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COMPARING BERTRAND AND COURNOT OUTCOMES IN THE PRESENCE OF PUBLIC FIRMS

ARGHYA GHOSH AND MANIPUSHPAK MITRA

Abstract. We revisit the classic comparison between Bertrand and Cournot outcomes in a mixed market with private and public firms. A departure from the standard setting, i.e., one where all firms maximize profits, provides new insights. A welfare-maximizing public firm's price is strictly lower while its output is strictly higher in Cournot competition. And whereas the private firm's quantity is strictly lower in Cournot (as in the standard setting), its price can be higher or lower. Despite this ambiguity, both firms, public and private, earn strictly lower profits in Cournot. The consumer surplus is strictly higher in Cournot under a linear demand structure. All these results also hold with more than two firms under a wide range of parameterizations. The ranking reversals also hold in a richer setting with a partially privatized public firm, where the extent of privatization is endogenously determined by a welfare-maximizing government. As a by-product of our analysis, we find that in a differentiated duopoly setting, partial privatization always improves welfare in Cournot but not necessarily in Bertrand competition.

JEL classification numbers: L13, H42

Keywords: Bertrand, Cournot, public firms, partial privatization

1. Introduction

It is now well known that Bertrand competition yields lower prices and profits and higher quantities, consumer surplus, and welfare than Cournot competition (see, for example, Singh and Vives [19], Cheng [4], Vives [22], and Okuguchi [16]). Subsequently, exploiting cost asymmetries, Dastidar [6], Qiu [17], Häckner [12], and Amir and Jin [1] have provided important counterexamples where at least one of these conclusions fails to hold. To date, however, the literature comparing Bertrand and Cournot outcomes has focused almost exclusively on environments where all firms maximize profits. We revisit this classic comparison, but in *mixed markets*, where profit-maximizing private firms coexist with public firms.

This coexistence is observed in several oligopolistic sectors including banking, insurance, telecommunications, and postal services in Europe, Canada, and Japan. In the United States, both public and private firms coexist in the packaging and overnight delivery

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industries (Matsumura and Matsushima [15]). While in the developed countries public firms are primarily present in the services and utilities sector, in the developing world their presence is pervasive. In developing countries, the share of public enterprises in manufacturing output and employment lies in the 30-70% range (Schmitz [18]).

So, how does a public firm differ from its private counterpart? We follow the mixed oligopoly literature in assuming that the difference lies in the objective function. Unlike private firms, which maximize profits, a public firm maximizes welfare (sum of consumer surplus and profits).[1] We characterize and compare Cournot and Bertrand outcomes in a differentiated duopoly where a private firm competes against a welfare maximizing public firm (sections 2 and 3). The results are often strikingly different from the ones obtained in the standard setting which has only profit-maximizing firms.

In the standard setting the equilibrium prices are strictly higher in Cournot. In contrast, we find that the welfare-maximizing *public firm's price is strictly lower in Cournot* than in Bertrand competition while the *private firm's price can be higher or lower in Cournot.* Despite the ambiguity in price ordering between Bertrand and Cournot for the private firm's price, comparison of quantities and profits (between the two) gives unambiguous results. The *public firm's quantity is strictly higher in Cournot* whereas the private firm's output, as Singh and Vives [19] show, is strictly lower. We find that, contrary to standard findings, *both firms' profits are strictly lower in Cournot.* All these results hold under general utility specifications satisfying standard assumptions used in the literature. We also find general sufficiency conditions under which consumer surplus is strictly higher and welfare is strictly lower under Cournot. Using these sufficiency results we show that for linear demand structure used in Singh and Vives [19], *consumer surplus (welfare) is indeed strictly higher (lower) under Cournot.* These results hold for more than two firms under a range of parameter values.

Comparison of Bertrand and Cournot outcomes is of fundamental importance in the industrial organization literature. The paper contributes to this literature by conducting the classic comparison in a richer setting where all firms do not necessarily maximize profits. As the results (mentioned above) suggest, the departure from the standard setting provide new insights. In addition, the paper draws from as well as contributes to the public economics literature on mixed oligopoly which examines the incentives and welfare consequences of maintaining/privatizing a public firm. While both modes of market competition, Cournot and Bertrand, have been employed in this literature to analyze privatization—see for example, De Fraja and Delbono [7], Matsumura [14] for Cournot analysis, and Cremer et. al. [5] and Anderson et al. [2], for Bertrand analysis—there

[1]See De Fraja and Delbono [8] and Basu [3, Ch.16] for surveys on mixed oligopoly. Also, see De Fraja and Delbono [7] and Anderson, de Palma, and Thisse [2] for a critical discussion on welfare maximization as an objective of public firms.

does not exist any systematic comparison between the two. The paper fills this gap. More importantly, the paper shows that in mixed oligopolies, the standard Bertrand-Cournot rankings obtained from an oligopoly analysis with profit-maximizing firms no longer hold.

In section 4, we relax the assumption that the public firms maximize welfare. We re-examine the comparison (between Bertrand and Cournot outcomes) in an environment with possible partial privatization of the public firm. In particular, we *endogenize* the degree of privatization by constructing a stylized two-stage game in which a public firm maximizes a weighted sum of its own profits and welfare in stage two, and the weights, indicating the extent of privatization, are chosen optimally by a welfare-maximizing government in stage one. The reversals also hold under this richer setting. As a by-product of our analysis, we find that the mode of market competition (i.e., Bertrand or Cournot) can have qualitatively different welfare implications for partial privatization.

2. The Basic Model

Consider an economy with two sectors: a competitive sector producing the numeraire good y and an imperfectly competitive sector with two firms, firm 1 and firm 2, each producing a differentiated good. Let p_i and q_i denote firm i's price and quantity respectively, where $i = 1, 2$. The representative consumer maximizes $V(q, y) \equiv U(q) + y$ subject to $p_1 q_1 + p_2 q_2 + y \leq I$ where $q \equiv (q_1, q_2) \in \Re^2_+$ and I denotes income. The utility function $U(q)$ is continuously differentiable as often as required on $\Re^2_+$. Furthermore, the following holds:

Assumption 1. For $i, j \in \{1, 2\}$, $i \neq j$, (i) $U_i(q) \equiv \frac{\partial U(q)}{\partial q_i} > 0$, (ii) $U_{ii}(q) \equiv \frac{\partial^2 U(q)}{\partial {q_i}^2} < 0$, (iii) $U_{ij}(q) \equiv \frac{\partial^2 U(q)}{\partial q_i \partial q_j} < 0$, (iv) $U_{ij}(q) = U_{ji}(q)$, and (v) $|U_{ii}(q)| > |U_{ij}(q)|$.

These assumptions are standard in the literature (see, for example, section 5 in Singh and Vives [19]).

Since $V(q, y)$ is separable and linear in y, there are no income effects, and consequently, for a large enough income, the representative consumer's optimization problem is reduced to choosing q to maximize $U(q) - p_1 q_1 - p_2 q_2$. Utility maximization yields the inverse demands: $p_i = \frac{\partial U(q)}{\partial q_i} \equiv P_i(q)$ for $q_i > 0$, $i = 1, 2$. Then, applying Assumption 1 gives two important properties: (a) demand slopes downward, since $\frac{\partial P_i(q)}{\partial q_i} \equiv U_{ii}(q) < 0$, and (b) two goods are substitutes, since $\frac{\partial P_i(q)}{\partial q_j} \equiv U_{ij}(q) < 0$, $(i \neq j)$.

Inverting the inverse demand system yields the direct demands: $q_i = D_i(p)$ where $i = 1, 2$ and $p = (p_1, p_2) \in \Re^2_+$. From the assumptions on $U(q)$, we have

$$\frac{\partial D_i(p)}{\partial p_i} = \frac{U_{jj}}{U_{11}U_{22} - U_{12}U_{21}} < 0, \frac{\partial D_i(p)}{\partial p_j} = -\frac{U_{ij}}{U_{11}U_{22} - U_{12}U_{21}} > 0, \tag{1}$$

where $i, j \in \{1, 2\}$, $i \neq j$ and U_{ij}, U_{ii} are evaluated at $q = (q_1, q_2) \equiv (D_1(p), D_2(p))$.

There are no fixed costs. Each firm has a constant marginal cost $m > 0$. We assume that unit costs are constant and symmetric for the public and private firm in order to highlight our source of reversal: the presence of public firms.[2] Qualitatively, all our propositions hold under asymmetric costs.

Firm 2 maximizes profit and we refer to firm 2 as the private firm. Firm 1 is a public sector enterprise, or, in short, a public firm. Following the mixed oligopoly literature, we assume that the public firm maximizes welfare. Profits and welfare, in terms of q and p, are precisely defined below.

2.1. **Cournot competition.** Corresponding to a quantity vector $q \equiv (q_1, q_2)$, profits of firm i, denoted by $\pi_i(q)$, and welfare, denoted by $W(q)$, are:

$$\begin{aligned} \pi_i(q) &= (p_i(q) - m)q_i, \\ W(q) &= U(q) - m(q_1 + q_2). \end{aligned}$$

From Assumptions 1 it follows that $\frac{\partial^2 W(q)}{\partial q_i^2} = U_{ii} < 0$, $\frac{\partial^2 W(q)}{\partial q_i \partial q_j} = U_{ij} < 0$. Since $|U_{ii}| > |U_{ij}|$, $|\frac{\partial^2 W(q)}{\partial q_i^2}| > |\frac{\partial^2 W(q)}{\partial q_i \partial q_j}|$. We also make the following assumptions which are standard in the oligopoly literature.

Assumption 2. For $i, j \in \{1, 2\}$, $i \neq j$, we have (i) $\frac{\partial^2 \pi_i(q)}{\partial q_i \partial q_j} < 0$, (ii) $\frac{\partial^2 \pi_i(q)}{\partial q_i^2} < 0$, and (iii) $|\frac{\partial^2 \pi_i(q)}{\partial q_i^2}| \geq |\frac{\partial^2 \pi_i(q)}{\partial q_i \partial q_j}|$.

Assumption 2(i) says that quantities are strategic substitutes. Assumption 2 (ii) says that firm i's profit is strictly concave in its own output. Assumption 2 (iii), together with $|\frac{\partial^2 W(q)}{\partial q_i^2}| > |\frac{\partial^2 W(q)}{\partial q_i \partial q_j}|$ implies $\left(\frac{\partial^2 W(q)}{\partial q_i^2}\right)\left(\frac{\partial^2 \pi_i(q)}{\partial q_i^2}\right) - \left(\frac{\partial^2 W(q)}{\partial q_i \partial q_j}\right)\left(\frac{\partial^2 \pi_i(q)}{\partial q_i \partial q_j}\right) > 0$ which is sufficient for the uniqueness of the Cournot solution in our set up.

