must include an integral.
individually optimise their respective payoff. Note that given the structure of both Bertrand and constrained coalitional price setting, a rationing rule must be dependant on the price vector $p$ (and $Q$), therefore we will always speak of a set of trades satisfying a rationing rule for a given $p$ (and $Q$).

**Definition 1.** The trades $q = \{q_{ij}\}_{i \in B, j \in S}$ satisfy *rational rationing* for price vector $p$ and coalition $Q$ if and only if;

i. $x_i(p, q) \geq x_i(p, (q_{-i}, \hat{q}_i))$ for all $0 \leq \hat{q}_i \leq q_i$ and $b_i \in Q$;\footnote{We adopt the convention $q_i = \{q_{ij}\}_{j \in S}$ and $q_j = \{q_{ij}\}_{i \in B}$.}

ii. $x_j(p, q) \geq x_j(p, (q_{-j}, \hat{q}_j))$ for all $0 \leq \hat{q}_j \leq q_j$ and $s_j \in Q$;

iii. given that condition i. holds, if there exists $b_i \in Q$ such that,

$$x_i(p, (q_{-i}, \hat{q}_i)) > x_i(p, q), \quad (1.2)$$

for some $\hat{q}_i \geq 0$, there must exist $s_j \in Q$ for whom $\hat{q}_{ij} > q_{ij}$ and $x_j(p, (q_{-j}, \hat{q}_j)) < x_j(p, q)$.

iv. given that condition ii. holds, if there exists $s_j \in Q$ such that,

$$x_j(p, (q_{-j}, \hat{q}_j)) > x_j(p, q), \quad (1.3)$$

for some $\hat{q}_j \geq 0$, there must exist $b_i \in Q$ for whom $\hat{q}_{ij} > q_{ij}$ and $x_i(p, (q_{-i}, \hat{q}_i)) < x_i(p, q)$.

The first two conditions have a straightforward interpretation. They state that under rational rationing neither a buyer (condition i.) nor a seller (condition ii.) can benefit by unilaterally deciding to trade less than their allotted quantity; an option available to all players in the second stage of the game. Condition iii. states that if a buyer can improve his payoff by varying his purchases, at least one of the sellers required to supply a greater quantity of her product (recall condition i.) must be worse off as a result. Stated another way, if some group of sellers are left with excess capacity as a result of the rational rationing rule $q$, it cannot be the case that there exists a buyer who can improve his payoff by rearranging his consumption to exploit this
excess capacity. The final condition is the analogue of iii. concerning excess demand.

The efficient rationing rule adopted by Kreps and Scheinkman (1983) always satisfies the rational rationing rule set out above. To see this note that conditions i. and ii. are implicit within efficient rationing while conditions iii. and iv. must be satisfied if the trades maximise the aggregate surplus that accrues to buyers. The following example illustrates the relationship between efficient rationing, rational rationing, and trades that do not satisfy either rationing rule.

**Example 1.1.** Consider a market in which \( B = \{b_1, b_2, b_3\} \) and \( S = \{s_1, s_2\} \). Suppose that both sellers produce identical products at a constant marginal cost of zero and that each seller has a capacity of two. Each buyer demands one unit, with buyers 1 and 2 willing to pay a price of 2 for the product while buyer 3 has a willingness to pay of 1.

Suppose that the prices produced by the first stage game are \( p_1 = 1/2 \) and \( p_2 = 3/2 \), and consider the following trades:

i. Buyers 1 and 2 each purchase a unit from seller 1 and buyer 3 does not trade.

ii. Buyers 1 and 3 each purchase a unit from seller 1 and buyer 2 purchases a unit from seller 2.

iii. Buyers 1 and 2 each purchase a unit from seller 2 and buyer 3 purchases a unit from seller 1.

The first set of trades satisfies both efficient rationing and rational rationing. The buyers with the highest valuation for the product purchase from the seller with the lowest price thereby maximising the aggregate surplus that accrues to buyers, creating an aggregate surplus of 3 for the buyers and 1 for the sellers. Buyer 3 does not receive any of the good and cannot profitably exploit seller 2’s excess capacity at the price \( p_2 = 3/2 \) as this price is greater than buyer 3’s valuation for the product.

The second set of trades sees buyer 3 trading with the low price seller while buyer 2 is left to trade with the high price seller. Such a situation
might arise if buyers 1 and 3 arrive at the store before buyer 2, exhausting the capacity of seller 1. By the time buyer 2 arrives at the store he has no choice but to purchase the good from seller 2 at the higher price. In this case the aggregate surplus that accrues to buyers has fallen to 5/2 while the total seller payoffs have risen to 5/2. This second set of trades clearly do not satisfy efficient rationing as they do not maximise the aggregate buyer surplus. The trades remain rational however, as no buyer can profitable exploit the excess capacity of seller 2 and the demand of every buyer is satisfied.

The final set of trades do not satisfy either rationing rule. The total surplus that accrues to buyers is 3/2; half of the maximum. Moreover, given that seller 1 has one unit of excess capacity, buyer 1 could reduce the quantity that he purchases from seller 2 to zero, instead purchasing one unit of the good from seller 1. This variation strictly improves buyer 1’s payoff without reduce seller 1’s payoff, violating condition iii. of rational rationing.

In the models of spatial competition developed in section 2 — in which each seller faces a constant marginal cost and unlimited capacity — rational rationing uniquely defines payoffs for all buyers, and is sufficient to generate the core.

In contrast where sellers face a more complex cost structure rational rationing will not generally be sufficient. In these cases the choice of $q$ becomes one of the choices in the coalitional game. However, in recognition of the unilateral options available to players in the second-stage game, coalitions are constrained to select $q$ such that it satisfies rational rationing. Henceforth we write,

$$q^*(p, Q) = \{ q \in \mathbb{R}_+^{R \times S} : q \text{ satisfies rational rationing for } (p, Q) \},$$

(1.4)

to indicate the set of trades that satisfy rational rationing for a given price vector $p$ and coalition $Q$. Intuitively, while the players are committing to a set of trades at the same time that they are negotiating prices, they recognise that the commitment to a given $q$ must robust against the unilateral options that they will posses in the second stage of the game.
1.2 Solving for Prices

At the centre of the constrained coalitional game is the set of allocations that can be feasibly generated by players. For a given coalition $Q \subseteq N$, the set of feasible allocations is defined as,

$$V(Q) = \{ x(p, q) |_{Q} : p \in \mathbb{R}^{S \cap Q}, q \in q^*(p, Q) \},$$

where $x|_Q = \{x_i\}_{i \in Q}$. Thus every allocation in the feasible set must be able to be implemented by a vector of linear prices $p$ and the trades $q$. Moreover, the quantities traded are individually rational given the prevailing prices and active coalition. It is important to note that $V(N)$ is exactly the set of outcomes that may be produced by the Bertrand price setting game. Moreover, the constraint created by the price mechanism means that the constrained coalitional price setting game is of the non-transferable utility (NTU) type.\(^7\)

The constrained coalitional approach differs from Bertrand only in the way in which a solution is extracted from $V(N)$. For the purposes of this model the core of an NTU coalitional game $(N, V)$ is defined as follows:

**Definition 2.** An allocation $x$ is in the core of $(N, V)$ if and only if for all $Q \subseteq N$ and $y \in \mathbb{R}^{|Q|}$, if $y_k \geq x_k$ for all $k \in Q$, with strict inequality for all $k \in P \subseteq Q$ where $P$ is a coalition of strictly positive measure, then $y \notin V(Q)$.

Myerson (1991) employs strict inequalities in defining the core of an NTU coalitional game. In contrast to the definition of the core of a transferable utility type coalitional (TU) game, the choice of strength for the inequalities is not innocuous.\(^8\) As example 1.2 illustrates, weak inequalities create a more stringent condition that is harder to satisfy than strict inequalities, giving us greater confidence in the robustness of the core to re-contracting. Intuitively

\(^7\)Hamilton et. al. (1991) incorrectly state that the price setting game they analyse is of the TU type.

\(^8\)In a TU game weak and strict inequalities are equivalent. If one member of a coalition in a TU game strictly benefits from a deviation that leaves the other members of the coalition no worse off, some portion of that benefit can be redistributed between the remaining players of the coalition to create a strict benefit for every player. In an NTU game such an arbitrary re-allocation may not be possible.
the use of weak inequalities can be seen as implying a weak preference for altruism.

Example 1.2. The difference between the two cores can be seen clearly in the two player NTU game \((N, V)\) where \(N = \{1, 2\}\), \(V(1) = V(2) = \{0\}\) and \(V(N) = \{(x_1, x_2) : \max\{x_1, x_2\} \leq 1\}\). The set of allocations that are feasible for the grand coalition are illustrated in figure 1 as the area \(ABC0\). By Myerson’s definition the core of \((N, V)\) is the border \(ABC\), whereas according to definition 2 the core of \((N, V)\) is the point \(B\). Invoking weak inequalities lowers the dimension of the core and substantially increases the precision of the model. ■

The constrained coalitional price setting game belongs to a special class of NTU coalitional that can be generated by applying a constraint to a TU coalitional game. Consider a real valued function \(W : 2^N \rightarrow \mathbb{R}\) satisfying,

\[
V(Q) \subseteq W(Q) = \left\{ x_Q : \sum_{k \in Q} x_k \leq W(Q) \right\}, \quad \forall Q \subseteq N. \tag{1.6}
\]

\((N, W)\) is the NTU representation of the TU coalitional game \((N, W)\) with free disposal. There is a sense in which the game \((N, V)\) can be generated by applying a constraint to the game \((N, W)\), hence the use of the term constrained in the name of the price setting game.\(^9\) The natural candidate

\(^9\)Not all NTU coalitional games can be represented as the product of such a constraint.
for $W(Q)$ in the price setting game is the maximum value that can be created by trades between the members of coalition $Q$.

Constrained coalitional games can be easier to work with than other NTU games thanks to the following useful result:

**Lemma 1.** Suppose that $(N, V)$ and $(N, W)$ satisfy (1.6),

$$(\text{Core}(N, W) \cap V(N)) \subseteq \text{Core}(N, V).$$

(1.7)

Proof. Consider an allocation $x \in \text{Core}(N, W)$. It must be the case that,

$$\sum_{k \in Q} x_k \geq W(Q), \quad \forall Q \subseteq N.$$  

(1.8)

From the construction of $W(Q)$ it follows that there does not exist a $Q \subseteq N$ and $y \in W(Q)$ such that $y_k \geq x_k$ for all $k \in Q$, with strict inequality for all $k \in P \subseteq Q$ where $P$ is a coalition of strictly positive measure. Moreover, given that $V(Q) \subseteq W(Q)$, no such $y$ exists in $V(Q)$ either.

Lemma 1 states that if an allocation lies in the core of a TU analysis of a market, and is feasible in the sense that it can be implemented via a linear price mechanism, that allocation will also lie in the core of the constrained coalitional price setting game. Moreover, given that all allocations in the core of a TU game are efficient, if $(\text{Core}(N, W) \cap V(N)) \neq \emptyset$, there exists an efficient allocation in $\text{Core}(N, V)$.

Lemma 1 is not generally sufficient to generate the core of a constrained coalitional game. Some allocations in $\text{Core}(N, W)$ may not be feasible in $(N, V)$. Moreover, as a consequence of constraining $(N, V)$, an allocation $y$ that lies outside of $\text{Core}(N, W)$ may still be found within $\text{Core}(N, V)$ if sufficient allocations have been removed from $V(\cdot)$ to prevent $y$ from being blocked. Nevertheless, lemma 1 is particularly useful for proving existence, and provides an important link between the outcomes of unconstrained bargaining and the constrained coalitional price setting game.

The following example illustrates the construction of a feasible set, and the implications of lemma 1, for a bilateral monopoly market.

Consider the two player NTU game in which $V(1, 2) = \{x : x_1x_2 \leq 1\}$. There does not exist a real valued function $W$ satisfying (1.6) as the sum $x_1 + x_2$ is unbounded.
Example 1.3. A seller $s$ with unlimited capacity and a constant marginal cost of zero, trades a quantity $q$ to a buyer $b$. The buyer’s payoff is,

$$x_b(p, q) = q - \frac{q^2}{2} - pq.$$  \hfill (1.9)

where $p$ is the market price, while the seller’s payoff is $x_s(p, q) = pq$.

Given this payoff structure rational rationing admits a unique quantity for every choice of price. In this example we have but one buyer and one seller — precluding competitive effects on either side of the market — while the seller has unlimited capacity. It follows that the only constraint on the quantity traded is the buyer’s demand at the prevailing price, specifically $q'(p, N) = \{1 - p\}$ for all $p \in [0, 1]$ and $q = 0$ otherwise. Substituting for these values we find that $x_b = (p^2 - 2p + 1)/2$ and $x_s = p - p^2$ when the players agree to some price $p \in [0, 1]$.

We are now in a position to generate the feasible sets. Substituting for $p$ we find that $V(b) = V(s) = \{0\}$, and $V(N) = \{x : x_s = \sqrt{2x_b} - 2x_b\}$. The set $V(N)$ is illustrated by the curve $ABC$ in figure 2. The greatest total surplus that can be created in this market is $\frac{1}{2}$, however this outcome can only be achieved when both the price and the seller’s payoff are zero. In order for the monopolist to appropriate value in this example, she must destroy some of this total surplus. This feature of monopoly behaviour survives in the constrained coalitional price setting game because the seller must charge the same unit price for every unit sold, and the seller is unable to force the buyer to purchase a sub-optimally large quantity at that price.

It is straightforward to see that a TU coalitional game $(N, W)$, with $W(b) = W(s) = 0$ and $W(N) = \frac{1}{2}$, satisfies the relationship set out in (1.6). The set $W(N) = \{x : x_b + x_s \leq \frac{1}{2}\}$ is illustrated in figure 2 as the area lying on and below the line $CD$.

In a two player NTU coalitional game of this type we can find the core by inspection. An allocation $x$ is in the core of $(N, V)$ if and only if $x \geq 0$ and there does not exist $y \in V(N)$ such that $y > x$. Applying this criteria to figure 2 we see that the core of $(N, V)$ is the curve $BC$, while the core of $(N, W)$ is the line $CD$. In accordance with lemma 1, the only point in $\text{Core}(N, W)$ that is feasible in the game $(N, V)$ — the point $C$ — also lies
in the core of \((N, V)\). No other point in Core\((N, W)\) can be achieved by
the players in the game \((N, V)\), illustrating the importance of the feasibility
condition in lemma 1. At the same time the remaining allocations on the
curve \(BC\) are only in Core\((N, V)\) because the constraints that reduce the set
\(W(N)\) to the set \(V(N)\) have removed all the allocation in \(W(N)\) capable of
blocking them.

Two features of Core\((N, V)\) are a product of the NTU structure of this
game, and cannot arise in a TU coalitional game. First, all but one point in
Core\((N, V)\) are inefficient. This is clearly the case as the set of payoffs that
sum to the maximum available surplus is illustrated by the line \(CD\). Second,
and as a purely technical matter, Core\((N, V)\) is not convex.

1.3 Interpreting the Core

The core of a constrained coalitional game can be very large — as is the
case in the switching costs example of the next section — or may converge
to a unique allocation as is the case in the convex costs model of section 3.
Where the core encompasses a wide range of allocations we are left with the
question: How can we refine the core in such a way as to imbue the model
with predictive power?