A quantity vector $q^{\mathcal{C}} \equiv (q_1^{\mathcal{C}}, q_2^{\mathcal{C}})$ is a Cournot equilibrium if and only if $W(q^{\mathcal{C}}) \geq W(q_1, q_2^{\mathcal{C}})$ for all $q_1 \neq q_1^{\mathcal{C}}$ and $\pi_2(q^{\mathcal{C}}) \geq \pi_2(q_1^{\mathcal{C}}, q_2)$ for all $q_2 \neq q_2^{\mathcal{C}}$. Corresponding to a Cournot equilibrium quantity vector $q^{\mathcal{C}}$, define

$$\begin{aligned} p_i^{\mathcal{C}} &= P_i(q^{\mathcal{C}}), \\ \pi_i^{\mathcal{C}} &= \pi_i(q^{\mathcal{C}}), \\ CS^{\mathcal{C}} &= U(q^{\mathcal{C}}) - p_1^{\mathcal{C}} q_1^{\mathcal{C}} - p_2^{\mathcal{C}} q_2^{\mathcal{C}}, \quad \text{and} \\ W^{\mathcal{C}} &= W(q^{\mathcal{C}}) = U(q^{\mathcal{C}}) - m(q_1^{\mathcal{C}} + q_2^{\mathcal{C}}), \end{aligned}$$

where $p_i^{\mathcal{C}}$, $\pi_i^{\mathcal{C}}$, $CS^{\mathcal{C}}$, and $W^{\mathcal{C}}$ respectively are firm i's price, firm i's profits, consumer surplus, and welfare in Cournot equilibrium.

[2]See Matsumura and Matsushima [15] for an analysis of privatization under endogenous cost differences.

2.2. **Bertrand competition.** Corresponding to a price vector $p \equiv (p_1, p_2)$, profits of firm i, denoted by $\tilde{\pi}_i(p)$, and welfare, denoted by $\tilde{W}(p)$, are:

$$\begin{aligned}\tilde{\pi}_i(p) &= (p_i - m)D_i(p),\\ \tilde{W}(p) &= U(D_1(p), D_2(p)) - m(D_1(p) + D_2(p)).\end{aligned}$$

We assume that p_1 and p_2 are strategic complements. More formally, the following holds:

Assumption 3. For $i, j \in \{1, 2\}$, $i \neq j$, we have (i) $\frac{\partial^2 \tilde{W}(p)}{\partial p_i \partial p_j} > 0$, (ii) $\frac{\partial^2 \tilde{\pi}_i(p)}{\partial p_i \partial p_j} > 0$, and (iii) $\left(\frac{\partial^2 \tilde{W}(p)}{\partial p_i^2}\right)\left(\frac{\partial^2 \tilde{\pi}_j(p)}{\partial p_j^2}\right) > \left(\frac{\partial^2 \tilde{W}(p)}{\partial p_i \partial p_j}\right)\left(\frac{\partial^2 \tilde{\pi}_j(p)}{\partial p_i \partial p_j}\right)$.

Strategic complementarity (of prices) is a standard assumption in a differentiated Bertrand duopoly with profit-maximizing firms. Assumption 3 (i) and (ii) states that strategic complementarity holds even when one firm maximizes welfare. Assumption 3 guarantees the uniqueness of the Bertrand equilibrium.

Assumptions 2 and 3 hold for the standard quadratic utility specification considered in the literature (see, for example, Dixit [9], Singh and Vives [19], Qiu [17], and Häckner [12]).

A price vector $p^{\mathcal{B}} \equiv (p_1^{\mathcal{B}}, p_2^{\mathcal{B}})$ is a Bertrand equilibrium if and only if $\tilde{W}(p^{\mathcal{B}}) \geq \tilde{W}(p_1, p_2^{\mathcal{B}})$ for all $p_1 \neq p_1^{\mathcal{B}}$ and $\tilde{\pi}_2(p^{\mathcal{B}}) \geq \tilde{\pi}_2(p_1^{\mathcal{B}}, p_2)$ for all $p_2 \neq p_2^{\mathcal{B}}$. Corresponding to a Bertrand equilibrium price vector $p^{\mathcal{B}}$, define

$$\begin{aligned}q_i^{\mathcal{B}} &= D_i(p^{\mathcal{B}}),\\ \pi_i^{\mathcal{B}} &= \tilde{\pi}_i(p^{\mathcal{B}}) \equiv \pi_i(q^{\mathcal{B}}),\\ CS^{\mathcal{B}} &= \tilde{CS}(p^{\mathcal{B}}) \equiv CS(q^{\mathcal{B}}) = U(q^{\mathcal{B}}) - p_1^{\mathcal{B}} q_1^{\mathcal{B}} - p_2^{\mathcal{B}} q_2^{\mathcal{B}}, \quad \text{and}\\ W^{\mathcal{B}} &= \tilde{W}(p^{\mathcal{B}}) \equiv W(q^{\mathcal{B}}) = U(q^{\mathcal{B}}) - m(q_1^{\mathcal{B}} + q_2^{\mathcal{B}}),\end{aligned}$$

where $q_i^{\mathcal{B}}$, $\pi_i^{\mathcal{B}}$, $CS^{\mathcal{B}}$, and $W^{\mathcal{B}}$ respectively are firm i's price, firm i's profits, consumer surplus, and welfare in Bertrand equilibrium.

3. Comparing Bertrand and Cournot Outcomes

Before we start comparing Bertrand and Cournot outcomes we introduce a technical lemma that will be used in proving the results of this section.

Lemma 1. Consider any differentiable function $f : \Re_+^2 \rightarrow \Re$. Define $a(t) = ta' + (1-t)a'' \in \Re_+^2$ where $a' \equiv (a_1', a_2')$ and $a'' \equiv (a_1'', a_2'') \in \Re_+^2$ and $t \in [0, 1]$. For all such $a', a'' \in \Re_+^2$,

$$f(a') - f(a'') = (a_1' - a_1'') \int_0^1 \left(\frac{\partial f(a(t))}{\partial a_1(t)}\right) dt + (a_2' - a_2'') \int_0^1 \left(\frac{\partial f(a(t))}{\partial a_2(t)}\right) dt. \tag{2}$$

Proof: Using $a_i' - a_i'' = \frac{\partial a_i(t)}{\partial t}$ for $i = 1, 2$ in the right hand side of (2) we get $\int_0^1 \left[\left(\frac{\partial f(a(t))}{\partial a_1(t)} \right) \left(\frac{\partial a_1(t)}{\partial t} \right) + \left(\frac{\partial f(a(t))}{\partial a_2(t)} \right) \left(\frac{\partial a_2(t)}{\partial t} \right) \right] dt = \int_{t=0}^{t=1} d[f(a(t))] = f(a(1)) - f(a(0)) = f(a') - f(a'')$. □

3.1. **Prices.** First we compare the pricing of the public firm under Cournot and Bertrand competition. In a differentiated duopoly with profit-maximizing firms only, equilibrium prices are higher under Cournot competition (Singh and Vives [19]; Vives[22]). Theorem 1 shows that the conclusion does not hold in the presence of a public firm. The public firm's price is *strictly lower* under Cournot competition irrespective of demand specification.

Theorem 1. Suppose $q_i^{\mathcal{C}} > 0$ and $q_i^{\mathcal{B}} > 0$ for $i = 1, 2$. Then $p_1^{\mathcal{B}} > p_1^{\mathcal{C}} = m$.

Proof: Under Cournot competition, $q_1 = q_1^{\mathcal{C}}$ maximizes $W(q_1, q_2^{\mathcal{C}})$. Since $q_i^{\mathcal{C}} > 0$, from first order conditions we get

$$\frac{\partial W(q^{\mathcal{C}})}{\partial q_1} = \frac{\partial U(q^{\mathcal{C}})}{\partial q_1} - m = 0.$$

We have $\frac{\partial U(q^{\mathcal{C}})}{\partial q_1} \equiv P_1(q^{\mathcal{C}})$ and by definition, $p_1^{\mathcal{C}} = P_1(q^{\mathcal{C}})$. Thus $p_1^{\mathcal{C}} = m$.

Noting that $p^{\mathcal{B}} = (p_1^{\mathcal{B}}, p_2^{\mathcal{B}})$ constitutes the Bertrand equilibrium, from first order conditions we get:

$$\frac{\partial \tilde{W}(p^{\mathcal{B}})}{\partial p_1} = (p_1^{\mathcal{B}} - m)\frac{\partial D_1(p^{\mathcal{B}})}{\partial p_1} + (p_2^{\mathcal{B}} - m)\frac{\partial D_2(p^{\mathcal{B}})}{\partial p_1} = 0, \tag{3}$$

$$\frac{\partial \tilde{\pi}_2(p^{\mathcal{B}})}{\partial p_2} = (p_2^{\mathcal{B}} - m)\frac{\partial D_2(p^{\mathcal{B}})}{\partial p_2} + D_2(p^{\mathcal{B}}) = 0. \tag{4}$$

From (1), we have (i) $\frac{\partial D_1(p^{\mathcal{B}})}{\partial p_1} < 0$ and (ii) $\frac{\partial D_2(p^{\mathcal{B}})}{\partial p_1} > 0$. Since $\frac{\partial D_2(p^{\mathcal{B}})}{\partial p_2} < 0$ and $D_2(p^{\mathcal{B}}) = q_2^{\mathcal{B}} > 0$, from (4) it follows that (iii) $p_2^{\mathcal{B}} - m > 0$. Together with (i)—(iii), equation (3) implies that $p_1^{\mathcal{B}} - m > 0$. □

To understand Theorem 1, consider an infinitesimally small increase in p_1 from $p_1 = m$. This reduces the public firm's output, q_1, and raises the private firm's output, q_2. The welfare loss from the reduction in q_1 is second order while the welfare gain from an increase in q_2 is first order (since $p_2^{\mathcal{B}} > m$). This logic implies that if the two goods are substitutes, and firms compete in prices, a welfare-maximizing public firm will set its price strictly higher than marginal cost. Thus $p_1^{\mathcal{B}} > m$. Such considerations are absent in Cournot competition. Under Cournot conjecture, the welfare maximizing public firm chooses its own output, taking the private firm's output as given. Thus, the public firm behaves like a welfare-maximizing monopolist, which in turn yields $p_1^{\mathcal{C}} \equiv \frac{\partial U(q^{\mathcal{C}})}{\partial q_1} = m$.

While $p_1^{\mathcal{C}} < p_1^{\mathcal{B}}$, $p_2^{\mathcal{C}}$ can be lower or higher than $p_2^{\mathcal{B}}$. On the one hand, as in the standard setting, the perceived elasticity of demand of a firm is smaller under the Cournot conjecture which raises $p_2^{\mathcal{C}}$.[3] On the other hand, a lower $p_1^{\mathcal{C}}$ (compared to $p_1^{\mathcal{B}}$) in our framework creates a downward pressure on the private firm's price, which lowers $p_2^{\mathcal{C}}$. As we will see later, for the linear demand system, these two effects exactly offset each other, which in turn yields $p_2^{\mathcal{C}} = p_2^{\mathcal{B}}$. However, in general, the overall effect is ambiguous and the precise comparison of private prices between the two (Cournot and Bertrand) depends on the demand system.

3.2. **Quantities.** Surprisingly, despite the ambiguity that $p_2^{\mathcal{C}}$ can be lower or higher than $p_2^{\mathcal{B}}$, the ranking of quantities is unambiguous. The public firm's quantity is *strictly higher* under Cournot while the private firm's quantity is strictly higher under Bertrand. Thus, the standard quantity ranking is reversed for the public firm but not for the private firm. See Theorem 2 below.