In their examination of location choice on the Hotelling line Hamilton et.
al. (1991) consider only the core allocation that delivers the highest payoff
to sellers. Intuitively, this allocation can be viewed as the core allocation
that arises when the sellers possess all the bargaining power. Given that the
Bertrand price setting game grants sellers the power to make final take-it or
leave-it offers, an allocation that is preferred by all sellers can be regarded
as the analogue of the Bertrand Nash equilibrium.

However, the remainder of the core has value as well. Such is the flexibility
of the constrained coalitional price setting game that it can be applied to
markets — such as intermediate markets — in which buyers are considered
to possess some degree of bargaining power. The example presented in section
3 shows the effect that both monopoly and monopsony power have on the
shape of the core. Exactly where in this core we would expect the market to
operate will depend upon the relative bargaining power of the two sides.

The remainder of this paper is devoted to illustrating techniques for applying the constrained coalitional price setting game to a number of prominent setting.

2 Spatial Competition

Hamilton et. al. (1991) developed the constrained coalitional methodology in order to solve the problem of location choice on the Hotelling line. They adopted this technique because when firms are closely spaced on the line the Hotelling model does not possess a pure strategy Bertrand Nash equilibrium. In this section we extend Hamilton et. al.’s model to a general spatial setting, in the process proving that the central result developed by Hamilton et. al. for the Hotelling line remain valid in this setting. Specifically, it is shown that the core of the spatial market is equivalent to the set of payoffs that can be generated by undercut-proof prices. This fact allows us to reduce the often complex problem of finding the core to a single condition on each seller’s price.

The essence of spatial competition can be captured in the form of each player’s payoff. Each seller $s_j$ is assumed to have unlimited capacity and face a constant marginal cost $c_j$ such that,

$$x_j(p, q) = (p_j - c_j) \sum_{i \in B} q_{ij}. \quad (2.1)$$

While the payoff of each buyer $b_i$ can be expressed as,

$$x_i(p, q) = \int_0^{\sum_{j \in S} q_{ij}} D_i^{-1}(\vartheta) d\vartheta - \sum_{j \in S} \left(1_{q_{ij} > 0} K_{ij} + g_i(p_j + k_{ij})\right). \quad (2.2)$$

Here $K_{ij} \in \mathbb{R}_+$ is defined as the fixed cost that $b_i$ faces when trading with $s_j$, $1_{q_{ij} > 0}$ is an indicator function that takes the value 1 if $q_{ij} > 0$ and 0 otherwise, and $k_{ij} \in \mathbb{R}$ is the per unit transport cost. Finally, $D_i^{-1}(\cdot)$ is buyer $b_i$’s inverse demand.

The structure of this spatial market is very general and allows for a continuum of buyers, discrete buyers with distinct demand functions or a combination of both. Importantly, while the values of $K_{ij}$ and $k_{ij}$ may imply a
geometric structure to the market, this need not be the case. The form of buyer payoffs set out in (2.2) admits the possibility of either buyers or sellers occupying multiple locations.

The only limits that need be placed on a buyer’s demand function are: (i) each demand function $D_i(\cdot)$ must be non-increasing; (ii) each $D_i(p_j)$ must be defined for all $p_j \in \mathbb{R}_+$; and (iii) $D_i(0) \in (0, \infty)$, and there must exist some $\bar{p}_j \in (0, \infty)$ such that $D_i(\bar{p}_j) = 0$.\footnote{The effect of these assumptions is to ensure that for any finite arbitrary combination of demand functions, there exists some price $p'_j$, bounded away from zero, such that,}

$$\frac{\partial}{\partial p_j} (p_j - c_j) \sum D_i(p_j) < 0, \quad \forall p_j \in [c_j, p'_j).$$

Throughout this section we use the switching costs model to illustrate step-by-step the way in which a spatial problem can be solved through the use of the constrained coalitional price setting game. The switching costs model is one of the most fundamental in industrial organisation. In its simplest form there are two locations with a buyer and seller collocated at each. Each buyer has unit demand and must decide between buying locally or incurring transport cost $t > 0$ to purchase from the seller at the other location.

Several authors have tackled the switching costs model and the model has spawned a number of innovative solutions. The mixed strategy Bertrand Nash equilibrium to this model was characterised by Shilony (1977), Eaton & Engers (1990) found Markov perfect equilibria of an infinitely repeated sequential moves Bertrand price setting game, while Shy (2001) proposed that an undercut-proof criteria is the appropriate solution concept to apply to the model. While it turns out that Shy’s solution (almost) coincides with the solution proposed here, Shy does not provide a theoretical justification undercut-proofness as an equilibrium concept. Rather, Shy justifies the undercut-proof criteria on the ground of existence and tractability.

In the first part of this example we show how the switching costs model can be parameterised in terms of the payoff functions set out in (2.1) and (2.2), and demonstrate that the model does not posses a pure strategy Bertrand Nash equilibrium.
Example 2.1. In the switching costs model we have $B = \{b_1, b_2\}$ and $S = \{s_1, s_2\}$, with $K_{ij} = t$ for $i \neq j$ and $K_{ij} = 0$ otherwise. To complete the characterisation we have $c_1 = c_2 = c$, $k_{ij} = 0$ for all $i$ and $j$, and,

$$D_i^{-1}\left(\sum_{j \in S} q_{ij}\right) = \begin{cases} v & \sum_{j \in S} q_{ij} \in [0,1] \\ 0 & \sum_{j \in S} q_{ij} \in (1,\infty) \end{cases} \quad \forall i \in B,$$  \hspace{1cm} (2.3)

where $v > 2t + c$ is the valuation that the buyers place on a unit of output.

One can readily check that this model does not possess a pure strategy Bertrand Nash equilibrium. It is straightforward to see that we must have $p_i \geq c$ for all $i \in \{1, 2\}$ in a Bertrand Nash equilibrium. We are left with three possible cases:

Where $p_i > p_j + t$, seller $s_i$ receives a payoff of zero as the buyer $b_i$ can get a better deal by trading with seller $s_j$. Given that $s_i$ can increase her payoff to $p_j + t - c > 0$ by matching $s_j$’s price we cannot have $p_i > p_j + t$ in a pure strategy Bertrand Nash equilibrium.

Regardless of the tie breaking rule, for $p_i = p_j + t$ we must have either $s_i$ or $s_j$ trading less than one unit to $b_i$. It follow that at least one seller must be able to strictly increase her payoff by lowering her price by sufficiently small $\varepsilon > 0$; this excludes $p_i = p_j + t$ as a possibility.

Finally, for $p_i \in (p_j - t, p_j + t)$ note that $s_i$ trades only with $b_i$ and receives the payoff $x_{si} = p_i - c$. Given that this interval of prices is open, $s_i$ can always find another price that is greater than $p_i$, but still lies within the interval $(p_j - t, p_j + t)$, thereby increasing her payoff. It follows that we cannot have $p_i \in (p_j - t, p_j + t)$ in a pure strategy Bertrand Nash equilibrium.

$\blacksquare$

2.1 Rational Rationing in a Spatial Market

Given the payoff structure of a spatial market, rational rationing uniquely defines each buyer’s payoff for a given $p$ and $Q$. To see this note that the sellers in $Q$ always have excess capacity and as such for all $q \in q^*(p, Q)$,

$$q_i = \arg\max_{q_i} \int_0^{\sum_{j \in S} q_{ij}} D_i^{-1}(\vartheta) d\vartheta - \sum_{j \in S} \left(1_{q_{ij} > 0}K_{ij} + q_{ij}(p_j + k_{ij})\right) \quad (2.4)$$
Any $q$ that violates this condition for a buyer $b_i$ must also violate either condition i. or iii. of definition 1.

Imposing the requirement for rational rationing delivers us the following general result:

**Lemma 2.** Consider a market in which buyer and seller payoffs satisfy (2.1) and (2.2). For all $p \in \mathbb{R}^{|S|}$, $Q \subseteq N$, $q \in q^*(p, Q)$ and $b_i \in Q$, if $q_{im} > 0$ then,

$$
\int_0^{\sum_{j \in S} q_{ij}} D_i^{-1}(\vartheta)d\vartheta - \sum_{j \in S} (\mathbb{1}_{q_{ij} > 0}K_{ij} + q_{ij}(p_j + k_{ij})) = \int_0^{\hat{q}_{im}} D_i^{-1}(\vartheta)d\vartheta - (K_{im} + \hat{q}_{im}(p_m + k_{im})),
$$

(2.5)

where $\hat{q}_{im} = \sum_{j \in S} q_{ij}$. Moreover, there exists a $\hat{q} \in q^*(p^*, q^*)$ such that $\hat{q}_i = \{0, \ldots, 0, \hat{q}_{im}, 0, \ldots, 0\}$.

Lemma 2 states that given a price vector $p$, each buyer in a coalition can pick any seller in the coalition with whom they would be willing to trade some strictly positive amount, and still maximise his payoff by trading only with that seller. This result is straightforward once you recognise that the products being traded in the market are homogeneous save for their spatial characteristics, and that the transport costs of dealing with any given seller are a (weakly) concave function of the quantity traded. The proof of lemma 2 can be found in appendix A.

The significance of this result is that we need only consider deviating coalitions with a single seller when constructing the core, which gets us half way to proving that the core is defined by undercut-proof prices. The full significance of lemma 2 is made clear in the proof of proposition 1 below.

**Example 2.2.** Returning to the switching costs example, rational rationing admits a very limited range of trades. Given the payoff structure $q \in q^*(p, N)$ implies,

$$
(q_{ii}, q_{ji}) \in \begin{cases} 
\{(0,0)\} & p_i > p_j + t \\
\{(0,0), (1,0)\} & p_i = p_j + t \\
\{(1,0)\} & p_j - t < p_i < p_j + t, \ \forall i \neq j \\
\{(1,0), (1,1)\} & p_i = p_j - t \\
\{(1,1)\} & p_i < p_j - t
\end{cases}
$$

(2.6)
In other words, when all players are party to a transaction trades are uniquely defined by price except where two sellers “tie” in the offer that they present to a buyer. In this case that buyer becomes indifferent between the two sellers but will not split his purchase as this would cause the buyer to incur the full transport cost \( t \) regardless of the quantity that he receives from the distant seller.

### 2.2 The Undercut-Proof Criteria

The core of a constrained coalitional game, in a spatial market, is remarkably well behaved. Hamilton et. al. (1991) have shown that on the Hotelling line with a continuum of buyers an allocation lies in the core of \((N, V)\) if and only if it can be generated by under-cut-proof prices. The following proposition extends this result to our generalised spatial setting.

**Proposition 1** (Undercut-Proof Criteria). Consider a market in which buyer and sellers payoffs satisfy (2.1) and (2.2). An allocation \( x^* \) lies in the core of \((N, V)\) if and only if \( x^* = x(p^*, q^*) \) where \( q^* \in q^*(p^*, N) \) and \( p^* \) satisfies,

1. \( p^*_j \geq c_j \) for all \( s_j \in S \);

2. And for all \( s_j \) and \( p_j < p^*_j \), if \( p^*_j > c_j \) and there exists \( \bar{p}_j > c_j \) such that \( q^*_j \neq 0 \) for some \( q_j \in q^*(p^* - p_j, \bar{p}_j, N) \), then,

\[
x(p^*, q^*) > (p_j - c_j) \sum_{i \in B} q_{ij},
\]

for all \( q \in q^*(p^* - p_j, p_j, N) \).

The proof of proposition 1 can be found in appendix B, however a brief outline is presented here. The proof consists of two elements: First, it is necessary to show that if a vector of payoffs cannot be implemented by an undercut-proof price vector then it does not lie in the core. This is straightforward as the coalition consisting of the deviating seller, and all buyers willing to purchase her product at the lower price, can replicate the gains that they achieve through a unilateral price drop in a deviating coalition that agrees to the same lower price.

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The second part of the proof is to demonstrate that if a coalition of players can block a vector of payoffs then that vector of payoffs cannot be implemented by an undercut-proof price vector. If a blocking coalition contains a single seller and at least one buyer, the logic is the reverse of that outline in the previous paragraph. Where a deviating coalition contains more than one seller we know from lemma 2 that we can reduce this coalition to a sub-coalition consisting of a single seller and all the buyers who traded with this seller in the original deviating coalition. This sub-coalition must also be able to block the vector of payoffs as every member will be at least as well off as they were in the original deviation.

The undercut-proof criteria readily extend to the case in which a seller occupies multiple locations and is able to set a different price at each location. Where a seller $s_j \in S$ occupies $m$ locations, $s_j$’s price vector can be expressed as $p_j = (p_{j1}, \ldots, p_{jm})$. In this case if we read $p_j < p_j^*$ as $p_{jn} \leq p_{j^*n}$ for all $n \in 1, \ldots, m$, with strict inequality for some $n$, proposition 1 continues to hold.

Lemma 2 plays a key role in the proof of proposition 1 as it allows us to focus only on deviating coalitions containing a single seller, and from there to reduce the definition of the core to a single condition on each seller’s price. However, lemma 2 only applies where each buyer’s payoff is independent of the quantities consumed by the other buyers. The undercut-proof criteria does not generalise to the case in which there are externalities from consumption.

The other critical assumption in the spatial model is the unlimited capacity of the sellers. Unlimited capacity eliminates competition between buyers, giving them a common preference for lower prices. In the capacity constrained model examined in the next section upward pressure on prices may arise where capacity is insufficient to satisfy demand at a given price.

Proposition 1 provides a rigorous technical justification for Shy’s (2001) approach. Moreover, the undercut-proof criteria dramatically simplifies the task of defining the core as it allows the problem to be solved without reference to outcomes that can be generated by the $2^N - 1$ potential deviating coalitions. This ease of use is apparent in the solution to the switching costs
Example 2.3. The core of the switching costs model is defined with respect to 15 distinct feasible sets corresponding to the 15 deviating coalitions that are possible in a four player game. By employing proposition 1 this process can be reduced to one of identifying the market share that can be supported in the core, before deriving the undercut-proof condition on the two prices. By proposition 1 prices that satisfy this condition must implement allocations in the core.

First note that proposition 1 allows us to assume that \( p_i \geq c \) for all \( i \in \{1, 2\} \). It is straightforward to see that each buyer must trade with their closest seller. Were \( s_i \) to trade less than one unit to \( b_i \) then either \( p_i = p_j + t \) or \( s_i \) receives a payoff of zero.\(^{[11]}\) In either case \( s_i \) can strictly increase her payoff by reducing her price to \( p'_i = p_j + t - \varepsilon \) for sufficiently small \( \varepsilon > 0 \).

To complete the characterisation of the core we need only determine the conditions under which prices are undercut-proof given this division of the market. Trading only with her collocated buyer the seller \( s_i \) receives the payoff,

\[
x_{s_i} = p_i - c. \tag{2.8}
\]

The greatest payoff that \( s_i \) can achieve through a deviation occurs when \( s_i \) undercuts \( s_j \)’s price by an amount \( t \) and \( q_i = (1, 1) \). In this case \( s_i \) receives the payoff,

\[
x_{s_i} = 2(p_j - t - c). \tag{2.9}
\]

Undercut-proofness requires the payoff in (2.8) to be strictly greater than the payoff in (2.9). Rearranging we get,

\[
p_j - c \leq \frac{p_i - c}{2} + t, \tag{2.10}
\]

with strict inequality where \( p_i > c \). Solving (2.10) simultaneously for all \( i \in \{1, 2\} \) yields \( \sup p = (2t + c, 2t + c) \). The range of prices that implement core allocations is illustrated in figure 3 as union of the areas \( A \) and \( B \). Note that both the upper and rightward boundaries are not included.