The proof of the theorem is shown in two steps. First, we establish two claims: $\frac{\partial W(q^{\mathcal{B}})}{\partial q_1} > 0$ and $\frac{\partial \pi_2(q^{\mathcal{B}})}{\partial q_2} < 0$. Then, using these claims together with Cournot first-order conditions, $\frac{\partial W(q^{\mathcal{C}})}{\partial q_1} = \frac{\partial \pi_2(q^{\mathcal{C}})}{\partial q_2} = 0$, and the sign of second partials of $W(.)$ and $\pi_2(.)$, we prove the theorem.

Theorem 2. Suppose $q_i^{\mathcal{C}} > 0$ and $q_i^{\mathcal{B}} > 0$ for $i = 1, 2$. Then $q_1^{\mathcal{C}} > q_1^{\mathcal{B}}$ and $q_2^{\mathcal{C}} < q_2^{\mathcal{B}}$.

Proof: First, we evaluate the functions $\frac{\partial W(q)}{\partial q_1}$ and $\frac{\partial \pi_2(q)}{\partial q_2}$ at $q = (q_1^{\mathcal{B}}, q_2^{\mathcal{B}})$. We have that

$$\frac{\partial W(q^{\mathcal{B}})}{\partial q_1} = p_1^{\mathcal{B}} - m > 0.$$

Also,

$$\begin{aligned} \frac{\partial \pi_2(q^{\mathcal{B}})}{\partial q_2} &= (p_2^{\mathcal{B}} - m) + q_2^{\mathcal{B}} \left(\frac{\partial P_2(q^{\mathcal{B}})}{\partial q_2} \right) \\ &= \left(\frac{-q_2^{\mathcal{B}}}{\frac{\partial D_2(p^{\mathcal{B}})}{\partial p_2}} \right) \left(1 - \frac{\partial D_2(p^{\mathcal{B}})}{\partial p_2} \frac{\partial P_2(q^{\mathcal{B}})}{\partial q_2} \right) \\ &< 0, \end{aligned}$$

where the second equality follows from applying (4), and the inequality follows from noting that $q_2^{\mathcal{B}} > 0$, $\frac{\partial D_2(p^{\mathcal{B}})}{\partial p_2} < 0$, $1 - \frac{\partial D_2(p^{\mathcal{B}})}{\partial p_2} \frac{\partial P_2(q^{\mathcal{B}})}{\partial q_2} = 1 - \frac{U_{11}(q^{\mathcal{B}})U_{22}(q^{\mathcal{B}})}{U_{11}(q^{\mathcal{B}})U_{22}(q^{\mathcal{B}}) - U_{12}(q^{\mathcal{B}})U_{21}(q^{\mathcal{B}})} = -\frac{U_{12}(q^{\mathcal{B}})U_{21}(q^{\mathcal{B}})}{U_{11}(q^{\mathcal{B}})U_{22} - U_{12}(q^{\mathcal{B}})U_{21}(q^{\mathcal{B}})} < 0.$

[3]See Proposition 6.1 in Vives [23] for a comparison of elasticities under Bertrand and Cournot.

Applying Lemma 1 and the Cournot first-order conditions $\frac{\partial W(q^{\mathcal{C}})}{\partial q_1} = \frac{\partial \pi_2(q^{\mathcal{C}})}{\partial q_2} = 0$ we get:

$$\frac{\partial W(q^{\mathcal{B}})}{\partial q_1} = \frac{\partial W(q^{\mathcal{B}})}{\partial q_1} - \frac{\partial W(q^{\mathcal{C}})}{\partial q_1} = (q_1^{\mathcal{B}} - q_1^{\mathcal{C}}) \int_0^1 \frac{\partial^2 W(q(t))}{\partial q_1(t)^2} dt + (q_2^{\mathcal{B}} - q_2^{\mathcal{C}}) \int_0^1 \frac{\partial^2 W(q(t))}{\partial q_1(t) \partial q_2(t)} dt,$$

$$\frac{\partial \pi_2(q^{\mathcal{B}})}{\partial q_2} = \frac{\partial \pi_2(q^{\mathcal{B}})}{\partial q_2} - \frac{\partial \pi_2(q^{\mathcal{C}})}{\partial q_2} = (q_1^{\mathcal{B}} - q_1^{\mathcal{C}}) \int_0^1 \frac{\partial^2 \pi_2(q(t))}{\partial q_1(t) \partial q_2(t)} dt + (q_2^{\mathcal{B}} - q_2^{\mathcal{C}}) \int_0^1 \frac{\partial^2 \pi_2(q(t))}{\partial q_2(t)^2} dt.$$

Applying Assumptions 1 — 3 it follows that [4]

$$\operatorname{sgn}\left[\frac{\partial^2 W(q(t))}{\partial q_1(t)^2}\right] = \operatorname{sgn}\left[\frac{\partial^2 W(q(t))}{\partial q_1(t)\partial q_2(t)}\right] = \operatorname{sgn}\left[\frac{\partial^2 \pi_2(q(t))}{\partial q_1(t)\partial q_2(t)}\right] = \operatorname{sgn}\left[\frac{\partial^2 \pi_2(q(t))}{\partial q_2(t)^2}\right] = -1.$$

Using these signs and that $\frac{\partial W(q^{\mathcal{B}})}{\partial q_1} > 0$ and $\frac{\partial \pi_2(q^{\mathcal{B}})}{\partial q_2} < 0$, it is straightforward to show that there are only two possibilities:

(a) $q_1^{\mathcal{B}} > q_1^{\mathcal{C}}$ and $q_2^{\mathcal{B}} < q_2^{\mathcal{C}}$, or
(b) $q_1^{\mathcal{B}} < q_1^{\mathcal{C}}$ and $q_2^{\mathcal{B}} > q_2^{\mathcal{C}}$.

Suppose (a) holds. Then

$$\frac{\partial W(q^{\mathcal{B}})}{\partial q_1} > 0 \Rightarrow \frac{\int_0^1 \frac{\partial^2 W(q(t))}{\partial q_1(t)^2} dt}{\int_0^1 \frac{\partial^2 W(q(t))}{\partial q_1(t)\partial q_2(t)} dt} < \frac{q_2^{\mathcal{C}} - q_2^{\mathcal{B}}}{q_1^{\mathcal{B}} - q_1^{\mathcal{C}}}. \tag{5}$$

Similarly,

$$\frac{\partial \pi_2(q^{\mathcal{B}})}{\partial q_2} < 0 \Rightarrow \frac{\int_0^1 \frac{\partial^2 \pi_2(q(t))}{\partial q_1(t)\partial q_2(t)} dt}{\int_0^1 \frac{\partial^2 \pi_2(q(t))}{\partial q_2(t)^2} dt} > \frac{q_2^{\mathcal{C}} - q_2^{\mathcal{B}}}{q_1^{\mathcal{B}} - q_1^{\mathcal{C}}}. \tag{6}$$

Combining (5) and (6) we get

$$\frac{\int_0^1 \frac{\partial^2 W(q(t))}{\partial q_1(t)^2} dt}{\int_0^1 \frac{\partial^2 W(q(t))}{\partial q_1(t)\partial q_2(t)} dt} < \frac{\int_0^1 \frac{\partial^2 \pi_2(q(t))}{\partial q_1(t)\partial q_2(t)} dt}{\int_0^1 \frac{\partial^2 \pi_2(q(t))}{\partial q_2(t)^2} dt}. \tag{7}$$

[4]For any $x \in \Re$,

$$\operatorname{sgn}[x] = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Inequality (7) is impossible given $|\frac{\partial^2 W(q)}{\partial q_1^2}| > |\frac{\partial^2 W(q)}{\partial q_1 \partial q_2}|$ (which follows from Assumption 1) and Assumption 2 (ii) which gives $|\frac{\partial^2 \pi_2(q)}{\partial q_2^2}| \geq |\frac{\partial^2 \pi_2(q)}{\partial q_1 \partial q_2}|$. Hence the only possibility is (b). □

3.3. **Profits, Consumer Surplus, and Welfare:** Equipped with findings from Theorems 1 and 2, we are now ready to compare Bertrand and Cournot for consumer surplus, profits, and welfare. In a standard differentiated duopoly setting, Cournot competition yields higher profits, lower consumer surplus, and lower welfare compared to Bertrand competition.

Part (A) of Theorem 3 below states that both firms earn strictly higher profits under Bertrand competition. Thus, compared to Singh and Vives [19], we have a reversal in profit rankings for both firms. Claims regarding consumer surplus and welfare, made in parts (B) and (C) respectively, are contingent on the Bertrand-Cournot ranking of the public firm's price. While (B) is relatively straightforward, (C) is subtle.

Theorem 3. Suppose $q_i^{\mathcal{B}} > 0$ and $q_i^{\mathcal{C}} > 0$ for $i = 1, 2$. Then we have the following results.

(A) $\pi_i^{\mathcal{B}} > \pi_i^{\mathcal{C}}$ for $i = 1, 2$.
(B) If $p_2^{\mathcal{C}} \leq p_2^{\mathcal{B}}$ then $CS^{\mathcal{B}} < CS^{\mathcal{C}}$.
(C) If $p_2^{\mathcal{C}} \geq p_2^{\mathcal{B}}$ then $W^{\mathcal{B}} > W^{\mathcal{C}}$.

Proof: We prove each part separately.

Proof of Part (A): First consider $\pi_1^{\mathcal{B}}$ and $\pi_1^{\mathcal{C}}$. Since $p_1^{\mathcal{C}} = m$, $\pi_1^{\mathcal{C}} = 0$. Also, since $p_1^{\mathcal{B}} > m$ and $q_1^{\mathcal{B}} > 0$, we have $\pi_1^{\mathcal{B}} > 0 = \pi_1^{\mathcal{C}}$. Next, consider $\pi_2^{\mathcal{B}}$ and $\pi_2^{\mathcal{C}}$. Note that $\pi_2^{\mathcal{B}} \equiv \tilde{\pi}_2(p^{\mathcal{B}})$ and $\pi_2^{\mathcal{C}} \equiv \pi_2(q^{\mathcal{C}}) = \tilde{\pi}_2(p^{\mathcal{C}})$. By writing

$$\pi_2^{\mathcal{B}} - \pi_2^{\mathcal{C}} = [\tilde{\pi}_2(p^{\mathcal{B}}) - \tilde{\pi}_2(p_1^{\mathcal{B}}, p_2^{\mathcal{C}})] + [\tilde{\pi}_2(p_1^{\mathcal{B}}, p_2^{\mathcal{C}}) - \tilde{\pi}_2(p^{\mathcal{C}})],$$

and then using Lemma 1 we get

$$\pi_2^{\mathcal{B}} - \pi_2^{\mathcal{C}} = \left[\tilde{\pi}_2(p^{\mathcal{B}}) - \tilde{\pi}_2(p_1^{\mathcal{B}}, p_2^{\mathcal{C}})\right] + (p_1^{\mathcal{B}} - p_1^{\mathcal{C}}) \int_0^1 \frac{\partial \tilde{\pi}_2(p_1(t), p_2^{\mathcal{C}})}{\partial p_1(t)} dt. \tag{8}$$

From the definition of Bertrand equilibrium it follows that (i) $\tilde{\pi}_2(p^{\mathcal{B}}) \geq \tilde{\pi}_2(p_1^{\mathcal{B}}, p_2^{\mathcal{C}})$. By Theorem 1, (ii) $p_1^{\mathcal{B}} > p_1^{\mathcal{C}} = m$. Finally, (iii) $\frac{\partial \tilde{\pi}_2(p_1(t), p_2^{\mathcal{C}})}{\partial p_1(t)} = (p_2^{\mathcal{B}} - m)\frac{\partial D_2(p_1(t), p_2^{\mathcal{C}})}{\partial p_1(t)} > 0$ since $p_2^{\mathcal{B}} - m > 0$ and $\frac{\partial D_2(p_1(t), p_2^{\mathcal{C}})}{\partial p_1(t)} > 0$ (see equation 1). Using (i)—(iii) in (8) we get $\pi_2^{\mathcal{B}} > \pi_2^{\mathcal{C}}$.

Proof of Part (B): From Theorem 1 we know $p_1^{\mathcal{C}} < p_1^{\mathcal{B}}$. Moreover, if $p_2^{\mathcal{C}} \leq p_2^{\mathcal{B}}$ then it is straightforward to show $CS^{\mathcal{B}} < CS^{\mathcal{C}}$ using Lemma 1 for the function $\tilde{CS}(p)$ (by considering the price vectors $p^{\mathcal{B}}$ and $p^{\mathcal{C}}$) and by noting that $\frac{\partial \tilde{CS}(p)}{\partial p_i} < 0$ for all $i = 1, 2$.

Proof of Part (C): Note that $W^{\mathcal{B}} \equiv \tilde{W}(p^{\mathcal{B}})$ and $W^{\mathcal{C}} \equiv W(q^{\mathcal{C}}) = \tilde{W}(p^{\mathcal{C}})$. By writing

$$W^{\mathcal{B}} - W^{\mathcal{C}} = \left[\tilde{W}(p^{\mathcal{B}}) - \tilde{W}(p_1^{\mathcal{C}}, p_2^{\mathcal{B}})\right] + \left[\tilde{W}(p_1^{\mathcal{C}}, p_2^{\mathcal{B}}) - \tilde{W}(p^{\mathcal{C}})\right],$$

and then using Lemma 1 we get

$$W^{\mathcal{B}} - W^{\mathcal{C}} = \tilde{W}(p^{\mathcal{B}}) - \tilde{W}(p_1^{\mathcal{C}}, p_2^{\mathcal{B}}) + (p_2^{\mathcal{B}} - p_2^{\mathcal{C}}) \int_0^1 \frac{\partial \tilde{W}(p_1^{\mathcal{C}}, p_2(t))}{\partial p_2(t)} dt. \tag{9}$$

Since $p_1^{\mathcal{B}}(> m)$ is the best response to $p_2^{\mathcal{B}}$ and $p_1^{\mathcal{C}}(= m)$ is not, we have that $\tilde{W}(p^{\mathcal{B}}) > \tilde{W}(p_1^{\mathcal{C}}, p_2^{\mathcal{B}})$. Note that for all $t \in (0, 1)$,

$$\begin{aligned} \frac{\partial \tilde{W}(p_1^{\mathcal{C}}, p_2(t))}{\partial p_2(t)} &= (p_1^{\mathcal{C}} - m)\frac{\partial D_1(p_1^{\mathcal{C}}, p_2(t))}{\partial p_2(t)} + (p_2(t) - m)\frac{\partial D_2(p_1^{\mathcal{C}}, p_2(t))}{\partial p_2(t)} \\ &= (p_2(t) - m)\frac{\partial D_2(p_1^{\mathcal{C}}, p_2(t))}{\partial p_2(t)} \\ &< 0, \end{aligned}$$

where the second equality follows from using $p_1^{\mathcal{C}} = m$ and the inequality follows from applying (1) and the fact that $p_2(t) \geq \min\{p_2(1), p_2(0)\} \equiv \min\{p_2^{\mathcal{B}}, p_2^{\mathcal{C}}\} > m$. Thus, if $p_2^{\mathcal{B}} \leq p_2^{\mathcal{C}}$, then $(p_2^{\mathcal{B}} - p_2^{\mathcal{C}}) \int_0^1 \frac{\partial \tilde{W}(p_1^{\mathcal{C}}, p_2(t))}{\partial p_2(t)} dt > 0$, which, together with $\tilde{W}(p^{\mathcal{B}}) > \tilde{W}(p_1^{\mathcal{C}}, p_2^{\mathcal{B}})$, gives the result. □

Theorem 3 has three parts. Part (A) shows an unambiguous profit ranking of each firm under Bertrand and Cournot outcomes. From the price ranking for the public firm ($p_1^{\mathcal{B}} > p_1^{\mathcal{C}} = m$) it is obvious that $\pi_1^{\mathcal{B}} > \pi_1^{\mathcal{C}}$. Surprisingly, despite the ambiguity in ranking between $p_2^{\mathcal{B}}$ and $p_2^{\mathcal{C}}$, we get $\pi_2^{\mathcal{B}} > \pi_2^{\mathcal{C}}$. We show this by making use of (i) the Nash equilibrium argument that $\tilde{\pi}_2(p^{\mathcal{B}}) \geq \tilde{\pi}_2(p_1^{\mathcal{B}}, p_2^{\mathcal{C}})$ and (ii) $\frac{\partial \tilde{\pi}_2(p_1(t), p_2^{\mathcal{C}})}{\partial p_1(t)} > 0$. Neither (i) nor (ii) depend on the sign of $p_2^{\mathcal{B}} - p_2^{\mathcal{C}}$. Theorem 3 (B) is easy to understand. Theorem 3 (C) states that if the private firm's price in Cournot is at least as high as that in Bertrand, then welfare reversal cannot occur. This is a strong result since one would expect that if $p_1^{\mathcal{C}} < p_1^{\mathcal{B}}$ (as in Theorem 1) and $p_2^{\mathcal{C}} \geq p_2^{\mathcal{B}}$, then there is a possibility of welfare reversal. Theorem 3 (C) rules out such a possibility.

Remark 1. The results obtained in Theorem 3 are not necessarily restricted to a duopoly. Consider, for example, the welfare comparison in part (C). In particular, let us consider a $n(\geq 2)$-firm oligopoly with n_1 public firms and $n_2(\equiv n - n_1)$ private firms. Assume $1 \leq n_1 < n$. Label these firms from 1 to n such that firms labeled 1 to n_1 are public while firms labeled n_1+1 to n are private. Let $G_1 = \{1, 2, ..., n_1\}$ and $G_2 = \{n_1+1, n_1+2, \ldots, n\}$ denote the group of public firms and the group of private firms respectively. Assume that an interior Bertrand equilibrium exists. Then there exists a Bertrand equilibrium with the following property: $p_i^{\mathcal{B}} = p_1^{\mathcal{B}}$ if $i \in G_1$, and $p_i^{\mathcal{B}} = p_2^{\mathcal{B}}$ if $i \in G_2$. Similarly, an interior Cournot equilibrium exists in which the following holds: $p_i^{\mathcal{C}} = m$ if $i \in G_1$, and $p_i^{\mathcal{C}} = p_2^{\mathcal{C}}$ if

$i \in G_2$. Then we find: $p_2^{\mathcal{C}} \geq p_2^{\mathcal{B}} \Rightarrow W^{\mathcal{B}} > W^{\mathcal{C}}$ provided $\sum_{j \in G_2} \frac{\partial D_i(\mathbf{m},\mathbf{p})}{\partial p_j} < 0$ for all $i \in G_2$, where $\mathbf{m}$ is a n_1-element vector with all elements m and $\mathbf{p}$ is a n_2-element vector with all elements p. Similar conditions are invoked in the differentiated oligopoly literature to ensure that own-price effects dominate the cross-price effects. See, for example, condition (A.3) in Vives [22] or pp. 157 in Vives [23].

To proceed further, that is, to provide a more precise comparison between Bertrand and Cournot for consumer surplus (or welfare), we need to compute the private firm's price which in turn requires us to assume specific utility functions. To this end, we consider a quadratic utility specification which gives a linear demand system. A comprehensive comparison between Bertrand and Cournot outcomes for linear demand systems is provided in the next subsection.

3.4. **Linear Demand.** Consider the quadratic utility specification proposed in Dixit [9] and subsequently used in Singh and Vives [19], Qiu[17], Häckner [12] and several other papers in this literature:

$$U(q) = a(q_1 + q_2) - \frac{1}{2}(q_1^2 + q_2^2) - bq_1q_2, \tag{10}$$

where $a > c$, and $b \in (0, 1)$. The goods are independent if $b = 0$ and perfect substitutes if $b = 1$. The restriction that b lie strictly between 0 and 1 implies that the goods are imperfect substitutes. The degree of substitutability increases, or equivalently, the extent of product differentiation declines, as b increases. It is easy to verify that utility specification in (10) satisfies Assumptions 1—3.

The inverse demands corresponding to (10) are linear and given by

$$p_1 = a - q_1 - bq_2, \quad p_2 = a - q_2 - bq_1. \tag{11}$$

Inverting the inverse demands yields the direct demands:

$$q_1 = \frac{a(1-b) - p_1 + bp_2}{1 - b^2}, \quad q_2 = \frac{a(1-b) - p_2 + bp_1}{1 - b^2}. \tag{12}$$

Proposition 1. Suppose $U(q)$ is given by (10). Then for all $b \in (0, 1)$ we have

(A) $p_1^{\mathcal{C}} < p_1^{\mathcal{B}}$, $p_2^{\mathcal{C}} = p_2^{\mathcal{B}}$.
(B) $q_1^{\mathcal{C}} > q_1^{\mathcal{B}}, q_2^{\mathcal{C}} < q_2^{\mathcal{B}}; q_1^{\mathcal{C}} + q_2^{\mathcal{C}} > q_1^{\mathcal{B}} + q_2^{\mathcal{B}}$.
(C) $\pi_i^{\mathcal{C}} < \pi_i^{\mathcal{B}}$, $i = \{1, 2\}$.
(D) $CS^{\mathcal{C}} > CS^{\mathcal{B}}$.
(E) $W^{\mathcal{C}} < W^{\mathcal{B}}$.

Sketch of the Proof: By Theorem 1, $p_1^{\mathcal{C}} < p_1^{\mathcal{B}}$. By Theorem 2, $q_1^{\mathcal{C}} > q_1^{\mathcal{B}}$ and $q_2^{\mathcal{C}} < q_2^{\mathcal{B}}$. That $\pi_i^{\mathcal{C}} < \pi_i^{\mathcal{B}}$, $i = \{1, 2\}$ follows from Theorem 3(A). Now suppose $p_2^{\mathcal{C}} = p_2^{\mathcal{B}}$ (as claimed in part (A)). Then parts (D) and (E) immediately follow from Theorem 3(B) and 3(C) respectively. Thus the only claims left to prove are (i) $q_1^{\mathcal{C}} + q_2^{\mathcal{C}} > q_1^{\mathcal{B}} + q_2^{\mathcal{B}}$ and (ii) $p_2^{\mathcal{C}} = p_2^{\mathcal{B}}$. Proof of these claims follow from routine computation and are provided in the Appendix.