\(^{[11]}\)This follows directly from the result on rational rationing developed in example 2.2.
The area $AB$ lies between the lines $p_2 = p_1 - t$ and $p_2 = p_1 + t$. These lines indicate the prices at which buyers become indifferent between the two sellers. The only core allocations that leave a buyer indifferent between the two sellers correspond to the prices $p = (c + t, c)$ and $p = (c, c + t)$. These allocations still require each buyer to trade exclusively with their local seller, however the prices remain undercut-proof as the low price seller is trading at cost and has no incentive to expand her market share.

The switching costs model provides a good illustration of the way in which constraining the set of feasible payoffs can expand the core. The core of a TU analysis of this game yields each seller a payoff in the interval $[0, t]$ as the marginal contribution of each seller to a TU analysis of this model is the travel cost $t$ that she eliminates for her collocated buyer. The set of prices that are implied by these payoffs are illustrated in figure 3 as the region $A$.

The prices in $B$ correspond to allocations that do not lie in the core of a TU analysis of the switching cost model, but do lie in the core of the constrained coalitional price setting game. Sellers benefit from their inability to selectively target discounts to individual buyers. A seller cannot attract the distant buyer without dropping her price, this in turn reduces the revenue that she receives from her local buyer. Sellers are less willing to discount in the constrained coalitional price setting game and this reluctance is illustrated by the inclusion of the region $B$ in the set of prices that implement
3 Price Competition with Convex Costs

Edgeworth (1925) established that the Bertrand price setting game may fail to possess a pure strategy Bertrand Nash equilibrium where goods are homogeneous and sellers face convex cost functions.\(^\text{12}\) This fact can be clearly seen in Kreps and Scheinkman’s (1983) proof that Cournot competition is equivalent to Bertrand competition with capacity pre-commitment. In Kreps and Scheinkman’s model a pure strategy Bertrand Nash equilibrium fails to exist where sellers commit to capacities that are greater than their respective Cournot best responses.

Price competition with convex costs poses a greater challenge for the constrained coalitional price setting game than was the case with the spatial model developed in the previous section. Convex costs may result in sellers facing an upper bound on the quantity that they are willing to supply at a given price and may result in demand exceeding supply at some prices. This forces us to consider a new factor in our analysis, that of competition between buyers for scarce supply.

Three results are developed in this section. To begin with we show that all sellers setting the market clearing price always implements a core outcome. In an associated example it is shown that both monopoly and monopsony power have the ability to admit inefficient outcomes into the core. Moreover, monopoly and monopsony power are treated symmetrically in this model. Finally, we demonstrate that where demand is coarse the difficulty of neatly matching buyers and sellers in a deviating coalition may admit inefficient outcomes even where there exists competition on both sides of the market. However, we are able to show that if demand is sufficiently fine the core

\(^{12}\text{To be precise Edgeworth (1925) — writing before the concept of a mixed strategy had been developed — argues that the equilibrium may be indeterminate where sellers face convex costs. Instead, Edgeworth conjectured that faced with convex costs sellers would progressively undercut one and other until the profits that can be gained by undercutting rivals are less than those available to a seller by setting the monopoly price against residual demand. This would give rise to a cycle in which each period of undercutting would be followed by a sharp price rise.}\)
collapses to the market clearing outcome. The proof of this final result is both inspired by, and similar to, the core equivalence proof of Debreu and Scarf (1963).

We continue to employ a very general structure in this section. Each seller’s cost function is assumed to conform to the conditions $C'_j(\cdot) \geq 0$ and $C''_j(\cdot) \geq 0$, while buyer payoffs are assumed to take the form,

$$x_i(p, q) = \int_{0}^{\sum_{j \in S} q_{ij}} D^{-1}_i(\vartheta) d\vartheta - \sum_{j \in S} p_j q_{ij}. \quad (3.1)$$

Inverse demand functions retain the structure that was assumed in the previous section.

For the purposes of analysing this model it is useful to aggregate demand and supply functions. For a coalition $Q \subseteq N$ let,

$$D_Q(\rho) = \sum_{i \in B \cap Q} D_i(\rho), \quad (3.2)$$

and let $D_Q^{-1}(\cdot)$ be the inverse of this aggregate demand. Moreover defining,

$$Y_j^{-1}(\vartheta) = \frac{d}{d\vartheta} C_j(\vartheta) \quad (3.3)$$

to be the inverse of $s_j$’s supply function we can write aggregate supply as,

$$Y_Q(\rho) = \sum_{j \in S \cap Q} Y_j(\rho), \quad (3.4)$$

and the inverse of this aggregate supply as $Y_Q^{-1}(\cdot)$.

### 3.1 Market Clearing Prices and the Core

A market clearing price is a price at which neither excess demand nor excess supply exists in a market. In this section we prove that each seller setting a price equal to the market clearing price always implements an allocation in the core of the constrained coalitional price setting game. However, inefficient allocations may also lie within the core where competition is absent on at least one side of the market.
Given the nature of our analysis it is necessary to define a market clearing price for every $Q \subseteq N$. Formally, define $p^c_Q$ implicitly as the set of prices that satisfies,

$$p^c_Q = \{\rho \in \mathbb{R} : D_Q(\rho) = Y(\rho)\}. \quad (3.5)$$

Given the assumption on the form of $D(\cdot)$ and $C(\cdot)$, $p^c_Q$ must be a convex set.

**Proposition 2.** Suppose that buyer payoffs satisfy (3.1), and that seller cost functions satisfy $C'_j(\cdot) \geq 0$ and $C''_j(\cdot) \geq 0$. If $p^*$ satisfies $p^*_i = p^*_j \in p^c_N$ for all $i, j \in S$, and $q^* \in q^r(p^*, N)$, the allocation $x(p^*, q^*)$ lies in the core of $(N, V)$.

The proof to proposition 2 can be found in appendix C. The structure of the proof is straightforward: It is shown that an allocation that can be implemented by each seller setting the market clearing price must lie in the core of the TU analysis of the convex costs model. Given that this allocation is feasible to the grand coalition, by lemma 1 this allocation must also lie in the core of the constrained coalitional price setting game.

A direct consequence of proposition 2 is that there must always exist at least one efficient allocation in the core of the constrained coalitional price setting when it is applied to a market in which the sellers face convex costs. This guarantees both that the core will be non-empty, and that there must exist at least one efficient outcome that is also stable.

However, proposition 2 does not exclude the possibility that we may also find inefficient outcomes in the core. The following example demonstrates that distortions may be created by both monopoly and monopsony power.

**Example 3.1.** Consider a market in which a monopoly seller is trading with a monopsony buyer. Let $N = \{b, s\}$ with,

$$x_b(p, q) = q(1 - p) - \frac{q^2}{2}, \quad (3.6)$$

and,

$$x_s(p, q) = qp - \frac{q^2}{2}. \quad (3.7)$$

These payoffs correspond to the demand function $D(p) = 1 - p$, and the supply function $Y(p) = p$. Given (3.6) and (3.7) rational rationing implies
that \( q^* \) is a singleton taking the value,
\[
q^*(p, N) = \min\{p, (1 - p)\},
\] (3.8)
which is to say the quantity traded is limited by the lesser of monopoly supply and monopsony demand at the prevailing price. The market clearing price is \( p^*_N = 1/2 \).

We claim that an allocation \( x(p^*, q^*) \) lies in the core of \((N, V)\) if and only if \( p^* \in [1/3, 2/3] \). To see this note that individually \( b \) and \( s \) can only guarantee themselves a payoff zero. Given that rational rationing also guarantees each player a non-negative payoff, it follows that an allocation \( x \) can only be blocked by the coalition \( N \). For \( q \in q^*(p, N) \), each player’s payoff is concave in \( p \) with (3.6) achieving a maximum at \( p = 1/3 \) while (3.7) is maximised at \( p = 2/3 \). For \( p < 1/3 \) and \( p > 2/3 \) players have a common interest in moving the price toward the interval \([1/3, 2/3]\), however for \( p \in [1/3, 2/3] \) there is a tension between the preferences of the buyer and seller and they will be unable to construct a mutually beneficial deviation.

This example is significant for two reasons. First, it demonstrates that when transactions are constrained to take place at linear prices, it is possible for inefficient outcomes to accompany the efficient outcome in the core. A player who controls one side of the market has the capacity to exploit this position by pushing the price away from the market clearing level. This captures the player a greater payoff, while at the same time diminishing the total value created in the market. The distortion is identical to that created by a monopoly in the Bertrand price setting game. That the inefficiency can be sustained in the core of the constrained coalitional price setting game illustrates the importance of those allocations that are possible in a TU analysis, but are absent from the feasible set \( V(N) \).

Second, this example demonstrates that market power is treated symmetrically regardless of on which side of the market it arises. The monopoly price is one possible outcome here, while at the same time if the monopsony buyer possesses a greater degree of bargaining power than the seller, we would expect the price to lie below a market clearing level.

\[\blacksquare\]
3.2 Collapsing the Core

The final result for this section concerns the conditions under which the core collapses to only admit allocations corresponding to market clearing prices. We begin with an example that demonstrates the way in which a coarse division of demand amongst buyers can give rise to inefficient core outcomes before proving that when demand is sufficiently fine the core only admits market clearing outcomes.

The core of the constrained coalitional game is defined with respect to outcomes that can be generated when some subset of players are excluded from the game. In the spatial model excluding a group of players was not problematic as buyers were happy to trade exclusively with their preferred seller, and sellers had unlimited capacity. However, in the presence of convex costs a single seller may not be able to satisfy the demand of a group of buyers, and as such the buyers may be unwilling to exclude every other seller from a deviating coalition.

Inefficient core outcomes emerge as possibility where there exists a difficulty in matching buyer demand to seller supply in a deviating coalition. This source of inefficiency is distinct from the monopoly distortion examined in example 3.1, and cannot be attributed to a lack of competition. The problem can be clearly seen in the following example:

**Example 3.2.** Consider a market with two buyers and two sellers. Suppose that buyer 1 faces the demand function \( D_1(p) = 1 - p \), while buyer 2’s demand is given by \( D_2(p) = 2(1 - p) \). Similarly, suppose that seller 1’s supply function is \( Y_1(p) = p \), while for the second seller \( Y_2(p) = 2p \). In this market \( p^N = 1/2 \).

Rational rationing allows for the price vector \( p = (2/3, 1/2) \) to result in \( s_1 \) trading 1/3 of a unit to \( b_1 \), while \( b_2 \) purchases one unit from \( s_2 \). This outcome is clearly not efficient as one of the trades does not take place at the market clearing price, however we claim that it is a core outcome. Proving this can be accomplished by demonstrating that no player can strictly benefit from a deviation.

**Seller 1:** Seller 1 has set the monopoly price relative to buyer 1’s demand. For seller 1 to improve her payoff she must sell some quantity to buyer 2 at
a price that is greater than $1/2$. However, buyer 2 is already satisfying his demand at the price $p_2 = 1/2$ and therefore would never be a party to this deviation.

**Buyer 2:** Buyer 2 cannot benefit unless one or both of the sellers lowers her price below $1/2$, however such a deviation must leave that seller strictly worse off.

**Seller 2:** Selling one unit at $p_2 = 1/2$ yields seller 2 a payoff of $1/4$, exhausting her capacity in the process. It follow that seller 2 cannot increase her payoff without raising her price. Buyer 2 would not be a party to a price increase and as such seller 2 can only trade with buyer 1 in a deviation that increases her payoff. Against buyer 1’s demand, seller 2’s monopoly price is $p_2 = 3/5$. While this price would be acceptable to buyer 2 as it is less that the price $p_1 = 2/3$ that he is currently paying, it would only yield seller 2 a payoff of $1/5$ which is less than her payoff under the original price vector.

**Buyer 1:** Buyer 1 cannot benefit without seller 1 lowering her price — which she would not be willing to do — or being able to purchase some part of seller 2’s output at $p_2 = 1/2$. This second option would reduce the allotment going to buyer 2, which would prevent his participation, and seller 2 will not deviate without buyer 2.

The distortion illustrated in this example is clearly not the result of a lack of competition. Rather, it occurs because buyer 2 and seller 2 are larger than their rivals, and cannot easily be matched with them in a deviating coalition. Many coaltional games experience similar difficulties when faced with asymmetric players and these asymmetries can substantially increase the size of the core. One method for eliminating surplus allocations from the core was developed by Debreu and Scarf (1963) in the development of the core equivalence theorem.

The **core equivalence theorem** compares the core of a TU coaltional analysis of an economy with its Walrasian allocation. It is readily shown that Walrasian allocation lies in the core of this TU game. However, in the presence of asymmetries the core can also contain many non-Walrasian allocations. Following Edgeworth’s (1925) conjecture, Debreu and Scarf replicate the economy $n$ times, showing that the only allocation that lies in the core
of the \( n \)-replica of the TU game, for all \( n \in \mathbb{N} \), is the Walrasian allocation.

The approach adopted in this paper differs from that adopted by Debreu and Scarf (1963) in one important respect: Instead of replicating the entire economy we divide each buyer into \( n \) equal parts, while leaving sellers untouched. The intuition that we wish to develop is that when there exists competition between sellers, as demand becomes fine relative to supply, the core of the constrained coalitional price setting game will converge to the market clearing outcome.

**Definition 3.** The \( n \)-slice of the game \((N, V)\) is defined as the game \((N_n, V)\) in which each buyer has been sliced into \( n \) equal parts. Formally, the grand coalition becomes \( N_n = S \cup B^n \) where \( B^n = \{b_{i1}, \ldots, b_{in}\}_{i \in B} \). The payoff to each buyer continues to satisfy (3.1) with \( D_{ik}(\rho) = \frac{1}{n} D_{i}(\rho) \) for all \( i \in B \) and \( k \in \{1, \ldots, n\} \).

To complete the notation we define \( q_{ik} \) as the quantity that buyer \( b_{ik} \) purchases from seller \( s_j \). Note that \( p_{N}^* = p_{N_n}^* \) for all \( n \in \mathbb{N} \).

In the interests of tractability will assume that buyer demand is twice continuously differentiable and strictly downward sloping, and that the set of buyers is finite. Moreover we assume that every seller is willing to supply a strictly positive quantity at the market clearing price and that removing any single seller would alter the market clearing price. These assumptions are not necessary but aid considerably in restricting the length of an already complex proof. Amongst other things, these additional assumptions guarantee that the market clearing price will be unique, and prevent the inclusion of sellers who have no impact upon the game.

**Proposition 3 (Core Convergence).** Suppose that \( B \) is finite and that buyer payoffs satisfy (3.1) with \( D_i(\cdot) \) twice continuously differentiable and strictly downward sloping for all \( i \in B \). Suppose further that \( |S| \geq 2 \) and that \( C'_j(\cdot) \geq 0, C''_j(\cdot) \geq 0, V_j(p_{N}^*) > 0 \) and \( p_{N}^* \neq p_{N-j}^* \) for all \( j \in S \).

If there exists \( j \in S \) such that \( p_j^* \neq p_N^* \), then there exists at least one \( n \in \mathbb{N} \) such that,

\[ x(p^*, q^*) \notin \text{Core}(N_n, V), \quad (3.9) \]

\footnote{Equivalently \( D_{ik}^{-1}(\frac{q^*}{n}) = D_i^{-1}(\vartheta) \).}
for all \( q^* \in q^*(p^*, N_n) \).\(^{14}\)

The proof of proposition 3 is set out in appendix D but a brief outline is presented here. The proof is divided into three steps: First it is shown that if a vector of prices can implement an allocation in the core of \((N_n, V)\) for all \( n \in \mathbb{N} \), all sellers must set the same price. Second we show that prices that leave excess demand cannot implement core allocations for all \( n \). And finally we show that prices that leave excess supply likewise cannot implement core allocations for all \( n \). As each price vector that violates proposition 3 must violate one of these conditions we have our proof.

Of the three steps the first is the most technically challenging. In order to prove that sellers cannot trade at different prices we take a coalition consisting of the lowest priced seller, a buyer who purchases from the highest priced seller, and other randomly assigned buyers. We require that the total demand of the buyers be at least as large as the seller’s willingness to supply at the low price, and that by subtracting the difference between the first buyer’s demand at the low and high prices we are left with a total demand that is strictly less than the seller’s willingness to supply. By agreeing to the low price this deviating coalition can strictly increase the payoff to the first buyer without leaving any other player worse off.

Proving that there always exists an \( n \)-slice for which such a coalition can be constructed turns out to be equivalent to proving that there always exists natural numbers \( m \) and \( n \) such that,

\[
\frac{1}{n} \varepsilon > \frac{m}{n} - \delta \geq 0,
\]

for arbitrary \( \varepsilon > 0 \) and \( \delta \in (0, 1) \). The problem is complicated by the fact that \( n \) appears in the first two terms of the equation. Consequently, as \( n \) is increased to allow a tighter fit, the size of the interval into which the term \( \frac{m}{n} - \delta \) must be confined, also contracts.

Once we are able to assume that all trades must occur at the same price the remaining two steps are straightforward. Prices cannot leave buyers with

\(^{14}\)In fact for any non-compliant price vector we can find an infinite and strictly increasing sequence of natural numbers for which the price vector cannot implement an allocation in the core of \((N_n, V)\).
excess demand as for a fine enough slice a successful deviation could be constructed by excluding a small group of buyers, redistributing the quantities that they would have received amongst buyers with excess demand. Excess supply can similarly be excluded as a possibility given that when demand is sufficiently fine a seller with excess capacity can gain by undercutting her rivals.

The following example illustrates this three step process. In this example we employ a continuum of buyers, thereby avoiding the matching problems that were illustrated in example 3.2.\textsuperscript{15}

\textit{Example} 3.3. Consider a market with a unit mass of buyers, each of whom demands one unit and has a willingness to pay of one. Suppose that there are two sellers present in this market and that both sellers have a marginal cost of zero and a common capacity of $\bar{q}$. We will consider cases in which $\bar{q}$ belongs to the open interval $(0, 1)$.\textsuperscript{15}

From proposition 2 we know that market clearing prices implement core allocations. In other words their exists $q^* \in q^r(p^*, N)$ such that the allocation $x(p^*, q^*)$ lies in the core if $p^*_1 = p^*_2$ and $p^*_1 \in p^*_N$. Given that the total capacity of sellers in the market is $2\bar{q}$ we have,

$$p^*_N = \begin{cases} \{1\} & \bar{q} \in (0, 1/2) \\ [0, 1] & \bar{q} = 1/2 \\ \{0\} & \bar{q} \in (1/2, 1) \end{cases}. \quad (3.11)$$

It only remains to show that no other allocation lies in the core. Three steps allow us to eliminate all other allocations as possibilities.

\textit{Step 1}: For $x(p^*, q^*)$ to lie in the core we must have $p^*_1 = p^*_2$. Suppose to the contrary that $p^*_1 < p^*_2$, and consider the coalition of seller 1 and a mass $\bar{q}$ of buyers containing a strictly positive mass of buyers who purchase strictly positive quantities from seller 2. By agreeing to the price $p^*_1$ this coalition strictly increases the payoff of the buyers who purchased from seller 2 without reducing the payoff to any other member of the coalition.

\textit{Step 2}: A core allocation cannot leave a buyer with excess demand, implying that the common price charged by sellers cannot drop below 1 where $\bar{q} \in$\textsuperscript{15}You can think of this model as having one buyer sliced into uncountably many parts.
To see this note that there must exist a positive mass of buyers who are purchasing less than one unit each. Consider a coalition consisting of both sellers and a mass $2\bar{q}$ of buyers containing a strictly positive mass of buyers who are purchasing less than one unit. By retaining the original price this coalition can strictly increase the payoff to the buyers who originally received less than one unit without reducing the payoff to any other member of the coalition. This deviation cannot block an allocation when $p_1^* = p_2^* = 1$ as buyers do not benefit from increasing the amount that they purchase when the price is equal to their valuation.

**Step 3:** A core allocation cannot leave a seller with excess capacity, consequently sellers cannot lift their price above zero where $\bar{q} \in (1/2, 1)$. Suppose to the contrary that $p_1^* = p_2^* > 0$ and note that at least one seller must be trading a quantity that is less than their capacity $\bar{q}$. Consider the coalition of one such seller and a mass $\bar{q}$ of buyers. For sufficiently small $\varepsilon > 0$ every member of this coalition can strictly increase their payoff by agreeing to the price $p_1^* - \varepsilon$. ■

This model inspired Edgeworth’s (1925) original criticism of the Bertrand price setting game. Edgeworth showed that where $\bar{q} \in (1/2, 1)$ the model does not possess a pure strategy Bertrand Nash equilibrium. Moreover, the support of the mixed strategy Bertrand Nash equilibrium is bounded away from the market clearing price where $\bar{q} \in (1/2, 1)$.

Contrast this with the constrained coalitional price setting game. Proposition 2 guarantees that we always get a market clearing outcome in the core, while proposition 3 and example 3.3 demonstrate that there is a sense in which these market clearing outcomes can be regarded as more robust than any other allocation that may lie in the core of a particular example. The only indeterminacy in the constrained coalitional price setting game arises where $p_N^*$ is itself an interval.

The presence of market clearing outcomes in the core of the constrained coalitional price setting game gains a particular significance when considering Cournot’s model of quantity competition. In Cournot competition it is assumed that sellers unilaterally select output levels before this output is auctioned off at the market clearing price. Kreps and Scheinkman (1983)
proved that the Cournot Nash equilibrium is equivalent to the Bertrand Nash equilibrium if sellers must unilaterally commit to a capacity in advance of the market. The result does not hold for out of equilibrium outcomes as where the capacities selected by sellers are greater than Cournot best responses a pure strategy Bertrand Nash equilibrium does not exist and expected payoffs from the mixed strategy equilibria are greater than market clearing payoffs.

Propositions 2 and 3 show that the equivalence between Cournot competition, and price competition with capacity pre-commitment, is completed if prices are determined in accordance with the constrained coalitional price setting game.

4 Vertical Contracting with a Linear Price Constraint

The final model considered in this paper is price formation in vertically related markets. Specifically, we examine the case in which the output produced by sellers must be transformed by an intermediary in order to be considered valuable by buyers. For the purposes of this section the products in both the upstream and downstream markets are assumed to be homogeneous, and transformation is assumed to occur at a constant 1:1 ratio.

Two results are developed here. The first, an extension of the market clearing result developed in section 3, shows that it is always a core outcome for both markets to operate at market clearing prices. The second demonstrates that double-marginalisation as generally understood in the literature is never supported as a core outcome. However, where the costs of both seller and intermediary are strictly convex, a vertical merger still has the potential to produce a small social benefit.

Intermediaries are at the same time buyers in the upstream market and sellers in the downstream market. Like sellers, each intermediary is constrained to set a single unit price for his output, and is unable to differentiate between buyers in the downstream market. Let $M$ be the set of intermediaries who are capable of transforming the upstream product, for sale in the downstream market. For each $m_k \in M$ let $C_k(\vartheta)$ denote the cost of trans-
forming the quantity $\vartheta$. We assume that $C_k'(\cdot) \geq 0$ and $C_k''(\cdot) \geq 0$ for all $k \in M$.

In order to completely specify transactions we must extend $p$ and $q$ to capture the transactions in both the upstream and downstream markets. Formally, let $p \in \mathbb{R}^{|M \cup S|}$ denote the unit price charged by each seller and intermediary for their output, and let $q \in \mathbb{R}^{|B \times M \cup M \times S|}_+$ denote the quantity exchanged between any buyer-intermediary and intermediary-seller pair. In order to distinguish sellers and intermediaries we define,

$$T_k^{-1}(\vartheta) = \frac{d}{d\vartheta} C_k(\vartheta)$$

(4.1)

to be the inverse of $m_k$’s supply function. The aggregate supply of transformations is expressed as,

$$T_Q(\rho) = \sum_{k \in M \cap Q} T_j(\rho),$$

(4.2)

and the inverse of this aggregate supply as $T_Q^{-1}(\cdot)$.

Intermediary payoffs take the same basic form as seller payoffs. For each intermediary $m_k \in M$ this payoff can be expressed as,

$$x_k(p, q) = \sum_{i \in B} p_k q_{ik} - C_k \left( \sum_{i \in B} q_{ik} \right) - \sum_{j \in S} p_j q_{kj},$$

(4.3)

such that,

$$\sum_{i \in B} q_{ik} \leq \sum_{j \in S} q_{kj}.$$  

(4.4)

This last inequality states that the total quantity purchased by an intermediary in the upstream market must be at least as large as the quantity that he produces for the downstream market. Rational rationing implies that (4.4) holds with equality.

### 4.1 Efficiency and the Core

In moving to a vertical setting with a fixed transformation ratio, it is convenient to switch our focus from prices to quantities traded. For example it is easiest to define the market clearing outcome in terms of aggregate quantity traded, only then backing out the market clearing prices.
The value created by a coalition $Q$ is maximised when the total quantity traded in each market $q^c_Q$ satisfies,

$$D^{-1}_Q(q^c_Q) = Y^{-1}_Q(q^c_Q) + T^{-1}_Q(q^c_Q). \quad (4.5)$$

We write $p^d_Q = D^{-1}_Q(q^c_Q)$ to denote the market clearing price in the downstream market, and $p^u_Q = p^d_Q - T^{-1}_Q(q^c_Q)$ to denote the upstream market clearing price.

**Proposition 4.** Suppose that buyer payoffs satisfy (3.1), and that intermediary and seller cost functions satisfy $C'_j(\cdot) \geq 0$ and $C''_j(\cdot) \geq 0$. For $p^* \text{ satisfying } p^*_j = p^*_N$ for all $s_j \in S$ and $p^*_k = p^d_N$ for all $m_k \in M$, and $q^* \in q^r(p^*, N)$, the allocation $x(p^*, q^*)$ lies in the core of $(N, V)$.

The proof of proposition 4 can be found in appendix E. The proof is a relatively straightforward extension of the proof of proposition 2 and we will not dwell on it here. Save to say that the logic of the proof can be extended iteratively to $n$ vertically related markets and demonstrates that efficiency is always a stable outcome in the constrained coalitional price setting game.

### 4.2 Double-Marginalisation

One of the most commonly quoted social welfare rationales for vertical mergers, in markets where both the seller and intermediary are monopolists, is that a vertical merger eliminates strong double-marginalisation. **Strong double-marginalisation** occurs in the Bertrand price setting game because both the seller and the intermediary act unilaterally, each imposing a distortion on the market in order to extract additional rents. From an anti-trust point of view, the irony of strong double-marginalisation is that both profit and social welfare would be increased if the seller and intermediary could co-ordinate their prices.

In this subsection we show that whilst the presence of a monopolist at any of the three levels of the market may admit inefficient outcomes to the core of the constrained price setting game, these inefficiencies cannot exceed that which would be created were any one of the monopoly players able to dictate
the price in both markets. This excludes the possibility of strong double-marginalisation arising as a stable outcome of the constrained coalitional price setting game. However, a weaker form of double-marginalisation may still occur where both the seller and intermediary face strictly convex cost functions.

We begin with a simple example that allows us to compare the maximum distortion that can arise in a vertically related market under both Bertrand and constrained coalitional price setting. Specifically, for a simple model we develop the strong double-marginalisation result that arises in Bertrand price competition and show that this outcome can be blocked in the constrained coalitional price setting game.

For the purposes of this subsection we assume that each level of the market is occupied by a monopolist and as such $N = \{b, m, s\}$. The prices in the up and downstream markets are denoted $p^u$ and $p^d$ respectively, and $q$ is the quantity traded in both markets.

**Example 4.1.** Consider a pair of vertically related markets in which both the seller and the intermediary face a constant marginal cost of zero, while the buyer’s demand is given by $D(p^d) = 1 - p^d$.

In the standard Bertrand treatment price setting is sequential with the seller setting a price for her output before the intermediary sets the price for the downstream market. In a sub-game perfect Bertrand Nash equilibrium the intermediary selects $p^d$ to maximise his profit given the upstream price:

$$p^d = \arg\max_{\rho} (\rho - p^u)(1 - \rho) = \frac{1 + p^u}{2}. \quad (4.6)$$

Similarly, the seller selects $p^u$ such that,

$$p^u = \arg\max_{\rho} \rho \left(1 - \frac{1 + \rho}{2}\right) = \frac{1}{2}, \quad (4.7)$$

which in turn implies $p^d = 3/4$ and that the total quantity traded $q = 1/4$. This outcome is illustrated in figure 4 and produces the payoffs $x_s = 1/8$, $x_m = 1/16$ and $x_b = 1/32$.

The strong double-marginalisation outcome is both highly inefficient and fails to maximise total industry profits; the sum of seller and intermediary
payoffs. All three players could be made better off by a co-ordinated price drop resulting in \( \hat{p}^d = 1/2 \) and \( \hat{p}^u = 1/3 \). At \( \hat{p}^d = 1/2 \) the quantity traded is twice the double-marginalisation level and industry profits are maximised. While still inefficient, the prices \( \hat{p}^d = 1/2 \) and \( \hat{p}^u = 1/3 \) deliver a strict Pareto improvement over the double-marginalisation outcome, resulting in the payoffs \( \hat{x}_s = 1/6 \), \( \hat{x}_m = 1/12 \) and \( \hat{x}_b = 1/8 \). This outcome is illustrated in figure 5.

The existence of a Pareto improvement that can be achieved via a co-ordinated price movement does not prevent the double-marginalisation outcome from being an equilibrium of the Bertrand price setting game as a Nash equilibrium is defined with respect to unilateral deviations only. However, the constrained coalitional price setting game does allow players to co-ordinate price movements and consequently the strong double-marginalisation outcome cannot be a core outcome in this example.

Where the seller and intermediary both face a constant marginal cost the upstream market price can be used to arbitrarily assign industry profits between these two players. In the context of the constrained coalitional price setting game this means that it is only the sum of seller and intermediary payoffs that matter as these players are not constrained in how they divide their joint surplus. Given that the buyer always benefits from a drop in the upstream price, it follows that an allocation cannot lie in the core if
the quantity traded is less than the quantity that maximises total industry
profits.

The situation is more complicated if either the seller or intermediary
face convex costs. The existence of convex costs limits the extent to which
value can be transferred between seller and intermediary via the upstream
market price. Seller and intermediary preferences diverge to the extent that
the best feasible outcome for each player may arise at different quantities.
Nevertheless, for relatively well behaved payoff functions we can demonstrate
that the quantity traded in a core allocation cannot be less than the quantity
that either the the seller or intermediary would set were they able to dictate
the price in both markets. This result is stated formally in proposition 5
below.