Needless to say, the equality, that is, $p_2^{\mathcal{C}} = p_2^{\mathcal{B}}$, is unlikely to hold for an arbitrary number of firms.[5] Underlying the equality, however, are two opposing effects which are quite general. On the one hand, as in the standard setting, a firm's perceived elasticity of demand is smaller under the Cournot set-up which raises $p_2^{\mathcal{C}}$.[6] On the other hand, a lower $p_1^{\mathcal{C}}$ (compared to $p_1^{\mathcal{B}}$) in our framework creates a downward pressure on the private firm's price, which lowers $p_2^{\mathcal{C}}$. When $U(q)$ is given by (10) these two effects offset each other, which in turn yields $p_2^{\mathcal{C}} = p_2^{\mathcal{B}}$.

For $n > 2$, we find a wide range of parameter values for which $p_2^{\mathcal{C}} < p_2^{\mathcal{B}}$. Together with Theorem 1, this finding implies that for a range of parameter values, all prices, public as well as private, are strictly lower under Cournot. See Remark 2 for details.

Remark 2. Consider an n-firm oligopoly with n_1 welfare-maximizing public firms and $n_2 \equiv n - n_1$ private firms. Then generalize the utility function as follows:

$$U(q) = a\sum_{i=1}^{n} q_i - \frac{1}{2}\sum_{i=1}^{n} q_i^2 - b\sum_{i}\sum_{j<i} q_i q_j, \tag{13}$$

where $a > m$ and $b \in (0,1)$ and each firm $i = 1, 2, \ldots, n$ produces exactly one variety. The equilibrium prices in Bertrand and Cournot set-ups exhibit within-group symmetry as mentioned in Remark 1. Let $p_1^{\mathcal{C}}(p_1^{\mathcal{B}})$ and $p_2^{\mathcal{C}}(p_2^{\mathcal{B}})$ denote the equilibrium prices charged by a public firm and a private firm respectively in Cournot (Bertrand) competition. Similarly, let $q_1^{\mathcal{C}}(q_1^{\mathcal{B}})$ and $q_2^{\mathcal{C}}(q_2^{\mathcal{B}})$ denote the equilibrium quantities produced by the public and private firm respectively in Cournot (Bertrand). We find:

(a) $p_1^{\mathcal{C}} = m < p_1^{\mathcal{B}}$, $\text{sgn}[p_2^{\mathcal{C}} - p_2^{\mathcal{B}}] = \text{sgn}[(n_2 - 1)(b(n-1) - (n_1 - 1))]$,
(b) $q_1^{\mathcal{C}} > q_1^{\mathcal{B}}$, $q_2^{\mathcal{C}} < q_2^{\mathcal{B}}$.

The output comparisons are the same as in Theorem 2. Regarding price comparison, note that we get $p_2^{\mathcal{C}} = p_2^{\mathcal{B}}$ as in Proposition 1 (A), if $n_2 = 1$. If $n_2 > 1$, $p_2^{\mathcal{C}} < p_2^{\mathcal{B}}$ for $b \in (0, \frac{n_1 - 1}{n-1})$. This interval is non-empty for all $n_1 > 1$.

That profits could be lower under Cournot competition has also been shown in Häckner [12]. However Häckner's findings relied on the presence of the following features: the presence of strictly more than two firms and cost/quality asymmetry. None of these features are present in our framework. In our framework it is the presence of public firms

[5]See the finding (a) in Remark 2 though, which implies that irrespective of the number of public firms, a private firm's price is the same in Cournot and Bertrand if there is only one private firm.
[6]See Proposition 6.1 in Vives [23] for a comparison of elasticities under Bertrand and Cournot.

that leads to the reversal of the profit and consumer surplus orderings (between Cournot and Bertrand competition).[7] These reversals also hold when the number of firms exceeds two for a range of $b \in (0,1)$.

The fact that welfare reversal does not occur under linear demand structure is surprising, since Cournot seems to be more competitive according to several other indicators of competition.[8] For example, all prices are weakly lower in a Cournot duopoly. Also, from Proposition 1 (B) we know that the aggregate output is higher under Cournot, i.e., $q_1^{\mathcal{C}} + q_2^{\mathcal{C}} > q_1^{\mathcal{B}} + q_2^{\mathcal{B}}$. Concerning welfare ordering in Proposition 1 (E) we find that if $U(q)$ is given by (10), $W(q) = s(q_1+q_2)+d(q_1-q_2)$, where $s(q_1+q_2) = (a-m)(q_1+q_2)-\frac{1}{2}(q_1+q_2)^2$ and $d(q_1-q_2) = -\frac{(1-b)}{4}(q_1-q_2)^2$. Though $s'(.) > 0$, $d'(.) < 0$. That is, while an increase in aggregate output increases welfare, an increase in output differences between the two firms decreases welfare (since both varieties enter symmetrically into the utility function). Compared to Bertrand, $q_1 + q_2$ is higher in Cournot but $q_1 - q_2$ is higher as well. It turns out that the latter effect dominates, preserving the standard welfare ordering.

4. Bertrand versus Cournot in the Presence of Partial Privatization

In our comparison between Bertrand and Cournot outcomes, we have so far assumed that firm 1 (i.e., the public firm) maximizes welfare. The assumption is not strictly necessary to obtain reversals. Our results hold even when firm 1 is partially privatized. To capture partial privatization, we modify firm 1's objective function as follows. Firm 1 maximizes $R_1(q;\theta) \equiv \theta\pi_1(q) + (1-\theta)W(q)$ under Cournot and $\tilde{R}_1(p,\theta) = \theta\tilde{\pi}_1(p) + (1-\theta)\tilde{W}(p)$ under Bertrand where $\theta \in [0,1]$, and $\pi_1(q)$,$W(q)$,$\tilde{\pi}_1(p)$ and $\tilde{W}(p)$ are as defined in the previous section.[9] For a welfare-maximizing public firm, $\theta = 0$. If $\theta = 1$, the public firm is fully privatized whereas if $\theta \in (0,1)$, the public firm is partially privatized. It is easy to show that there exists $\tilde{\theta} > 0$ such that Proposition 1 (C) and (D), i.e., $\pi_i^{\mathcal{C}} < \pi_i^{\mathcal{B}}$ for $i = 1,2$ and $CS^{\mathcal{C}} > CS^{\mathcal{B}}$, holds for $\theta < \tilde{\theta}$.

Although the above discussion suggests that the reversal of Bertrand-Cournot ordering can occur in the presence of a partially privatized public firm, a limitation is that the degree of privatization, captured by the parameter θ, is exogenous. Consequently, the comparison between Bertrand and Cournot implicitly assumes that the degree of privatization is the same under the two modes of competition. This is not satisfactory since the incentives for privatization are typically different for Bertrand and Cournot.

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[7] See López and Naylor [13] for reversal of profit ordering in a unionized oligopoly setting.

[8] Although there are parameterizations such that all prices, public as well as private, are lower under Cournot, welfare reversal does not occur under any of those parameterizations.

[9] Similar formulations exist in the mixed oligopoly literature with homogeneous goods. See Matsumura [14]. Also see Fershtman [11] for a related formulation.

To endogenize the degree of privatization we now construct a stylized two-stage game, where a welfare-maximizing government chooses $\theta \in [0,1]$ in stage 1, after which firms 1 and 2 compete in the product market in stage 2.[10]

4.1. **The Cournot game.** We consider a two-stage game. In stage 1, the social planner chooses $\theta \in [0,1]$ to maximize welfare. Given a stage 1 choice of $\theta \in [0,1]$, in stage 2, firm 1 chooses q_1 to maximize $R_1(q_1, q_2; \theta)$ and firm 2 chooses q_2 to maximize $\pi_2(q_1, q_2)$.

For any given $\theta \in [0,1]$, let $q^{\mathcal{C}}(\theta) = (q_1^{\mathcal{C}}(\theta), q_2^{\mathcal{C}}(\theta))$ denote output vector in stage 2 Cournot equilibrium. Then the following first order conditions must hold:

$$\frac{\partial R_1(q^{\mathcal{C}}(\theta);\theta)}{\partial q_1} = (p_1^{\mathcal{C}}(\theta) - m) + \theta q_1^{\mathcal{C}}(\theta)\frac{\partial P_1(q^{\mathcal{C}}(\theta))}{\partial q_1} = 0, \tag{14}$$

$$\frac{\partial \pi_2(q^{\mathcal{C}}(\theta))}{\partial q_2} = (p_2^{\mathcal{C}}(\theta) - m) + q_2^{\mathcal{C}}(\theta)\frac{\partial P_2(q^{\mathcal{C}}(\theta))}{\partial q_2} = 0, \tag{15}$$

where $p_i^{\mathcal{C}}(\theta) \equiv P_i(q^{\mathcal{C}}(\theta)), i = 1, 2$. Lemma 2 records the effect of privatization on outputs for later reference.

Lemma 2. Suppose $q_i^{\mathcal{C}}(\theta) > 0, i = 1, 2$. Then $\frac{\partial q_1^{\mathcal{C}}(\theta)}{\partial \theta} < 0$ and $\frac{\partial q_2^{\mathcal{C}}(\theta)}{\partial \theta} > 0$.

Proof: See Appendix.

When $\theta = 0$, the public firm's price equals marginal cost and hence the public firm earns zero profits. As θ increases, that is, as the weight attached to profits increases, the public firm finds it optimal to cut back production which raises its price above marginal cost. Lemma 2 says that indeed starting from any $\theta = \theta_0$, the public firm's output declines as θ increases. Since outputs are strategic substitutes, the private firm's output increases with an increase in θ.

Now consider the stage 1 choice of θ by a welfare-maximizing government. Define $W^{\mathcal{C}}(\theta) = W(q^{\mathcal{C}}(\theta))$. Using Lemma 2 it is straightforward to establish the following:

Proposition 2. Suppose $\theta^{\mathcal{C}}$ maximizes $W^{\mathcal{C}}(\theta)$. Then $\theta^{\mathcal{C}} > 0$.

Proof: Since $W(q)$ is continuous in q and $q^{\mathcal{C}}(\theta)$ is continuous in θ, $W^{\mathcal{C}}(\theta)$ is continuous in θ over the compact interval $\theta \in [0,1]$. Therefore, there exists $\theta^{\mathcal{C}} \in [0,1]$ such that $W^{\mathcal{C}}(\theta)$ attains its maximum at $\theta = \theta^{\mathcal{C}}$. Differentiating $W^{\mathcal{C}}(\theta)$ with respect to θ yields:

$$\frac{dW^{\mathcal{C}}(\theta)}{d\theta} = (p_1^{\mathcal{C}}(\theta) - m)\left(\frac{\partial q_1^{\mathcal{C}}(\theta)}{\partial \theta}\right) + (p_2^{\mathcal{C}}(\theta) - m)\left(\frac{\partial q_2^{\mathcal{C}}(\theta)}{\partial \theta}\right). \tag{16}$$

[10]Using the parameter θ to capture the degree of privatization is simplistic. Nevertheless, this is in line with the mixed oligopoly literature, which also uses the importance of the profit motive of the public firm to capture the degree of privatization.