Before proceeding to the proof it is worth considering what exactly is the
best outcome for the seller and intermediary. If the seller could dictate prices
in both markets she would choose to trade a quantity satisfying,

\[ q_s^* = \arg\max_q \left( p^u_q - \int_q^1 Y^{-1}(\theta)d\theta \right), \]

where \( p^u = D^{-1}(q) - T^{-1}(q) \). Similarly, were the intermediary able to dictate
the price in both markets me would choose to trade the quantity,

\[ q_m^* = \arg\max_q \left( (p^d - p^u)q - \int_q^1 T^{-1}(\theta)d\theta \right), \]

where \( p^d = D^{-1}(q) \) and \( p^u = Y^{-1}(q) \). Given that strong double-marginalisation
arises as a result of the seller being unable to dictate the terms of trade in
the downstream market, these quantities are greater than that which would
result from strong double-marginalisation in the Bertrand price setting game.

If \( D, T \) and \( Y \) are twice continuously differentiable, and (4.8) and (4.9)
are strictly concave, \( q_s^* \) and \( q_m^* \) are defined implicitly by,

\[ D^{-1}(q^*_s) - T^{-1}(q^*_s) - Y^{-1}(q^*_s) = q^*_s \frac{d}{dq^*_s} \left( T^{-1}(q^*_s) - D^{-1}(q^*_s) \right), \]

and

\[ D^{-1}(q^*_m) - T^{-1}(q^*_m) - Y^{-1}(q^*_m) = q^*_m \frac{d}{dq^*_m} \left( Y^{-1}(q^*_m) - D^{-1}(q^*_m) \right). \]
The terms of the RHS of both equations are positive implying that both $q^*_s$ and $q^*_m$ are less than the market clearing quantity.

**Proposition 5.** Suppose that $x_b$ satisfies (3.1) and that $D(\cdot)$ is twice continuously differentiable and strictly downward sloping. Suppose further that the cost functions of the seller and intermediary satisfy $C'_j(\cdot) \geq 0$ and $C''_j(\cdot) \geq 0$, and are likewise twice continuously differentiable. Finally, suppose that (4.8) and (4.9) are strictly concave. For $\hat{p}^d = D^{-1}(\hat{q})$, the allocation $x(\hat{p}, \hat{q})$ is not in Core($N, V$) if $\hat{q} < \min\{q^*_m, q^*_s\}$.

The proof is a straightforward extension of example 4.1. The trick lies in demonstrating that the upstream market price can transfer value between the seller and intermediary over a wide enough range of values that they gain a common interest in increasing the quantity traded when the quantity is less than $\min\{q^*_m, q^*_s\}$.

Proposition 5 establishes that in well behaved cases the quantity traded cannot be less than the quantities that the either the seller or the intermediary would specify if they could dictate the price in both markets, nevertheless it may be the case that vertical integration can deliver efficiency gains. To see this note that if a vertically integrated monopoly were able to dictate a the price for its output it would choose to trade the quantity,

$$q^*_v = \arg\max_\hat{q} \left( p^d \hat{q} - \int_0^\hat{q} \left( Y^{-1}(\hat{\vartheta}) + T^{-1}(\hat{\vartheta}) \right) d\hat{\vartheta} \right),$$

(4.12)

where $p^d = D^{-1}(\hat{q})$. If (4.12) is strictly concave $q^*_v$ must satisfy,

$$D^{-1}(q^*_v) - T^{-1}(q^*_v) - Y^{-1}(q^*_v) = -q^*_v \frac{d}{dq^*_v} D^{-1}(q^*_v).$$

(4.13)

Comparing (4.13) to (4.10) and (4.11) we see that $q^*_v > q^*_s$ if $T^{-1}(q^*_s) > 0$ and $q^*_v > q^*_m$ if $Y^{-1}(q^*_m) > 0$. In words, the quantity that maximises the vertically integrated monopoly’s payoff will be strictly greater, and more efficient, than quantity preferred by either the seller or intermediary, if both $C_s(\cdot)$ and $C_m(\cdot)$ are strictly convex.

Weak double-marginalisation arises because a seller (or intermediary), with the ability to dictate the price in both markets simultaneously, does not
take into account the profit that accrues to the intermediary (seller) when the intermediary’s markup (seller’s price) is set at marginal cost. This profit cannot be transferred between players via the upstream market price and therefore does not enter into the decision of the price setter. Conversely, a merged entity is motivated by total industry profit, resulting in an optimal quantity that is greater than the quantity that a price setting seller (or intermediary) would choose.

5 Discussion

The constrained coalitional price setting game presents a viable alternative to Bertrand price setting. Its predictions are precise, robust and remarkably well behaved in both a generalised spatial setting and in a market for homogeneous goods with convex production costs. The results developed in this paper reinforce the intuition of Shy (2001) that an undercut proof-criteria is an appropriate solution concept to apply to spatial models with finite buyers. While in markets for homogeneous goods the constrained coalitional price setting game restores the primacy of market clearing outcomes, even where sellers face strictly convex costs.

At first glance the constrained coalitional price setting game may seem like a radical departure from Bertrand price setting. However, in reality it is nothing more than another means of selecting a stable outcome (or outcomes) from a set of outcomes that are feasible given the technology of the market. The only distinction lies in the way in which we assume that interaction between the players in a market brings about the vector of prevailing prices. Bertrand price setting requires each seller to make a unilateral and irrevocable commitment to a price, while the constrain coalitional approach view prices as arising from unstructured interaction amongst all market participants.

The importance of the constraints imposed on the feasible set by the price mechanism can be clearly seen in the existence of inefficient outcomes within the core. In many cases, such as the multilateral monopoly examples of sections 3 and 4, inefficient outcomes lie in the core because of the
inability of players to arbitrarily reassign the value that they create. Moreover, as shown in the discussion of double-marginalisation, this inability to arbitrarily reassign value may lead to a weak form of double-marginalisation, although the distortion cannot be as large as that which arises in Bertrand price competition.

In this final section we consider why one might prefer the constrained coalitional price setting game to Bertrand price setting, compare and contrast the results produced by both models, and discuss some implications for antitrust policy.

5.1 Why Prefer a Coalitional Approach?

Thus far we have said little about why one might prefer constrained coalitional price setting over Bertrand. One factor in favour of the constrained coalitional approach is certainly tractability. The results developed in this paper demonstrate that in a number of important settings the constrained coalitional price setting game is considerably easier to work with than Bertrand price setting, and produces precise results that can be readily tested.

Despite belonging to the NTU family of coalitional games — a type of game that is notoriously difficult to solve given the arbitrary complexity of the feasible sets — the constrained coalitional price setting game is remarkably well behaved. The structure provided by the price constraint allows us to use straightforward conditions such as undercut-proofness (proposition 1) and market clearing (proposition 2) to construct the core.

Yet convenience alone cannot justify the use of a particular modelling technique. We must also be mindful of the assumptions that motivate the structure of the model. The Bertrand price setting game does not perform well in this regard.

When noncooperative game theory is first presented to undergraduate students the motivating example is usually the prisoners’ dilemma. The prisoners’ dilemma is a wonderfully simple game that engages the attention of students with its counter intuitive solution. More importantly from our point of view, the circumstances of the motivating example precisely match
the structure of a one shot simultaneous moves game. Each player in the prisoners’ dilemma is locked in a cell, ignorant of the action being taken by the other player. Moreover, the action — to confess or not confess — can only be taken once, and cannot be altered once the player has access to more information.

The same cannot be said of a Bertrand price setting game when the prices being modelled will persist for more than an instant. A seller will often observe the prices set by her rivals before the majority of buyers. Given the ease with which most sellers can change their respective prices, the seller has the opportunity to alter her price before the majority of transactions occur.

The structure of the Bertrand price setting game simply does not match the reality that it is attempting to model. The attraction of Bertrand price setting lies in the ex-post stability of a pure strategy Bertrand Nash equilibrium. Where an equilibrium exists in pure strategies a seller who observes all the prices set by her rivals has no incentive to unilaterally alter her price; all prices are mutual best responses. This is a stability based on complete knowledge rather than mutually correct anticipation. Mixed strategy Bertrand Nash equilibria do not share this property. Once every seller is able to observe the prices that result from her rivals employing their equilibrium strategy mix, at least one seller will have an incentive to unilaterally alter her price.

A one shot noncooperative game is unsatisfactory for modelling prices that will persist for more than an instant when an equilibrium only exists in mixed strategies. For this reason many authors prefer to employ a dynamic structure when addressing problems such as the switching costs model. Yet these models are not without their difficulties.

Whether we are considering a game with repeated simultaneous moves, or repeated sequential moves, dynamic models of price formation rely on the fact that sellers can rapidly change their prices in order to produce discount factors that are sufficiently close to one to support the full range of equilibria. The possibility of rapidly changing prices creates two distinct problems for a dynamic Bertrand price setting game: First, the more rapid are the price

\[\text{See for example Maskin and Tirole (1988).}\]
movements the worse will be the quality of information in the market. In
the limit no buyer will be able to observe more than one price at a time and
each seller becomes a local monopolist relative to the demand of those buyers
observing only her price in each period. The second difficulty arises on the
demand side. Over a short enough time horizon almost all consumption can
either be delayed or stored. This gives buyers an ability to act strategically
in a manner that could have a substantial impact on the payoffs to the var-
ious price choices. Buyers could provide disproportionate rewards to sellers
posting a low price in a particular period, and conversely punish sellers who
set high prices. This contrast with the standard models in which sellers face
a static demand in each period.

Limits on information and the strategic options of buyers could, of course,
be incorporated into the structure of a noncooperative game. However, im-
posing more structure upon a noncooperative game does tend to reduce the
range of real world situation to which the model may be applied. Moreover,
the structure itself may be a factor that can be manipulated by the players
in a market.

As is the case in models of bargaining, the alternative to adding more
structure is to eliminate structure entirely. The set of feasible outcomes in
an unstructured game must be constrained to account for the “technology”
employed in transactions — in this case the price mechanism — but within
these bounds we allow any player to propose any outcome. The core then
represents the set of outcomes that are stable. This stability always persists
ex-post. By definition when the prevailing prices implement a core allocation
no coalition of players can agree a deviation in which they would all (weakly)
benefit.

Ex-post stability allows the constrained coalitional price setting game to
model prices that are expected to persist for some period of time. For so long
as the identity of players and the form of their payoffs do not change, core
outcomes remain ex-post stable and no coalition has an incentive to deviate.

Moreover, the constrained coalitional prices setting game does not disen-
franchise buyers. Buyers take an active role in the game. This allows con-
strained coalitional price setting to be applied to markets with few buyers,
such as intermediate markets, where we would expect to see buyers playing a role in determining the terms of trade.

5.2 Unstructured Interaction

It is worth taking a moment to consider what is meant by an unstructured interaction. When referring to a bargaining problem — possibly the most common and least contentious application of coalitional game theory — players interact by issuing offers, counter offers and threats amongst other things. This interpretation cannot generally be extended to the interaction between buyers and sellers in an end product market.

Nevertheless one can view the prices in a market as being the product of an implicit bargaining process. There are many ways in which buyers and sellers can communicate with one and other. Consider, for example, a period of time in which sellers makes successive offers to the market by posting prices, and buyers indicate their acceptance of these terms through the quantities that they purchase. In this way we can view periods of price instability in a market as a process of negotiation over prices, and the subsequent period of price stability as the outcome of these negotiations. Given the ex-post stability of a core outcome, we would only expect price instability when one of the market fundamentals changes.

Buyers always have the option to use their consumption to send a message to sellers, employing the quantities they purchase to reward or punish a seller. The most extreme example of this behaviour is the consumer boycott. While often employed to bring about a change to the quality of a seller’s product, a consumer boycott could equally be invoked as an extreme sanction in a negotiation over price. This is particularly true where a seller is perceived to have acted egregiously; for example by price gouging.

Buyers have other means of communicating with and punishing sellers. A topical example is consumer reaction to the increasing price of petrol in recent years. Representatives of buyers in this market — motoring groups and politicians — have issued warnings against profiteering and called for anti-trust authorities to monitor retail prices closely. In extreme cases these
calls have resulted in enquiries that are costly to the sellers involved, and
damaging to their public image. Similar pressures have been placed on in-
surance companies following major disasters. In most cases the sellers are
not acting illegally, however vocal representatives clearly state the bargaining
position of those buyers active in the market.

5.3 Interpreting a Deviation

The constrained coalitional approach to price setting is not without its diffi-
culties. Throughout this paper we have glibly assumed that when a deviating
coalition forms it leaves “the market” and operates entirely independently.
But can two groups of buyers and sellers truly operate independently?

Where consumption is associated with externalities or network effects
the answer is clearly no. A decision by a coalition to separate itself from
the remainder of the market need not also sever the links through which the
externalities flow. It follows that in the presence of externalities or network
effects, the viability of a deviation will depend both upon the terms of trade
that can be negotiated within the deviating coalition, and any externalities
that will be produced by the remaining players.

Determining how the actions of the remaining players affect the payoffs
of a deviating coalition is a common problem in coalitional game theory.
Should we assume that the remaining players will act to minimise the payoffs
of deviating players, or will they seek to maximise their own payoffs? To
be consistent with the structure of the constrained coalitional price setting
game we would require the choice between adopting an aggressive or defensive
stance to be made with the choice of price, leaving each player to unilaterally
optimise their subsequent consumption.

It is also worth considering how the members of a deviating coalition
might come together. In an end product market in which the buyers are
consumers we would imagine that sellers would take a lead role in assembling
a deviating coalition. Sellers could use direct mail and loyalty schemes to
target a particular subset of buyers with whom they would like to defect.
Other deviations evolve in a more organic way, sometimes with the aid of a
third party. Schemes like Bartercard\textsuperscript{17} allow member buyers and sellers to trade with one and other on terms that are not available to non-members.

Pulling together a deviating coalition is not without its costs. For this reason it may be more appropriate to use the $\varepsilon$-core to define the set of stable market outcomes. The $\varepsilon$-core only permits an allocation to be blocked if there exists a deviation that benefits each participating player by an amount that is at least equal to $\varepsilon$. In the NTU framework of the constrained coalitional price setting game we could attribute different base $\varepsilon$ to each player acknowledging the asymmetric role that they play in bringing a coalition together. Nevertheless, the core always lies within the $\varepsilon$-core, and can be regarded as containing the set of super-stable outcomes.

5.4 Contrasting Results

In many places the predictions of Bertrand and constrained coalitional price setting are in direct conflict. Two prominent cases can be found in this paper: Kreps and Scheinkman (1983) demonstrated that where sellers have capacities that are greater than Cournot best responses, the support of the mixed strategy Bertrand Nash equilibrium may be bounded away from the market clearing price. In contrast proposition 3 suggest that there is a sense in which the market clearing price is the only viable outcome when competition exists on both sides of the market.

The intuition behind this result is straightforward. Buyers are completely passive in the Bertrand price setting game, unable to exercise any strategic option during the determination of prices. Sellers are able to unilaterally increase their price and take for granted that buyers will follow them. For this reason a seller’s payoff in the mixed strategy Bertrand Nash equilibrium must be at least as great as the payoff that the seller can achieve by setting the monopoly price relative to residual demand. In the Constrained coalitional price setting game all players are involved in the process of price formation. Sellers cannot benefit as a result of a deviation unless they can convince buyers to come with them. As a buyer will not agree to a price rise unless

\textsuperscript{17}See bartercard.com.
it is accompanied by an increase in the quantity allocated to the buyer, a deviation to a monopoly price will not generally be possible for a seller.

The second notable point of conflict concerns the possibility that strong double-marginalisation may arise in vertically related markets. Strong double-marginalisation cannot be a core outcome of the constrained coalitional price setting game because the allocation can be blocked by a co-ordinated price movement that benefits all players. We would expect similar results in parallel markets for complementary goods.

In both examples considered here the core outcomes are more efficient than the corresponding Bertrand Nash equilibria. In the case of convex costs this is because the active participation of buyers acts as a limit on the distortions that sellers can impose on the market. In the case of double-marginalisation increased efficiency results from the possibility of co-operation and co-ordination.

5.5 Implication for Antitrust Regulations

In many jurisdictions simply discussing pricing with a rival is considered to be *per se* anti-competitive. The structure of the Bertrand price setting game has reinforced this view. Regulators reason that Bertrand outcomes tend to be more efficient than monopoly outcomes. Efficiency is therefore served if sellers are prevented from colluding to set these monopoly outcomes.

The results generated by the constrained coalitional price setting game tend to contradict this view. The unstructured interaction that produces prices in this game permits unlimited communication between sellers and specifically allow seller to co-ordinate their prices. Collusion is prevented because buyers are also active players in this interaction.

But we can go further still. The following example demonstrates that even if sellers are permitted to make side payments, sellers will still not be able to implement a collusive outcome in the constrained coalitional price setting game.

*Example* 5.1. Consider a market for a homogeneous good with two buyers $b_1$ and $b_2$, and two sellers $s_1$ and $s_2$. Suppose that the sellers have unlimited
capacity at a constant marginal cost of zero, and that each buyer faces the demand function \( D_i(\rho) = (1 - \rho)/2 \).

This example satisfies the assumptions of the generalised spatial model developed in section 2 and therefore we may employ undercut-proofness to find the solution to a baseline case in which sellers cannot make side payments. Specifically, the core contains a single allocation that delivers each buyer a payoff of 1/4 while both sellers receive a payoff of zero.\(^{18}\)

Now suppose that sellers can make side payments to one and other. The form of each buyer’s payoff remains,

\[
x_{b_1} = x_{b_2} = (p, q) = \frac{1}{4}(1 - \min\{p_1, p_2\})^2,
\]

while the payoffs to the two sellers must satisfy,

\[
x_{s_1} + x_{s_2} = \min\{p_1, p_2\}(1 - \min\{p_1, p_2\}),
\]

which is too say that sellers are constrained only insofar as the sum of seller payoffs must equal the total surplus that accrues to that side of the market. Importantly, sellers are now free to pay one and other in return for setting a high price.

Despite this additional freedom sellers remain unable extract rents from this market. To see this suppose that \( \min\{p_1, p_2\} > 0 \) and without loss of generality suppose that \( x_{s_1} \leq x_{s_2} \). By (5.2) the payoff to \( s_1 \) must satisfy,

\[
x_{s_1} \leq \frac{1}{2} \min\{p_1, p_2\}(1 - \min\{p_1, p_2\}).
\]

We wish to show that this outcome cannot arise in the core. Consider a deviation of the coalition \( Q = \{b_1, b_2, s_1\} \) agreeing to the price \( p_1 = \frac{1}{2} \min\{p_1, p_2\} \). This deviation strictly increases the payoff to both the buyers and seller 1.

While side payments do provide sellers with a common incentive to maximise industry profits, they fail to facilitate collusion because the sellers remain in conflict over how to divide the surplus that accrues to their side of the market. This result readily generalises and demonstrates that even if

\(^{18}\)These payoffs can only be implemented by a price vector with at least one zero entry.
the sellers are allowed to pay one and other to participate in price fixing, the market clearing outcome will always remain in the core. Indeed in many cases, such as the one outlined above, the market clearing outcome remains the only outcome in the core.

So how then can collusion be facilitated? One obvious answer is through a market sharing agreement. If the sellers can merge in advance of the market each seller gets a fixed share of industry income, thereby removing the contest for this income from the price sharing game.\(^{19}\) Alternatively, and again in advance of the market, the sellers could agree to divide the market. In the example above seller 1 could commit not to trade with buyer 2 and vice versa. This would leave each seller as a local monopolist. In either case the collusive agreement would only hold if the share determined in the preliminary stage is binding.

Far from facilitating collusion, communication in the constrained coalitional price setting game tends to generate efficient outcomes and certainly performs better in this regard than does the Bertrand price setting game. If it is truly the case that buyers are completely passive participants in a market — as is assumed in the Bertrand model — the results of this paper suggest that a considerable social gain could be effected by encouraging buyers to take a more active role in the determination of prices. However, if price formation has more in common with the unstructured interaction assumed by the constrained coalitional price setting game then we must reconsider the nature of collusion and the practices that facilitate it.

### 5.6 Concluding Remarks

That a coalitional approach should find a pure strategy solution where the Bertrand price setting game failed, was anticipated by Edgeworth (1925). Edgeworth describes an equilibrium market outcome as being,

\[
\text{“...defined by the condition that no individual in any group,}
\]

\(^{19}\)For \(\alpha_1 \in (0, 1)\) and \(\alpha_2 = 1 - \alpha_1\) a merger or joint venture would cause seller payoffs to become,

\[
x_{s_j} = \alpha_j \min\{p_1, p_2\} (1 - \min\{p_1, p_2\}),
\]

where \(\alpha_j\) is seller \(j\)’s share of the joint venture.

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whether of buyers or sellers, can make a new contract with individuals in the other groups, such that all the re-contracting parties should be better off than they were under the preceding system of contracts.”

This is one of the earliest references to a core like concept of stability.

Possibly the greatest contribution of this paper is that it formalises the model developed by Hamilton et. al. (1991) in such a way that it can readily be applied to a wide range of market transactions. Armed with this formalism — and with the addition of the rational rationing rules also developed in this paper — application of the constrained coalitional price setting game is relatively straightforward.

The ease with which the results of this paper are developed is due in large part to the assumption that products are perfect substitutes (up to the limit of a spatial characteristic), and that each buyer’s payoff is independent of the consumption of other buyers. That being said, the constrained coalitional price setting game does produce precise results in cases where the Bertrand price setting game is at best ambiguous in its predictions.

If the constrained coalitional price setting game is to emerge as a viable alternative to Bertrand price setting this ease of use must extend to more complex settings. The game is yet to be tested where the products being produced by different sellers are compliments or partial substitutes. Various forms of externalities, including network externalities, likewise warrant investigation.

A Proof of Lemma 2

The integrals on either side of the equality are identical by construction. If \( q_i = (0, \ldots, 0, q_{im}, 0, \ldots, 0) \), (2.5) holds trivially. We are left with the case in which \( q_i \) has at least two strictly positive entries.

Let \( q_{ij} \) and \( q_{ik} \) be any two strictly positive entries in \( q_i \), it must be the case that \( p_j + k_{ij} = p_k + k_{ik} \). Suppose to the contrary that \( p_j + k_{ij} > p_k + k_{ik} \). But \( b_i \) could increase \( x_i \) by reducing \( q_{ij} \) to zero and increasing \( q_{ik} \) to \( q_{ij} + q_{ik} \),
thereby lowering the effective unit price paid by $b_i$ and contradicting the assumption that $q_i$ maximises $x_i$.

Again letting $q_{ij}$ and $q_{ik}$ be any two strictly positive entries in $q_i$, it must also be the case that $K_{ij} = K_{ik} = 0$. Suppose to the contrary that $K_{ij} > 0$, and without loss of generality that $K_{ij} \geq K_{ik}$. But $b_i$ could increase $x_i$ by reducing $q_{ij}$ to zero and increasing $q_{ik}$ to $q_{ij} + q_{ik}$, thereby foregoing the need for $b_i$ to incur the fixed cost $K_{ij}$ and contradicting the assumption that $q_i$ maximises $x_i$.

It follows that if $q_i$ has more than one strictly positive entry, for any $s_m \in S \cap Q$ such that $q_{im} > 0$, (2.5) can be satisfied by setting $\hat{q}_{im} = \sum_{j \in S} q_{ij}$. Moreover, the transactions $\hat{q}_i = (0, \ldots, 0, \hat{q}_{im}, 0, \ldots, 0)$ maximise $x_i$ given $p$ and $Q$, and therefore there exists a $\hat{q} \in q^*(p^*, q^*)$ with $\hat{q}_i$ as its $i$th component.

## B Proof of Proposition 1

$i.$ follows directly from the individual rationality of the sellers.

For $ii.$ the only if part is straightforward. Suppose to the contrary that there exists $\hat{p}_j < p_j^*$ such that,

$$x_j(p^*, q^*) \leq (\hat{p}_j - c_j) \sum_{i \in B} q_{ij}. \quad \text{(B.1)}$$

for some $q \in q^*((p^*_{-j}, \hat{p}_j), N)$. Define $B_j(q) = \{b_i \in B : q_{ij} > 0\}$. It follows from lemma 2 that the coalition $s_j \cup B_j(q)$ could deviate, agreeing to the price $\hat{p}_j$, without leaving any player in the coalition worse off. To complete the contradiction we need only show that at least one member of the coalition will be strictly better off as a result of the deviation.

If there exists $b_i \in B_j(q)$ such that $q_{ij}^* > 0$, $b_i$ strictly benefits as a result of paying a lower price. We are left with the case in which $q_{ij}^* = 0$ for all $b_i \in B_j(q)$ and as a consequence $x_j(p^*, q^*) = 0$. By assumption there exists a $c_j < \hat{p}_j < p_j^*$ and $q \in q^*((p^*_{-j}, \hat{p}_j), N)$ such that $q_{ij} > 0$, thus the allocation $x((p^*_{-j}, \hat{p}_j), q)$ with $x_j > 0$ is feasible in $(N, V)$.

Now suppose that a coalition $Q \subseteq N$ agreeing to a price vector $\hat{p}$ and transactions $\hat{q} \in q^*(\hat{p}, Q)$, can block the allocation $x(p^*, q^*)$. To prove the
if part of ii. we need only show that there exists a seller $s_j \in Q$, and $\tilde{q} \in q^r((p^*_j, \hat{p}_j), N)$ such that the coalition $s_j \cup B_j(\tilde{q})$ agreeing to the pair $(\hat{p}_j, \tilde{q})$, can likewise block $x(p^*, q^*)$.

First, and trivially, we note that $Q$ cannot consist entirely of either buyers or sellers. Rational rationing ensures that both buyers an sellers receive non-negative payoffs. A coalition consisting entirely of either buyers or sellers will generate a payoff of zero for each of its members and as such no player can strictly benefit.

Next we show that if there exists $s_j \in Q$ such that $\hat{p}_j > p^*_j$, it must be the case that $s_j$ makes no sales and receives a payoff of zero as a result of the deviation. Suppose to the contrary that a buyer $b_i \in Q$ purchases $q_{ij}$ units from $s_j$ as a result of the deviation. But $b_i$ can purchase from $s_j$ when the prevailing price vector is $p^*$ implying that the deviation must leave $b_i$ strictly worse off. It follows that $s_j$ makes no sales and receives no benefit from the deviation. Moreover, it must be the case that the coalition $Q \setminus s_j$, agreeing to the price vector $\hat{p}$, is also capable of blocking the allocation.

Finally we show that if the coalition $Q$ can block an allocation $x(p^*, q^*)$, there exists a seller $s_j \in Q$ such that the coalition $s_j \cup B_j(\hat{q})$, agreeing to the price $\hat{p}_j$, can likewise block the allocation. We can confine our attention to deviating coalitions in which $\hat{p}_j \leq p^*_j$ for all $s_j \in Q$.

Given that $\hat{p}_j \leq p^*_j$ for all $s_j \in Q$, and $x_i(\hat{p}, \hat{q}) \geq x_i(p^*, q^*)$ for all $b_i \in Q$, it must be the case that there exists a $\tilde{q} \in q^r((p^*_j, \hat{p}_j), N)$ such that $B_j(\tilde{q}) \subseteq B_j(\hat{q})$. In other words, if a buyer in $Q$ is willing to buy from $s_j$ where the prevailing prices are $\hat{p}$, he must be willing to buy from $s_j$ where $s_j$ retains the lower price $\hat{p}_j$ but every other seller sets their respective price from the vector $p^*$. Moreover the trades $\tilde{q}$ grant $s_j$ a weakly greater volume of sales than $\hat{q}$, thereby weakly increasing $s_j$’s payoff.

One player in $Q$ must strictly benefit from the deviation. If that player is the seller $s_j$, it follows from lemma 2 that the coalition $s_j \cup B_j(\tilde{q})$, agreeing to the pair $(\hat{p}_j, \tilde{q})$, can block the allocation $x(p^*, q^*)$. If the player that strictly benefits is the buyer $b_i$, pick a seller $s_j \in Q$ such that $\hat{q}_{ij} > 0$, and once again by lemma 2 the coalition $s_j \cup B_j(\tilde{q})$, agreeing to the pair $(\hat{p}_j, \tilde{q})$, can block the allocation $x(p^*, q^*)$. 53
C Proof of Proposition 2

From the aggregate measures of demand and cost it is possible to construct a function \( W : 2^N \to \mathbb{R} \), such that \( W(Q) = 0 \) for any coalition that does not contain at least one buyer and one seller, and,

\[
W(Q) = \int_0^{D_Q(p_Q^c)} \left( D_Q^{-1}(\vartheta) - Y_Q^{-1}(\vartheta) \right) d\vartheta, \quad (C.1)
\]

otherwise. Notice that \( W(Q) \) is the greatest total value that can be generated by trades between the members of the coalition \( Q \). It follows that \((N, V)\) and \( W(\cdot) \) satisfy the relationship set out in (1.6).

It follows from lemma 1 that in order to prove proposition 2 it is only necessary to show that the allocation \( x(p^*, q^*) \) lies in the core of \((N, W)\). We proceed via a series of lemmas.

**Definition 4.** Let \( x \) be an allocation in the game \((N, W)\). The effective unit price \( \bar{p}_i \), paid by a buyer \( b_i \in Q \) is defined as,

\[
\bar{p}_i = \frac{\int_0^{D_i(p_Q^c)} D_i^{-1}(\vartheta) d\vartheta - x_i}{D_i(p_Q^c)}. \quad (C.2)
\]

Similarly, the effective unit price \( \bar{p}_j \), received by a seller \( s_j \in Q \) is defined as,

\[
\bar{p}_j = \frac{x_j + \int_0^{Y_j(p_Q^c)} Y_j^{-1}(\vartheta) d\vartheta}{Y_j(p_Q^c)}. \quad (C.3)
\]

For a coalition \( Q \), \( \bar{p}_i \) is uniquely defined by \( x_i \) and vice versa. Moreover given that,

\[
\sum_{i \in B \cap Q} D_i(p_Q^c) = D_Q(p_Q^c) = Y_Q(p_Q^c) = \sum_{j \in S \cap Q} Y_j(p_Q^c) \quad (C.4)
\]

we have,

\[
\frac{\sum_{i \in B \cap Q} D_i(p_Q^c) \bar{p}_i}{D_Q(p_Q^c)} = \frac{\sum_{j \in S \cap Q} Y_j(p_Q^c) \bar{p}_j}{Y_Q(p_Q^c)}. \quad (C.5)
\]

In words the weighted average of effective unit prices paid by buyers is equal to the weighted average of effective unit prices received by sellers.
Lemma 3. Suppose that buyer payoffs satisfy (3.1). For all \( b_i \in B \) and \( q \in \mathbb{R}_+ \),
\[
\int_0^{D_i(p_N^c)} D_i^{-1}(q) d\theta - D_i(p_N^c)p_N^c < \int_0^q D_i^{-1}(q) d\theta - q\bar{p}_i, \tag{C.6}
\]
implies \( \bar{p}_i < p_N^c \).