We have $p_1^{\mathcal{C}}(0) - m = 0$, $p_2^{\mathcal{C}}(0) - m > 0$, and by Lemma 2, $\frac{\partial q_2^{\mathcal{C}}(0)}{\partial \theta} > 0$. Then it follows from (16) that $\frac{dW^{\mathcal{C}}(0)}{d\theta} > 0$, which in turn implies $\theta^{\mathcal{C}} > 0$. □

The intuition for Proposition 2 is simple. Consider an infinitesimally small increase in θ from $\theta = 0$. As the weight on its own profits increases, the public firm lowers its output, q_1, while the rival firm raises its output, q_2. The welfare loss from a reduction in q_1 is second order since $p_1^{\mathcal{C}}(0) - m = 0$ while the welfare gain from an increase in q_2 is first order since $p_2^{\mathcal{C}}(0) - m > 0$. This implies that there always exists a certain degree of privatization, which improves welfare when the second stage game is Cournot.

Note that although government maximizes welfare, it instructs the public firm to maximize something different: a weighted sum of its profits and welfare. This idea is familiar from the strategic delegation literature where managers are given incentives (by owners) to maximize a weighted sum of profits and sales even though the owners only care about profits (see, for example, Vickers [21], Fershtman and Judd [10], and Sklivas [20]). By assigning a strictly positive weight to sales in managers' incentive contracts, the profit-maximizing owner credibly commits to a *higher* output which in turn raises profits. Similarly, by assigning a strictly positive weight to profits in the public firm's objective function, the welfare-maximizing government credibly commits to a *lower* output (to be produced by the public firm) which in our framework raises welfare by partially correcting the underproduction by the private firm.

In a Cournot duopoly with homogenous products, Matsumura [14] also found that partial privatization can improve welfare. The key driving force behind Matsumura's finding as well as ours is that the outputs are strategic substitutes. For partial privatization to be strictly welfare improving in a homogeneous goods setting, either the public firm has to be relatively inefficient or the marginal cost has to be strictly increasing in output. In our framework, product differentiation alone, no matter how small, is sufficient to generate welfare-improving partial privatization under Cournot competition.

4.2. **The Bertrand game.** In stage 1, the government chooses $\theta \in [0,1]$ to maximize $\tilde{W}(p)$. In stage 2, firm 1 chooses p_1 to maximize $\tilde{R}_1(p_1, p_2; \theta) \equiv \theta\tilde{\pi}_1(p) + (1-\theta)\tilde{W}(p)$ and firm 2 chooses p_2 to maximize $\tilde{\pi}_2(p_1, p_2)$.

For any given $\theta \in [0,1]$, let $p^{\mathcal{B}}(\theta) = (p_1^{\mathcal{B}}(\theta), p_2^{\mathcal{B}}(\theta))$ denote the price vector in a stage 2 Bertrand equilibrium. Then the following first order conditions must hold:

$$\frac{\partial \tilde{R}_1(p^{\mathcal{B}}(\theta);\theta)}{\partial p_1} = (p_1^{\mathcal{B}}(\theta) - m)\frac{\partial D_1(p^{\mathcal{B}}(\theta))}{\partial p_1} + \theta q_1^{\mathcal{B}}(\theta) + (1-\theta)(p_2^{\mathcal{B}}(\theta) - m)\frac{\partial D_2(p^{\mathcal{B}}(\theta))}{\partial p_1} = 0, \tag{17}$$

$$\frac{\partial \tilde{\pi}_2(p^{\mathcal{B}}(\theta))}{\partial p_2} = (p_2^{\mathcal{B}}(\theta) - m)\frac{\partial D_2(p^{\mathcal{B}}(\theta))}{\partial p_2} + q_2^{\mathcal{B}}(\theta) = 0, \tag{18}$$

where $q_i^{\mathcal{B}}(\theta) \equiv D_i(p^{\mathcal{B}}(\theta))$, $i = 1, 2$.

Consider an infinitesimally small increase in θ from $\theta = 0$. Recall that in the Cournot game, introduction of this profit motive induced the public firm to reduce output. Similarly, here, suppose the profit motive induces the public firm to raise its price, p_1. Then the private firm's price, p_2, increases as well since prices are strategic complements (Assumption 3). Given $p_i^{\mathcal{B}}(0) > m$ for both $i = 1, 2$, a further increase in prices triggered by partial privatization reduces welfare in Bertrand competition. While this conclusion seems natural and holds under linear demand, note that we started with the supposition that the introduction of the profit motive induces the public firm to raise its price, p_1, above $p_1^{\mathcal{B}}(0)$. Assumptions 1—3 do not guarantee that. A sufficient condition for the supposition to hold is that the public firm produces more than the private firm in the absence of privatization. Lemma 3 and Proposition 3 summarize our discussion.

Lemma 3. Suppose $q_i^{\mathcal{B}}(0) > 0$ and furthermore $q_1^{\mathcal{B}}(0) > q_2^{\mathcal{B}}(0)$. Then $\frac{\partial p_i^{\mathcal{B}}(0)}{\partial \theta} > 0$, $i = 1, 2$.

Proof: See Appendix.

Proposition 3. Define $W^{\mathcal{B}}(\theta) = \tilde{W}(p^{\mathcal{B}}(\theta)) = W(q^{\mathcal{B}}(\theta))$. Now, suppose $q_i^{\mathcal{B}}(0) > 0$. Then $\frac{dW^{\mathcal{B}}(0)}{d\theta} < 0$ if $q_1^{\mathcal{B}}(0) > q_2^{\mathcal{B}}(0)$.

Proof: Differentiating $\tilde{W}(p^{\mathcal{B}}(\theta))$ with respect to θ and evaluating at $\theta = 0$ gives

$$\frac{d\tilde{W}(p^{\mathcal{B}}(0))}{d\theta} = \frac{\partial \tilde{W}(p^{\mathcal{B}}(0))}{\partial p_1} \frac{\partial p_1^{\mathcal{B}}(0)}{\partial \theta} + \frac{\partial \tilde{W}(p^{\mathcal{B}}(0))}{\partial p_2} \frac{\partial p_2^{\mathcal{B}}(0)}{\partial \theta}. \tag{19}$$

From first order conditions we get $\frac{\partial \tilde{W}(p^{\mathcal{B}}(0))}{\partial p_1} = 0$. Also, if $q_1^{\mathcal{B}}(0) > q_2^{\mathcal{B}}(0)$, $\frac{\partial p_2^{\mathcal{B}}(0)}{\partial \theta} > 0$ (Lemma 3). Hence it suffices to show that $\frac{\partial \tilde{W}(p^{\mathcal{B}}(0))}{\partial p_2} < 0$. We have

$$\frac{\partial \tilde{W}(p^{\mathcal{B}}(0))}{\partial p_2} = (p_1^{\mathcal{B}}(0) - m)\frac{\partial D_1(p^{\mathcal{B}}(0))}{\partial p_2} + (p_2^{\mathcal{B}}(0) - m)\frac{\partial D_2(p^{\mathcal{B}}(0))}{\partial p_2}. \tag{20}$$

From $\frac{\partial \tilde{W}(p^{\mathcal{B}}(0))}{\partial p_1} = (p_1^{\mathcal{B}}(0) - m)\frac{\partial D_1(p^{\mathcal{B}}(0))}{\partial p_1} + (p_2^{\mathcal{B}}(0) - m)\frac{\partial D_2(p^{\mathcal{B}}(0))}{\partial p_1} = 0$, we get $(p_1^{\mathcal{B}}(0) - m) = -\frac{(p_2^{\mathcal{B}}(0) - m)\frac{\partial D_2(p^{\mathcal{B}}(0))}{\partial p_1}}{\frac{\partial D_1(p^{\mathcal{B}}(0))}{\partial p_1}}$. Substituting this in (20), using (1), and then simplifying further gives $\frac{\partial \tilde{W}(p^{\mathcal{B}}(0))}{\partial p_2} = \frac{p_2^{\mathcal{B}}(0) - m}{U_{22}(q^{\mathcal{B}}(0))}$. Since $p_2^{\mathcal{B}}(0) - m > 0$ and $U_{22} < 0$, it follows that $\frac{\partial \tilde{W}(p^{\mathcal{B}}(0))}{\partial p_2} < 0$. □

Proposition 3 posits that small increments in θ from $\theta = 0$ reduce welfare under Bertrand competition. But what about large increments? For a linear demand system we find that they are not welfare improving either. Using CES preferences, Anderson et al. [2] have shown that full privatization, that is, a change from $\theta = 0$ to $\theta = 1$, reduces welfare in Bertrand competition. Using the standard quadratic utility specification, Proposition 4 shows that not only full privatization, but no extent of privatization can improve welfare in Bertrand competition.

4.3. **Linear demand:** Before stating our finding more formally, let us define $\theta^{\mathcal{B}}$ to be the value of $\theta \in [0,1]$ that maximizes $\tilde{W}^{\mathcal{B}}(\theta)$. Recall $\theta^{\mathcal{C}}$ denotes the optimal degree of privatization under Cournot. Proposition 4 below compares $\theta^{\mathcal{B}}$ and $\theta^{\mathcal{C}}$ for linear demand.

Proposition 4. Suppose $U(q)$ is given by (10). Then $\theta^{\mathcal{C}} \in (0,1)$ while $\theta^{\mathcal{B}} = 0$.

Proof: See Appendix.

Finally, we turn to the comparison between Bertrand and Cournot outcomes in this two-stage game with an endogenous degree of privatization. Define $\pi_i^{\mathcal{C}}(\theta) \equiv \pi_i(q^{\mathcal{C}}(\theta))$,$\pi_i^{\mathcal{B}}(\theta) \equiv \tilde{\pi}_i(p^{\mathcal{B}}(\theta)) \equiv \pi_i(q^{\mathcal{B}}(\theta)), CS^{\mathcal{C}}(\theta) \equiv CS(q^{\mathcal{C}}(\theta))$, and $CS^{\mathcal{B}}(\theta) \equiv CS(q^{\mathcal{B}}(\theta))$. As before, we find that the public firm's price (quantity) is strictly higher (lower) in Bertrand competition. While the private firm's price is now higher in Cournot, we still find a reversal of standard Bertrand-Cournot ordering for consumer surplus and profits under a range of parameterizations.

Proposition 5. Suppose $U(q)$ is given by (10). Then

(A) $\theta^{\mathcal{C}} = \frac{b(1-b)}{(4-3b)} > 0, \theta^{\mathcal{B}} = 0$;
(B) $p_1^{\mathcal{C}}(\theta^{\mathcal{C}}) < p_1^{\mathcal{B}}(\theta^{\mathcal{B}})$, $p_2^{\mathcal{C}}(\theta^{\mathcal{C}}) > p_2^{\mathcal{B}}(\theta^{\mathcal{B}})$;
(C) $q_1^{\mathcal{C}}(\theta^{\mathcal{C}}) > q_1^{\mathcal{B}}(\theta^{\mathcal{B}}), q_2^{\mathcal{C}}(\theta^{\mathcal{C}}) < q_2^{\mathcal{B}}(\theta^{\mathcal{B}})$;
(D) $\pi_2^{\mathcal{C}}(\theta^{\mathcal{C}}) < \pi_2^{\mathcal{B}}(\theta^{\mathcal{B}})$;
(E) $\pi_1^{\mathcal{C}}(\theta^{\mathcal{C}}) < \pi_1^{\mathcal{B}}(\theta^{\mathcal{B}})$ and $CS^{\mathcal{C}}(\theta^{\mathcal{C}}) > CS^{\mathcal{B}}(\theta^{\mathcal{B}})$ if $b \in (0, 0.84)$, and finally
(F) $W^{\mathcal{C}}(\theta^{\mathcal{C}}) < W^{\mathcal{B}}(\theta^{\mathcal{B}})$.

Proof: See Appendix.

5. Summary and Concluding Remarks

The paper provides a first systematic and comprehensive comparison between Bertrand and Cournot outcomes in *mixed markets* where profit-maximizing private firms coexist with public firms. The results, are strikingly different, often opposite to the ones obtained from similar comparison in the standard setting with only profit maximizing firms. The standard Bertrand-Cournot ranking is reversed for the public firm's price and quantity. On the other hand, ranking reversal never occurs for the private firm's quantity. The private firm's price can be higher or lower in Cournot. In contrast to the findings in the standard setting we find that both firms earn strictly lower profits in Cournot. In addition, consumer surplus is strictly higher in Cournot competition if the demand is linear. These results hold under a richer set-up with a partially privatized public firm, where the extent of privatization is endogenously determined by a welfare-maximizing government. It is also shown that partial privatization can have different welfare implications for Bertrand and Cournot competition. In particular, partial privatization (to a certain extent) always improves welfare under Cournot competition but not necessarily so under Bertrand competition.

Note that throughout the analysis we have assumed that the two goods are imperfect substitutes as this is the predominant case considered in the literature. If the two goods are complements, however, then the ranking reversals are unlikely. As before, the public firm's price equals marginal cost in the case of Cournot competition. However, we can no longer claim that the public firm's price is strictly lower in Cournot, since the public firm's price is either equal to or strictly less than marginal cost under Bertrand competition. To see why, consider an infinitesimally small decline in the public firm's price from marginal cost. This increases the public firm's output and since the two goods are complements, the private firm's output increases as well. As there is an underproduction of the private good, the increase in the private firm's output generates a first order welfare gain. Therefore, in Bertrand competition, a welfare-maximizing public firm will set its price strictly lower than marginal cost if it is allowed to make losses, and equal to marginal cost otherwise.

6. Appendix

Proof of Proposition 1: If $U(q)$ is given by (10), then under Cournot competition, the following first order conditions hold at $(q_1, q_2) = (q_1^{\mathcal{C}}, q_2^{\mathcal{C}})$:

$$\frac{\partial W}{\partial q_1} = a - q_1 - bq_2 - m = 0, \qquad \frac{\partial \pi_2}{\partial q_2} = a - 2q_2 - bq_1 - m = 0.$$

Solving the two first order conditions gives

$$q_1^{\mathcal{C}} = \frac{(2-b)(a-m)}{2-b^2}, \qquad q_2^{\mathcal{C}} = \frac{(1-b)(a-m)}{2-b^2}.$$

Substituting q_i by $q_i^{\mathcal{C}}$, $i = 1, 2$ in (11) yields equilibrium prices under Cournot:

$$p_1^{\mathcal{C}} = m, \qquad p_2^{\mathcal{C}} = m + \frac{(1-b)(a-m)}{2-b^2}.$$

The first order conditions under Bertrand competition are given by (3) and (4). Substituting $\frac{\partial D_i}{\partial p_i} = \frac{-1}{1-b^2}$ and $\frac{\partial D_i}{\partial p_j} = \frac{b}{1-b^2}$ $(i \neq j)$ and then solving (3) and (4) gives

$$p_1^{\mathcal{B}} = m + \frac{b(1-b)(a-m)}{2-b^2}, \qquad p_2^{\mathcal{B}} = m + \frac{(1-b)(a-m)}{2-b^2}.$$

Observe that $p_2^{\mathcal{C}} = p_2^{\mathcal{B}} = m + \frac{(1-b)(a-m)}{2-b^2}$. Substituting p_i by $p_i^{\mathcal{B}}$, $i = 1, 2$ in (12) yields equilibrium quantities under Bertrand:

$$q_1^{\mathcal{B}} = \frac{a-m}{1+b}, \qquad q_2^{\mathcal{B}} = \frac{(a-m)}{(1+b)(2-b^2)}.$$

Then routine calculation gives $(q_1^{\mathcal{C}} + q_2^{\mathcal{C}}) - (q_1^{\mathcal{B}} + q_2^{\mathcal{B}}) = \frac{2b(a-m)}{(1+b)(2-b^2)} > 0$. □

Proof of Lemma 2: Totally differentiating (14) and (15) with respect to θ and then solving for $\frac{\partial q_1^{\mathcal{C}}(\theta)}{\partial\theta}$ and $\frac{\partial q_2^{\mathcal{C}}(\theta)}{\partial\theta}$ we get

$$\frac{\partial q_1^{\mathcal{C}}(\theta)}{\partial\theta} = \frac{-q_1^{\mathcal{C}}(\theta)\left(\frac{\partial P_1(q^{\mathcal{C}}(\theta))}{\partial q_1}\right)\left(\frac{\partial^2\pi_2(q^{\mathcal{C}}(\theta))}{\partial q_2^2}\right)}{\Delta}, \tag{21}$$

$$\frac{\partial q_2^{\mathcal{C}}(\theta)}{\partial\theta} = \frac{q_1^{\mathcal{C}}(\theta)\left(\frac{\partial P_1(q^{\mathcal{C}}(\theta))}{\partial q_1}\right)\left(\frac{\partial^2\pi_2(q^{\mathcal{C}}(\theta))}{\partial q_2\partial q_1}\right)}{\Delta}. \tag{22}$$

where $\Delta = \left(\frac{\partial^2 R_1(q^{\mathcal{C}}(\theta);\theta)}{\partial q_1^2}\right)\left(\frac{\partial^2\pi_2(q^{\mathcal{C}}(\theta))}{\partial q_2^2}\right) - \left(\frac{\partial^2 R_1(q^{\mathcal{C}}(\theta);\theta)}{\partial q_2\partial q_1}\right)\left(\frac{\partial^2\pi_2(q^{\mathcal{C}}(\theta))}{\partial q_2\partial q_1}\right)$. The following second order conditions must hold: $\frac{\partial^2\pi_2(q^{\mathcal{C}}(\theta))}{\partial q_2^2} < 0$ and $\Delta > 0$. Then the result follows from noting that $q_1^{\mathcal{C}}(\theta) > 0$, $\frac{\partial P_1(q^{\mathcal{C}}(\theta))}{\partial q_1} = U_{11}(q^{\mathcal{C}}(\theta)) < 0$ and $\frac{\partial^2\pi_2(q^{\mathcal{C}}(\theta))}{\partial q_1\partial q_2} < 0$ (Assumption 2 (i)). □

Proof of Lemma 3: Totally differentiating conditions (17) and (18) with respect to θ and then solving for $\frac{\partial p_i^{\mathcal{B}}(\theta)}{\partial\theta}$, we get

$$\frac{\partial p_1^{\mathcal{B}}(\theta)}{\partial\theta} = \frac{\left[(p_2^{\mathcal{B}}(\theta)-m)\frac{\partial D_2(p^{\mathcal{B}}(\theta))}{\partial p_1} - q_1^{\mathcal{B}}(\theta)\right]\left(\frac{\partial^2\tilde{\pi}_2(p^{\mathcal{B}}(\theta))}{\partial p_2^2}\right)}{\tilde{\Delta}} \tag{23}$$

$$\frac{\partial p_2^{\mathcal{B}}(\theta)}{\partial\theta} = \frac{-\left[(p_2^{\mathcal{B}}(\theta)-m)\frac{\partial D_2(p^{\mathcal{B}}(\theta))}{\partial p_1} - q_1^{\mathcal{B}}(\theta)\right]\left(\frac{\partial^2\tilde{\pi}_2(p^{\mathcal{B}}(\theta))}{\partial p_2\partial p_1}\right)}{\tilde{\Delta}} \tag{24}$$

where $\tilde{\Delta} = \left(\frac{\partial^2\tilde{R}_1(p^{\mathcal{B}}(\theta);\theta)}{\partial p_1^2}\right)\left(\frac{\partial^2\tilde{\pi}_2(p^{\mathcal{B}}(\theta))}{\partial p_2^2}\right) - \left(\frac{\partial^2\tilde{R}_1(p^{\mathcal{B}}(\theta);\theta)}{\partial p_2\partial p_1}\right)\left(\frac{\partial^2\tilde{\pi}_2(p^{\mathcal{B}}(\theta))}{\partial p_2\partial p_1}\right)$. Substituting $p_2^{\mathcal{B}}(\theta) - m = \frac{q_2^{\mathcal{B}}(\theta)}{\left(-\frac{\partial D_2^{\mathcal{B}}(\theta)}{\partial p_2}\right)}$ (from (18)) and $\frac{\partial D_i(p^{\mathcal{B}}(\theta))}{\partial p_j} = \frac{U_{ij}(q^{\mathcal{B}}(\theta))}{U_{11}(q^{\mathcal{B}}(\theta))U_{22}(q^{\mathcal{B}}(\theta)) - U_{12}(q^{\mathcal{B}}(\theta))U_{21}(q^{\mathcal{B}}(\theta))}$ in (24) and then evaluating at $\theta = 0$ we get

$$\frac{\partial p_1^{\mathcal{B}}(0)}{\partial\theta} = \frac{q_2^{\mathcal{B}}(0)\left(\frac{q_1^{\mathcal{B}}(0)}{q_2^{\mathcal{B}}(0)} - \frac{U_{12}(q^{\mathcal{B}}(0))}{U_{22}(q^{\mathcal{B}}(0))}\right)\left(-\frac{\partial^2\tilde{\pi}_2(p^{\mathcal{B}}(0))}{\partial p_2^2}\right)}{\tilde{\Delta}}, \tag{25}$$

$$\frac{\partial p_2^{\mathcal{B}}(0)}{\partial\theta} = \frac{q_2^{\mathcal{B}}(0)\left(\frac{q_1^{\mathcal{B}}(0)}{q_2^{\mathcal{B}}(0)} - \frac{U_{12}(q^{\mathcal{B}}(0))}{U_{22}(q^{\mathcal{B}}(0))}\right)\left(\frac{\partial^2\tilde{\pi}_2(p^{\mathcal{B}}(0))}{\partial p_2\partial p_1}\right)}{\tilde{\Delta}}. \tag{26}$$

Note $q_2^{\mathcal{B}}(0) > 0$. By Assumption 3 (ii), $\frac{\partial^2\tilde{\pi}_2(p^{\mathcal{B}}(0))}{\partial p_2\partial p_1} > 0$. By Assumption 1 (v), $\frac{U_{12}(q^{\mathcal{B}}(0))}{U_{22}(q^{\mathcal{B}}(0))} < 1$. Since second-order conditions are satisfied at $p = p^{\mathcal{B}}(\theta)$, $\tilde{\Delta} > 0$ and $-\frac{\partial^2\tilde{\pi}_2(p^{\mathcal{B}}(0))}{\partial p_2^2} > 0$. Now, if $\frac{q_1^{\mathcal{B}}(0)}{q_2^{\mathcal{B}}(0)} > 1$ we have $\frac{q_1^{\mathcal{B}}(0)}{q_2^{\mathcal{B}}(0)} - \frac{U_{12}(q^{\mathcal{B}}(0))}{U_{22}(q^{\mathcal{B}}(0))} > 0$. Thus all expressions in the right-hand side of (25) and (26) are strictly positive which in turn implies $\frac{\partial p_i^{\mathcal{B}}(0)}{\partial\theta} > 0$, $i = 1, 2$. □

Proof of Proposition 4: Routine calculations show that, if $U(q)$ is given by (10), then for any given $\theta \in [0, 1]$, stage 2 Cournot equilibrium quantities are:

$$q_1^{\mathcal{C}}(\theta) = \frac{(2-b)(a-m)}{(2+2\theta-b^2)},\quad q_2^{\mathcal{C}}(\theta) = \frac{(1+\theta-b)(a-m)}{(2+2\theta-b^2)}.$$

Using $W^{\mathcal{C}}(\theta) = W(q^{\mathcal{C}}(\theta)) = U(q_1^{\mathcal{C}}(\theta), q_2^{\mathcal{C}}(\theta)) - m(q_1^{\mathcal{C}}(\theta) + q_2^{\mathcal{C}}(\theta))$ we get

$$W^{\mathcal{C}}(\theta) = \frac{[(7-6b-2b^2+2b^3)+\theta(14-10b)+3\theta^2](a-m)^2}{2(2+2\theta-b^2)^2}. \tag{27}$$

Existence of $\theta^{\mathcal{C}}$ follows from continuity of $W^{\mathcal{C}}(\theta)$ in θ over the compact interval $[0,1]$. Proposition 2 gives $\theta^{\mathcal{C}} > 0$. Since $\frac{dW^{\mathcal{C}}(1)}{d\theta} = -\frac{(a-m)^2}{(2+b)^3} < 0$, $\theta^{\mathcal{C}} < 1$. Thus $\theta^{\mathcal{C}} \in (0,1)$.