Proof. The expression on the RHS of (C.6) is (weakly) concave in \( q \). When \( \bar{p}_i = p_N^c \) the RHS is maximised where \( q = D_i(p_N^c) \). It follows that in order to satisfy the inequality we must have \( \bar{p}_i < p_N^c \). \hfill \Box

Lemma 4. Suppose that seller cost functions satisfy \( C_j'(\cdot) \geq 0 \) and \( C_j''(\cdot) \geq 0 \). For all \( s_j \in S \) and \( q \in \mathbb{R}_+ \),
\[
Y_j(p_N^c)p_N^c - \int_0^{Y_j(p_N^c)} Y_j^{-1}(q) d\theta < q\bar{p}_j - \int_0^q Y_j^{-1}(q) d\theta, \tag{C.7}
\]
implies \( \bar{p}_j > p_N^c \).

Proof. Follows directly from the proof of lemma 3. \hfill \Box

Lemma 5. Suppose that buyer payoffs satisfy (3.1), and that seller cost functions satisfy \( C_j'(\cdot) \geq 0 \) and \( C_j''(\cdot) \geq 0 \). The allocation \( x^* \) is in \( \text{Core}(N, W) \) if the corresponding effective unit price \( \bar{p}_i^* = p_N^c \) for all \( i \in N \).

Proof. Buy way of contradiction suppose that there exists a coalition \( Q \) such that,
\[
\sum_{i \in Q} x_i^* < W(Q). \tag{C.8}
\]
Let \( x \) be an allocation satisfying \( \sum_{i \in Q} x_i = W(Q) \) and \( x_i > x_i^* \) for all \( i \in Q \). Moreover define \( \bar{p}_i \) as the effective unit price corresponding to the payoff \( x_i \).

From lemma 3 we know that \( \bar{p}_i < p_N^c \) for all \( i \in B \cap Q \), while from lemma 4 we know that \( \bar{p}_j > p_N^c \) for all \( i \in S \cap Q \). This implies,
\[
\frac{\sum_{i \in B \cap Q} D_i(p_Q^c)\bar{p}_i}{D_Q(p_Q^c)} < p_N^c < \frac{\sum_{j \in S \cap Q} Y_j(p_Q^c)\bar{p}_j}{Y_j(p_Q^c)}, \tag{C.9}
\]
contradicting (C.5). \hfill \Box
D Proof of Proposition 3

The proof proceeds via a series of lemmas. Lemma 6 presents an existence result that is employed by lemma 8 to prove that if $p^*$ induces an allocation that lies in the core of $(N_n, V)$ for all $n \in \mathbb{N}$ then all trading sellers must set the same price. Lemma 9 illustrates that if $p^*$ induces an allocation that lies in the core of $(N_n, V)$ for all $n \in \mathbb{N}$ then $p^*$ cannot leave any buyer with excess demand, similarly lemma 10 demonstrates that $p^*$ cannot leave any seller with excess capacity.

**Lemma 6.** For all $\varepsilon > 0$ and $\delta \in (0, 1)$ we can select $m, n \in \mathbb{N}$ with $n \geq n \in \mathbb{N}$ such that $\frac{1}{n} \varepsilon > \frac{m}{n} - \delta \geq 0$.

**Proof.** If $\delta$ is rational then set $\frac{m}{n} = \delta$ and we are done. We are left with the case in which $\delta$ is not rational. For all $n \in \mathbb{N}$ we can define an $m_0$ such that,

$$m_0 \geq n \geq m_0 - \delta > 0.$$ (D.1)

Now define $\alpha_i$ such that,

$$\alpha_i = \min \left\{ \beta \in \mathbb{N} : \frac{\beta m_{i-1} - 1}{\beta \prod_{\phi=1}^{i-1} \alpha_{\phi} n} > \delta \right\},$$ (D.2)

for all $i \in \mathbb{N}$, where $m_i = \alpha_i m_{i-1} - 1$.\footnote{We follow the convention that $\prod_{\phi=1}^{0} \alpha_{\phi} = 1$.} From the construction of $\alpha_i$ we can see that,

$$\frac{\alpha_i m_{i-1} - 1}{\prod_{\phi=1}^{i} \alpha_{\phi} n} > \delta > \frac{(\alpha_i - 1)m_{i-1} - 1}{\prod_{\phi=1}^{i-1} \alpha_{\phi}(\alpha_i - 1)n},$$ (D.3)

which in turn implies,

$$\frac{m_i}{\prod_{\phi=1}^{i} \alpha_{\phi} n} - \delta < \frac{1}{\prod_{\phi=1}^{i} \alpha_{\phi}(\alpha_i - 1)n}.$$ (D.4)

To complete the proof we need only show that there exists an $i \in \mathbb{N}$ such that if we set $n = \prod_{\phi=1}^{i} \alpha_{\phi} n$ and $m = m_i$ then,

$$\frac{m}{n} - \delta < \frac{1}{n(\alpha_i - 1)} < \frac{\varepsilon}{n}.$$ (D.5)
Combining (D.2) and (D.4) we see that for all \( i \in \mathbb{N} \), it must be the case that \( \alpha_{i+1} \geq \alpha_i \). It is sufficient to show that for all \( i \) there exist an \( i' > i \) such that \( \alpha_{i'} > \alpha_i \) and as a consequence for all \( \frac{1}{\varepsilon} \in \mathbb{R}_+ \) there exists an \( i \in \mathbb{N} \) such that \( \alpha_i - 1 > \frac{1}{\varepsilon} \).

Suppose to the contrary that there exists an \( i^* \) such that \( \alpha_i = \alpha_{i^*} \) for all \( i > i^* \). The LHS and RHS of (D.3) can now be written as,

\[
\frac{\alpha_i m_{i-1} - 1}{\prod_{\phi=1}^i \alpha_{\phi n}} = \frac{1}{n} \left( m_0 - \sum_{\phi=1}^{i-1} \frac{1}{\prod_{\theta=1}^\phi \alpha_\theta} - \frac{1}{\prod_{\theta=1}^{i-1} \alpha_\theta} \sum_{\phi=i^*}^i \frac{1}{\alpha_{\phi}^{i^*-1} + 1} \right), \tag{D.6}
\]

and

\[
\frac{(\alpha_i - 1)m_{i-1} - 1}{\prod_{\phi=1}^i \alpha_{\phi}(\alpha_i - 1)n} = \frac{1}{n} \left[ m_0 - \sum_{\phi=1}^{i-1} \frac{1}{\prod_{\theta=1}^\phi \alpha_\theta} \right. \\
- \sum_{\phi=i^*}^{i-1} \frac{1}{\alpha_\theta} \left( \sum_{\phi=i^*}^{i-1} \frac{1}{\alpha_{\phi}^{i^*-1} + 1} + \frac{1}{\alpha_{i^*}^{i^*-1} + 1} \right), \tag{D.7}
\]

respectively. The limit of both (D.6) and (D.7) is,

\[
\lim_{i \to \infty} \frac{\alpha_i m_{i-1} - 1}{\prod_{\phi=1}^i \alpha_{\phi n}} = \lim_{i \to \infty} \frac{(\alpha_i - 1)m_{i-1} - 1}{\prod_{\phi=1}^i \alpha_{\phi}(\alpha_i - 1)n} = \frac{1}{n} \left( m_0 - \sum_{\phi=1}^{i-1} \frac{1}{\prod_{\theta=1}^\phi \alpha_\theta} - \frac{1}{\prod_{\theta=1}^{i-1} \alpha_\theta} \right). \tag{D.8}
\]

The term on the RHS of (D.8) must be equal to \( \delta \) given that it is the limit of \( \delta \)'s upper and lower bounds, and is unambiguously a rational number: A contradiction.

**Lemma 7.** Suppose that \( B \) is finite and that buyer payoffs satisfy (3.1) with \( D_i(\cdot) \) twice continuously differentiable and strictly downward sloping for all \( i \in B \). Suppose further that \( |S| \geq 2 \) and that \( C'_j(\cdot) \geq 0 \), \( C''_j(\cdot) \geq 0 \), \( Y_j(p_N) > 0 \) and \( p_N \neq p_{N-j} \) for all \( j \in S \).

Relabelling if necessary let \( j < k \) imply \( p_j^* \leq p_k^* \), and assume that \( Y_j(p_j^*) > 0 \) for all \( j \in S \).\(^{21}\) Suppose that for all \( n \in \mathbb{N} \) there exists a \( q^* \in q^*(p^*, N_n) \)

\(^{21}\)This assumption is made without loss of generality as given the assumptions placed on the form of each player's demand function a seller can guarantee herself a payoff of zero by setting an arbitrarily high price.
such that \( x(p^*, q^*) \) lies in the core of \((N_n, V)\). It must be the case that for all \( j \in S \) we can find an \( n \in \mathbb{N} \) such that for all \( n \geq n \),

\[
\sum_{k \in B} q^*_{ikj} > 0. \tag{D.9}
\]

**Proof.** Suppose to the contrary that there exists \( \hat{j} \in S \) such that for all \( n \in \mathbb{N} \) there exists an \( n \geq n \) for which (D.9) holds with equality. Set \( n \) such that \( D_i(p\hat{c}_N)/n < Y_{\hat{j}}(p\hat{c}_N) \) and select \( n \geq n \) and \((p^*, q^*)\) such that \( x(p^*, q^*) \in \text{Core}(N, V) \) and \( \sum_{ik \in B^n} q^*_{ikj} = 0 \). Note that \( x_{\hat{j}}(p^*, q^*) = 0 \). There are two cases to consider: \( p^*_1 > p\hat{c}_N \) and \( p^*_1 \leq \hat{p} \).

**Case 1:** \( p^*_1 > p\hat{c}_N \). Select an \( i \in B \) such that \( 0 < D_i(p\hat{c}_N) \). The coalition \{\( b_{11}, s_{\hat{j}} \}\} can agree to trade the quantity \( \frac{1}{n} D_i(p\hat{c}_N) \) at price \( p\hat{c}_N \), thereby leaving \( s_{\hat{j}} \) no worse off and strictly improving the payoff of \( b_{11} \): A contradiction.

**Case 2:** \( p^*_1 \leq \hat{p} \). By assumption we can find a buyer, whom we will designate \( b_{11} \) relabelling if necessary, who either experiences excess demand or purchases some quantity at a price that is greater than market clearing. For this buyer,

\[
\sum_{j \in S} q_{11j} < D_{11}(p\hat{c}_N). \tag{D.10}
\]

Define \( S' = \{ j \in S : p^*_j \leq \hat{p} \} \) and for all \( b_{ik} \in B^n \) let,

\[
\tilde{q}_{ik} = \max \left\{ 0, \sum_{j \in S'} q^*_{ikj} - \frac{1}{n} D_i(p\hat{c}_N) \right\}. \tag{D.11}
\]

The scalar \( \tilde{q}_{ik} \) represents that portion of a buyer’s demand that is not satisfied by sellers setting a price no greater than market clearing. Finally, find a set of buyers \( B' \in B^n \) such that \( b_{11} \in B' \) and,

\[
\sum_{j \in S'} Y_j(p^*_j) < \sum_{ik \in B'} \left( \tilde{q}_{ik} + \sum_{j \in S'} q^*_{ikj} \right) \leq Y_j(p\hat{c}_N) + \sum_{j \in S'} Y_j(p^*_j). \tag{D.12}
\]

The choice of \( n \) ensures that at least one such \( B' \) must exist.

Consider a deviation by the coalition \( Q = B' \cup S' \cup \{ \hat{j} \} \), agreeing to the prices \( \hat{p}_k = p\hat{c}_N \) for all \( k \in S' \) and \( \hat{p}_j = \hat{p} \), and the trades,

\[
\tilde{q}_{ikj} = q^*_{ikj} + \frac{\tilde{q}_{ik}}{\sum_{\phi \in B'} \tilde{q}_{i\phi}} \left( Y_j(p^*_j) - \sum_{\phi \in B'} q^*_{i\phi} \right), \tag{D.13}
\]

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for all $ik \in B'$ and $j \in S'$, and,
\[ \hat{q}_{ikj} = \max\left\{ 0, \frac{1}{n} D_i(p_N^c) - \sum_{j \in S'} \hat{q}_{ij} \right\}, \tag{D.14} \]
for all $ik \in B'$. This deviation leaves no player worse off and strictly increases the payoff to $b_{11}$: A contradiction.

Lemma 8. Suppose that $B$ is finite and that buyer payoffs satisfy (3.1) with $D_i(\cdot)$ twice continuously differentiable and strictly downward sloping for all $i \in B$. Suppose further that $|S| \geq 2$ and that $C_j'(\cdot) \geq 0$, $C_j''(\cdot) \geq 0$, $Y_j(p_N^c) > 0$ and $p_N^c \neq p_{N-j}^c$ for all $j \in S$.

Relabelling if necessary let $j < k$ imply $p_j^* \leq p_k^*$. If $p_1^* \neq p_{|S|}^*$, then there exists at least one $n \in \mathbb{N}$ such that,
\[ x(p^*, q^*) \notin \text{Core}(N_n, V), \tag{D.15} \]
for all $q^* \in q^*(p^*, N_n)$.

Proof. Suppose to the contrary that $p_1^* \neq p_{|S|}^*$ and that for all $n \in \mathbb{N}$ there exists $q^* \in q^*(p^*, N_n)$ such that $x(p^*, q^*)$ lies in the core of $(N_n, V)$. By lemma 7 if $x(p^*, q^*)$ lies in the core of $(N_n, V)$ for all $n \geq n_0$ we must have,
\[ \sum_{i \in B \setminus \{1, \ldots, n\}} \hat{q}_{ik|S|} > 0. \tag{D.16} \]

Given that the seller $s_{|S|}$ trades a positive quantity, rational rationing implies that there must exist at least one buyer, whom we will designate $b_{11}$, relabelling if necessary, for whom $\sum_{j \in S} q_{11j} \leq D_{11}(p_{|S|}^*) < D_{11}(p_1^*)$. Moreover, given that $p^*_1 > p_1^*$, rational rationing implies that $D_N(p_1^*) > Y_1(p_1^*)$ as no buyer would purchase the product at the higher price if excess capacity remained at the lower price.