The Bertrand equilibrium prices, given any stage 1 choice of θ, are:

$$p_1^{\mathcal{B}}(\theta) = m + \frac{(1-b)(2\theta+b)(a-m)}{(2+2\theta-b^2)}, \quad p_2^{\mathcal{B}}(\theta) = m + \frac{(1-b)(1+\theta+b\theta)(a-m)}{(2+2\theta-b^2)}.$$

Then using $q_i^{\mathcal{B}}(\theta) = D_i(p^{\mathcal{B}}(\theta))$ and $W^{\mathcal{B}}(\theta) \equiv W^{\mathcal{B}}(q^{\mathcal{B}}(\theta)) = U(q_1^{\mathcal{B}}(\theta), q_2^{\mathcal{B}}(\theta)) - m(q_1^{\mathcal{B}}(\theta) + q_2^{\mathcal{B}}(\theta))$ we get

$$q_1^{\mathcal{B}}(\theta) = \frac{(2+b\theta-b^2(1-\theta))(a-m)}{(1+b)(2+2\theta-b^2)}, \quad q_2^{\mathcal{B}}(\theta) = \frac{(1+\theta+b\theta)(a-m)}{(1+b)(2+2\theta-b^2)},$$

and

$$W^{\mathcal{B}}(\theta) = \frac{f(\theta)(a-m)^2}{2(1+b)(2+2\theta-b^2)^2}, \tag{28}$$

where $f(\theta) = 7+b-7b^2-b^3+2b^4+\theta(14-4b^2-2b^4)+\theta^2(3+7b+b^2-3b^3)$. Existence of $\theta^{\mathcal{B}}$ follows from continuity of $W^{\mathcal{B}}(\theta)$ in θ over the compact interval [0,1]. Observe that $q_i^{\mathcal{B}}(0) > 0$, for $i=1,2$ and $q_1^{\mathcal{B}}(0) = \frac{a-m}{1+b} > \frac{a-m}{(1+b)(2-b^2)} = q_2^{\mathcal{B}}(0)$. Then applying Proposition 3 we get $\frac{dW^{\mathcal{B}}(0)}{d\theta} < 0$. Indeed $\frac{dW^{\mathcal{B}}(\theta)}{d\theta} = -\frac{(2+b)(1-b)^2(b(1+b)+\theta(4+3b))(a-m)^2}{(2+2\theta-b^2)^3} < 0$ for all $\theta \in [0,1]$, which implies $\theta^{\mathcal{B}} = 0$. □

Proof of Proposition 5: Differentiating (27) gives $\frac{dW^{\mathcal{C}}(\theta)}{d\theta} = -\frac{(2-b)(b(1-b)-\theta(4-3b))(a-m)^2}{(2+2\theta-b^2)^3}$ which equals zero at $\theta = \frac{b(1-b)}{(4-3b)}$. Since $\frac{d^2W^{\mathcal{C}}(\theta)}{d\theta^2} = -\frac{(2-b)(4+4b+3b^3-10b^2)(a-m)^2}{(2+2\theta-b^2)^4} < 0$ at $\theta = \frac{b(1-b)}{(4-3b)}$, and $\theta^{\mathcal{C}} \in (0,1)$ by Proposition 2, it follows that $W^{\mathcal{C}}(\theta)$ attains its maximum at $\theta = \frac{b(1-b)}{(4-3b)}$. Thus $\theta^{\mathcal{C}} = \frac{b(1-b)}{(4-3b)}$. By Proposition 4, $\theta^{\mathcal{B}} = 0$. This proves (A).

Using $\theta^{\mathcal{B}}$ and $\theta^{\mathcal{C}}$ it is quite easy to verify that $p_1^{\mathcal{B}}(\theta^{\mathcal{B}}) = m + \frac{b(1-b)(a-m)}{(2-b^2)}$, $p_2^{\mathcal{B}}(\theta^{\mathcal{B}}) = m + \frac{(1-b)(a-m)}{(2-b^2)}$, $p_1^{\mathcal{C}}(\theta^{\mathcal{C}}) = m + \frac{b(1-b)(a-m)}{(4-3b^2)}$ and $p_2^{\mathcal{C}}(\theta^{\mathcal{C}}) = m + \frac{2(1-b)(a-m)}{(4-3b^2)}$. Result (B) follows by noting that (a) $p_1^{\mathcal{B}}(\theta^{\mathcal{B}}) - p_1^{\mathcal{C}}(\theta^{\mathcal{C}}) = \frac{2b(1+b)(1-b)^2(a-m)}{(2-b^2)(4-3b^2)} > 0$ and (b) $p_2^{\mathcal{C}}(\theta^{\mathcal{C}}) - p_2^{\mathcal{B}}(\theta^{\mathcal{B}}) = \frac{b^2(1-b)(a-m)}{(2-b^2)(4-3b^2)} > 0$. Moreover, one can also verify that $q_1^{\mathcal{B}}(\theta^{\mathcal{B}}) = \frac{(a-m)}{(1+b)}$, $q_2^{\mathcal{B}}(\theta^{\mathcal{B}}) = \frac{(a-m)}{(1-b)(2-b^2)}$, $q_1^{\mathcal{C}}(\theta^{\mathcal{C}}) = \frac{(4-3b)(a-m)}{(4-3b^2)}$ and $q_2^{\mathcal{C}}(\theta^{\mathcal{C}}) = \frac{2(1-b)(a-m)}{(4-3b^2)}$. Result (C) follows by noting that (a) $q_1^{\mathcal{C}}(\theta^{\mathcal{C}}) - q_1^{\mathcal{B}}(\theta^{\mathcal{B}}) = \frac{b(a-m)}{(1+b)(4-3b^2)} > 0$ and (b) $q_2^{\mathcal{B}}(\theta^{\mathcal{B}}) - q_2^{\mathcal{C}}(\theta^{\mathcal{C}}) = \frac{b^2(3-2b^2)(a-m)}{(1+b)(2-b^2)(4-3b^2)} > 0$.

In what follows we prove (D), (E) and (F). Using the values of $q_i^{\mathcal{C}}(\theta)$, $i=1,2$ from the proof of Proposition 3 we find that $\pi_1^{\mathcal{C}}(\theta^{\mathcal{C}}) = \frac{b(1-b)(4-3b)(a-m)^2}{(4-3b^2)^2}$, $\pi_2^{\mathcal{C}}(\theta^{\mathcal{C}}) = \frac{4(1-b)^2(a-m)^2}{(4-3b^2)^2}$, $CS^{\mathcal{C}}(\theta^{\mathcal{C}}) = \frac{(5-4b)(a-m)^2}{2(4-3b^2)}$ and $W^{\mathcal{C}}(\theta^{\mathcal{C}}) = \frac{(7-6b)(a-m)^2}{2(4-3b^2)}$. Since $\theta^{\mathcal{B}} = 0$, $p_i^{\mathcal{B}}(\theta^{\mathcal{B}})$ and $q_i^{\mathcal{B}}(\theta^{\mathcal{B}})$, $i=1,2$ are same as in the proof of Lemma 2. Using those values we get $\pi_1^{\mathcal{B}}(\theta^{\mathcal{B}}) = \frac{b(1-b)(a-m)^2}{(1+b)(2-b^2)}$, $\pi_2^{\mathcal{B}}(\theta^{\mathcal{B}}) = \frac{(1-b)(a-m)^2}{(1+b)(2-b^2)^2}$, $CS^{\mathcal{B}}(\theta^{\mathcal{B}}) = \frac{(5-b-3b^2+b^3)(a-m)^2}{2(1+b)(2-b^2)^2}$ and $W^{\mathcal{B}}(\theta^{\mathcal{B}}) = \frac{(7+b-7b^2-b^3+2b^4)(a-m)^2}{2(1+b)(2-b^2)^2}$. We now compare Bertrand and Cournot outcomes. It is easy to verify that $\pi_2^{\mathcal{C}}(\theta^{\mathcal{C}}) - \pi_2^{\mathcal{B}}(\theta^{\mathcal{B}}) = -\frac{(1-b)b^2[1+(1-b^2)(7-4b^2)](a-m)^2}{(1+b)(2-b^2)^2(4-3b^2)^2} < 0 \ \forall\ b \in (0,1)$ which proves (D). Moreover, proof of (E) follows by noting that (i) $\pi_1^{\mathcal{C}}(\theta^{\mathcal{C}}) - \pi_1^{\mathcal{B}}(\theta^{\mathcal{B}}) = -\frac{b(1-b)[8-2b-14b^2+b^3+6b^4](a-m)^2}{(1+b)(2-b^2)(4-3b^2)^2}$ which is strictly negative for $b \in (0, 0.84)$ and (ii) $CS^{\mathcal{C}}(\theta^{\mathcal{C}}) - CS^{\mathcal{B}}(\theta^{\mathcal{B}}) = \frac{b(1-b)(8-b-12b^2+4b^4)(a-m)^2}{2(1+b)(2-b^2)^2(4-3b^2)}$, which is strictly positive for $b \in (0, 0.9)$. Thus for all $b \in (0, 0.84)$ we have $\pi_1^{\mathcal{C}}(\theta^{\mathcal{C}}) < \pi_1^{\mathcal{B}}(\theta^{\mathcal{B}})$ and $CS^{\mathcal{C}}(\theta^{\mathcal{C}}) > CS^{\mathcal{B}}(\theta^{\mathcal{B}})$. Finally the welfare comparison between Bertrand and Cournot shows that $W^{\mathcal{C}}(\theta^{\mathcal{C}}) - W^{\mathcal{B}}(\theta^{\mathcal{B}}) = -\frac{b^2(1-b)(3-2b^2)(a-m)^2}{2(1+b)(2-b^2)^2(4-3b^2)} < 0$ which proves (F). □

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