From lemma 6 we know that we can select $m, n \in \mathbb{N}$, with $n \geq n$ such that,
\[ \frac{m}{n} D_N(p_1^*) \geq Y_1(p_1^*) > \frac{m}{n} D_N(p_1^*) - \frac{1}{n} \left( D_1(p_1^*) - D_1(p_{|S|}^*) \right), \tag{D.17} \]
and note that \( m \leq n \). Suppose that the coalition \( Q = s_1 \cup \{b_{i1}, \ldots, b_{im}\}_{i \in B} \) deviated, agreeing to the price \( p_1 = p_1^* \) and quantities \( q_{ik} = D_{ik}(p_1^*) \) for all \( b_{ik} \in Q \setminus b_{11} \) and,

\[
q_{11} = \frac{1}{n} D_1(p_1^*) + Y_1(p_1^*) - \frac{m}{n} D_{N_n}(p_1^*) > \frac{1}{n} D_1(p_{|S|}^*). \tag{D.18}
\]

This deviation satisfies the demand of every member of \( \{b_{i1}, \ldots, b_{im}\}_{i \in B} \setminus b_{11} \) at the lowest available price while strictly increasing the quantity assigned to \( b_{11} \). Moreover the capacity of \( s_1 \) is exhausted at her original price. It follows that every member of \( Q \setminus b_{11} \) is at least as well off and \( b_{11} \)'s payoff strictly increases: A contradiction. \( \square \)

**Lemma 9.** Suppose that \( B \) is finite and that buyer payoffs satisfy (3.1) with \( D_i(\cdot) \) twice continuously differentiable and strictly downward sloping for all \( i \in B \). Suppose further that \( |S| \geq 2 \) and that \( C_j^1(\cdot) \geq 0 \), \( C_j^2(\cdot) \geq 0 \), \( Y_j(p_N^*) > 0 \) and \( p_N^* \neq p_{N-j}^* \) for all \( j \in S \).

Let \( p_j^* = \hat{p}^* \) for all \( j \in S \). If \( Y_N(\hat{p}^*) < D_N(\hat{p}^*) \) then there exists \( n \in \mathbb{N} \) such that for all \( q^* \in q^*(p^*, N_n) \), \( x(p^*, q^*) \notin \text{Core}(N_n, V) \).

**Proof.** The proof proceeds in three parts. First we prove that if \( x(p^*, q^*) \) lies in the core of \((N_n, V)\) and there exists \( k \) and \( j \) such that \( q_{ikj}^* > 0 \) then \( \sum_{j \in S} q_{ikj}^* > 0 \) for all \( k \in \{1, \ldots, n\} \). Next we prove that for all \( i \in B \) if \( x(p^*, q^*) \) lies in the core of \((N_n, V)\) and there exists \( k \) and \( j \) such that \( q_{ikj}^* > 0 \), then,

\[
\sum_{k \in \{1, \ldots, n\}} q_{ikj}^* > \frac{n-1}{n} D_i(\hat{p}^*). \tag{D.19}
\]

Finally we prove that for sufficiently large \( n \in \mathbb{N} \), if there exists \( q^* \in q^*(p^*, N_n) \) such that \( x(p^*, q^*) \) lies in the core of \((N_n, V)\), then for all \( i \in B \) such that \( D_i(\hat{p}^*) > 0 \),

\[
\sum_{k \in \{1, \ldots, n\}} q_{ikj}^* > 0. \tag{D.20}
\]

**Step 1.** Suppose to the contrary that there exists an \( r \in B \) and \( n_1, n_2 \in \{1, \ldots, n\} \) such that \( q_{rn_{1j}}^* > 0 \) for some \( j \in S \) and \( \sum_{j \in S} q_{rn_{2j}}^* = 0 \). Consider the coalition \( N_n \setminus b_{rn_1} \) agreeing to the price vector \( p = p^* \) and the trades \( q^* \).
where \( q_{ikj} = q^*_{ikj} \) for all \( ik \neq rn_2 \) and \( j \in S \), and \( q_{rn_2j} = q^*_{rn_1j} \) for all \( j \in S \).

By agreeing to the pair \((p, q)\) the coalition \( N_n \setminus b_{rn_1} \) can strictly increase the payoff to \( b_{rn_2} \) while leaving the payoffs to every other member of the coalition unchanged, contradicting the assumption that \( x(p^*, q^*) \) lies in the core of \((N_n, V)\).

**Step 2.** Now suppose to the contrary that \( x(p^*, q^*) \) lies in the core of \((N_n, V)\), and (D.19) does not hold for some \( i \in B \) for whom there exists \( k \) and \( j \) such that \( q^*_{ikj} > 0 \). Select \( m \in \{1, \ldots, n\} \). We can redistribute the quantities \( \{q^*_{imj}\}_{j \in S} \) between the buyers \( \{b_{i1}, \ldots, b_{im}\} \setminus b_{im} \) in such a way that quantities are only added to buyers for whom,

\[
\sum_{j \in S} q^*_{ikj} < \frac{1}{n} D_i(\hat{p}^*), \tag{D.21}
\]

and then only up to the point at which (D.21) holds with equality. The failure of (D.19) to hold guarantees that there will be sufficient excess demand amongst the buyers in \( \{b_{i1}, \ldots, b_{im}\} \setminus b_{im} \) to completely reassign the quantity \( \sum_{j \in S} q^*_{imj} \) at the price \( \hat{p}^* \) without violating rational rationing. Moreover, the members of the coalition \( N_n \setminus b_{im} \) are no worse off, and any buyer who is allocated a portion of \( \sum_{j \in S} q^*_{imj} \) receives a strictly greater payoff as a result of the deviation: A contradiction.

**Step 3.** Suppose to the contrary that for all \( n \in \mathbb{N} \) there exists an \( n \in \mathbb{N} \) with \( n \geq m \), \( q^* \in q'(p^*, N_n) \) and \( i \in B \) such that \( D_i(\hat{p}^*) > 0 \), (D.20) holds with equality and \( x(p^*, q^*) \) lies in the core of \((N_n, V)\). It is straightforward to see that at least one buyer \( b_{hk} \in B_n \) must trade if value can be created, and it follows from step 2 that,

\[
D_h(\hat{p}^*) = \sum_{k \in S} q^*_{hkj}. \tag{D.22}
\]

Set \( n \) such that,

\[
\frac{1}{n} D_h(\hat{p}^*) \leq D_i(\hat{p}^*). \tag{D.23}
\]

By (D.23) we can redistribute the quantities \( \{q^*_{hkj}\}_{j \in S} \) equally amongst the members of \( \{b_{i1}, \ldots, b_{im}\} \). The members of the coalition \( N_n \setminus b_{hk} \) are no worse off, and all buyers in \( \{b_{i1}, \ldots, b_{im}\} \) receive a strictly greater payoff as a result of the deviation: A contradiction. \( \Box \)
Lemma 10. Suppose that $B$ is finite and that buyer payoffs satisfy (3.1) with $D_i(\cdot)$ twice continuously differentiable and strictly downward sloping for all $i \in B$. Suppose further that $|S| \geq 2$ and that $C_j''(\cdot) \geq 0$, $Y_j(p_N^j) > 0$ and $p_N^j \neq p_{N-j}^e$ for all $j \in S$.

Let $p_j^* = \hat{p}^*$ for all $j \in S$ and let there be at least two sellers for whom $Y_j(p_N^j) > 0$. If $D_N(\hat{p}^*) < Y_N(\hat{p}^*)$ then there exists $n \in \mathbb{N}$ such that for all $q^* \in q^*(p^*, N_n)$, $x(p^*, q^*) \notin \text{Core}(N_n, V)$.

Proof. Suppose to the contrary that $D_N(\hat{p}^*) < Y_N(\hat{p}^*)$ and for all $n \in \mathbb{N}$ there exists a $q^* \in q^*(p^*, N_n)$ such $x(p^*, q^*) \in \text{Core}(N_n, V)$. It must be the case that $\hat{p}^* > p_N^e$. Given that $D_N(\cdot)$ is strictly downward sloping and there exist at least two sellers willing to supply positive quantities at $p_N^e$, we can select $j \in S$ such that $Y_j(p_N^j) > 0$ and,

$$\sum_{k \in \{1, \ldots, n\}} q^*_{ikj} < \min\{D_N(\hat{p}^*), Y_j(\hat{p}^*)\}. \quad \text{(D.24)}$$

Select $m, n \in \mathbb{N}$ such that,

$$\sum_{k \in \{1, \ldots, n\}} q^*_{ikj} < \frac{m}{n} D_N(\hat{p}^*) \leq \min\{D_N(\hat{p}^*), Y_j(\hat{p}^*)\}. \quad \text{(D.25)}$$

The coalition $Q = s_j \cup \{b_{i1}, \ldots, b_{im}\}_{i \in B}$ can deviate, agreeing to the price $\hat{p}^*$ and the quantities $q_{ik} = D_{ik}(\hat{p}^*)$ for all $i \in B$ and $k \in \{1, \ldots, m\}$. The deviation leaves all buyers in $Q$ at least as well off and strictly increase $s_j$’s payoff: A contradiction. \qed

E Proof of Proposition 4

Again we call on lemma 1 to provide the proof. Define $W : 2^N \rightarrow \mathbb{R}$, such that $W(Q) = 0$ for any coalition that does not contain at least one buyer, one intermediary and one seller, and,

$$W(Q) = \int_{0}^{q_Q} (D_Q^{-1}(\vartheta) - T_Q^{-1}(\vartheta) - Y_Q^{-1}(\vartheta)) d\vartheta, \quad \text{(E.1)}$$

otherwise. Once again $(N, V)$ and $W(\cdot)$ satisfy the relationship set out in (1.6).
**Definition 5.** Let \( x \) be an allocation in the game \((N,W)\). The effective markup \( \Delta \bar{p}_k \) of a intermediary \( m_k \in Q \) is defined as,

\[
\Delta \bar{p}_k = \frac{x_k + \int_0^{T_j(p^d_Q - p^u_Q)} T_j^{-1}(\vartheta) d\vartheta}{T_j(p^d_Q - p^u_Q)}.
\]

(E.2)

By construction,

\[
\sum_{i \in B \cap Q} D_i(p^d_Q) \bar{p}_i = \sum_{j \in M \cap Q} T_j(p^d_Q - p^u_Q) = \sum_{j \in S \cap Q} Y_j(p^u_Q),
\]

(E.3)

which in turn implies,

\[
\sum_{i \in B \cap Q} \frac{D_i(p^d_Q) \bar{p}_i}{D_Q(p^d_Q)} = \frac{\sum_{j \in S \cap Q} Y_j(p^u_Q) \bar{p}_j}{Y_Q(p^u_Q)} + \frac{\sum_{j \in M \cap Q} T_k(p^d_Q - p^u_Q) \Delta \bar{p}_k}{T_Q(p^d_Q - p^u_Q)},
\]

(E.4)

which is to say the weighted average of effective prices paid by the buyers in \( Q \) is equal to the sum of the weighted averages of the seller effective prices and intermediary effective markups.

**Lemma 11.** Suppose that intermediary cost functions satisfy \( C'_j(\cdot) \geq 0 \) and \( C''_j(\cdot) \geq 0 \). For all \( m_k \in M \) and \( q \in \mathbb{R}_+ \),

\[
T_j(p^d_Q - p^u_Q) \cdot (p^d_N - p^u_N) - \int_0^{T_j(p^d_Q - p^u_Q)} T_j^{-1}(\vartheta) d\vartheta < q \Delta \bar{p}_j - \int_0^{q} T_j^{-1}(\vartheta) d\vartheta,
\]

(E.5)

implies \( \Delta \bar{p}_j > p^d_N - p^u_N \).

Proof. Follows directly from the proof of lemma 3. \(\square\)

**Lemma 12.** Suppose that buyer payoffs satisfy (3.1), and that seller and middleman cost functions satisfy \( C'_j(\cdot) \geq 0 \) and \( C''_j(\cdot) \geq 0 \). The allocation \( x^* \) is in Core\((N,W)\) if the corresponding effective unit price \( \bar{p}_i^* = p^d_N \) for all \( i \in B \), \( \Delta \bar{p}_j = p^d_N - p^u_N \) for all \( k \in M \) and \( \bar{p}_j^* = p^u_N \) for all \( j \in S \).

Proof. By way of contradiction suppose that there exists a coalition \( Q \) such that,

\[
\sum_{i \in Q} x_i^* < W(Q).
\]

(E.6)
Let $x$ be an allocation satisfying $\sum_{i \in Q} x_i = W(Q)$ and $x_i > x_i^*$ for all $i \in Q$. Moreover define $\bar{p}_i$ (or $\Delta \bar{p}_i$) as the effective unit price (or markup) corresponding to the payoff $x_i$.

From lemma 3 we know that $\bar{p}_i < \bar{p}_i^* = p_{d}$ for all $i \in B \cap Q$, from lemma 4 we know that $\bar{p}_j > \bar{p}_j^* = p_{u}$ for all $i \in S \cap Q$, and $\Delta \bar{p}_k < \Delta \bar{p}_k^* = p_{d} - p_{u}$ for all $k \in M \cap Q$. This implies,

$$\frac{\sum_{i \in B \cap Q} D_i(p_{d}^Q)\bar{p}_i}{D_Q(p_{d}^Q)} < p_N^d \quad (\text{E.7})$$

contradicting (E.4).

\[\square\]

### F Proof of Proposition 5

Given rational rationing, the allocation $x(\hat{p}, \hat{q})$ can only be blocked by the coalition $N$. Consider the allocation $x(p, q)$ with $q = \min\{q_m^*, q_s^*\}$, $p^d = D^{-1}(q)$ and $p^u \in (Y^{-1}(q), p^d - T^{-1}(q))$; there are two possibilities $\min\{q_m^*, q_s^*\} = q_m^*$ and $\min\{q_m^*, q_s^*\} = q_s^*$. In either case $x_b(p, q) > x_b(\hat{p}, \hat{q})$.

Suppose that $\min\{q_m^*, q_s^*\} = q_m^*$. Note that,

$$\hat{q}(p^d - Y^{-1}(\hat{q})) - \int_0^\hat{q} T^{-1}(\vartheta)d\vartheta < (p^d - Y^{-1}(q))q - \int_0^q T^{-1}(\vartheta)d\vartheta, \quad (F.1)$$

by assumption and,

$$\hat{q}Y^{-1}(\hat{q}) - \int_0^\hat{q} Y^{-1}(\vartheta)d\vartheta \leq Y^{-1}(q)q - \int_0^q Y^{-1}(\vartheta)d\vartheta. \quad (F.2)$$

It follows that there is sufficient value at the pair $(p, q)$ to leave both the seller and intermediary better off, the only remaining question is whether this value can be allocated in such a way as to improve the payoff to both seller and intermediary. If,

$$x_s(\hat{p}, \hat{q}) \leq Y^{-1}(q)q - \int_0^q Y^{-1}(\vartheta)d\vartheta, \quad (F.3)$$
then set \( p^u = Y^{-1}(q) \) and from (4.9) we can see that \( x_m(\hat{p}, \hat{q}) < x_m(p, q) \). For the case in which,

\[
x_s(\hat{p}, \hat{q}) > Y^{-1}(q)q - \int_0^q Y^{-1}(\vartheta)d\vartheta,
\]

it follows from the concavity of (4.8) and the fact that \( q_\ast^s \geq q > \hat{q} \), that,

\[
x_s(\hat{p}, \hat{q}) < (p^d - T^{-1}(q))q - \int_0^q Y^{-1}(\vartheta)d\vartheta.
\]

Therefore we can select \( p^u \in (Y^{-1}(q), p^d - T^{-1}(q)) \) such that \( x_s(p, q) = x_s(\hat{p}, \hat{q}) \) which leaves \( x_m(p, q) > x_m(\hat{p}, \hat{q}) \).

If \( \min\{q_m^*, q_\ast^s\} = q_\ast^s \) the same result can be developed by reversing the roles of the seller and intermediary.

**Bibliography**


