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Abstract

Dmitruk and Koshevoy [1991 *JET*] provided a complete characterization of the class of technologies for which there exists an efficiency index satisfying the Färe-Lovell [1978 *JET*] axioms. The technologies implicit in the standard mathematical-programming methods of measuring efficiency, data envelopment analysis (DEA) and free-disposal-hull (FDH) analysis, belong to this class. We assess the ability of three well-known indexes, the Debreu-Farrell index, the Färe-Lovell index, and the Zieschang index, to satisfy not only the Färe-Lovell axioms but also continuity axioms (for technologies as well as input quantities), on this restricted class of technologies. Our principal conclusions are that (a) restriction to these data-based technologies adds continuity in input quantities to the properties satisfied by the Färe-Lovell and the Zieschang indexes (thus eliminating a salient advantage of the Debreu-Farrell index), but (b) none of the indexes satisfies all Färe-Lovell axioms (nor all continuity axioms) on either DEA or FDH technologies, and hence (c) trade-offs among the indexes remain. These findings provide motivation for the search for an index that does satisfy these axioms on DEA and FDH technologies.

JEL classification: C43; C61; D24.

Keywords: Technical efficiency indexes; technical efficiency axioms

I. Introduction.

The use of technical efficiency indexes to identify and quantify the (in)efficiency of production units, introduced by Farrell [1957], has become a standard tool for studying productivity and efficiency. Over the last two to three decades, a staggering number of papers in economics and management science journals have used these methods to assess the efficiency of a wide array of diverse organizations (*e.g.*, school systems, hospitals, insurance firms, banks, airports and seaports, professional sports leagues, and transit systems[1]) and to study economic issues as diverse as environmental regulation, public vs. private firms, and international macroeconomic convergence.[2]

Despite the widespread use of efficiency indexes, there have been comparatively few studies of the properties of the different formulations. Axiomatization of (input) efficiency measurement was introduced by Färe and Lovell [1978], who proposed three axioms: *indication of efficient bundles* (the efficiency index equals one if and only if the input vector is Koopmans [1951] efficient), *monotonicity* (increasing input quantities reduces the value of the index), and *homogeneity* (proportionate changes of all input quantities reduces the index proportionately). The attractiveness of each of these axioms is self-evident.

The most extensively used efficiency index is the Debreu [1951] - Farrell [1957] index. Färe and Lovell [1978] pointed out that this index satisfies neither indication nor monotonicity for the general class of technologies satisfying minimal regularity conditions. They introduced the Färe-Lovell [1978] index[3] to overcome these deficiencies, but, as shown by Färe, Lovell, and Zieschang [1983] and Russell [1985], their measure does not satisfy monotonicity or homogeneity. Zieschang [1984] proposed an index that satisfies homogeneity and indication, but it fails to satisfy monotonicity (and, in fact, violates a weaker monotonicity condition).

The search for an efficiency index satisfying the Färe-Lovell axioms was halted by an impossibility result of Bol [1986] demonstrating that there exists no efficiency index satisfying indication, monotonicity, and homogeneity for all technologies (satisfying minimal regularity conditions). As indicated by Bol [1986], there exist two approaches to resolving this problem: (1) weakening the axioms and (2) restricting the set of technologies to which the index is to be applied.

Russell [1985, 1987] suggested a weakening of monotonicity (increasing input quantities does not increase the value of the index) and a modification of the indication condition

[1] These studies are far too numerous to attempt to provide even prototypical examples. Suffice it to say that the (nine-year-old) Berger and Humphrey [1997] survey of efficiency studies of banks alone, using only mathematical-programming methods of calculating efficiency, encompassed some 67 papers.

[2] Examples: Färe, Grosskopf, Noh, and Weber [2005], Pollitt [1996], Kumar and Russell [2002], and Henderson and Russell [2005].

[3] The Färe-Lovell index is often referred to as the "Russell measure," but it was formulated (and mischievously named) by Färe and Lovell [1978].

(indication of weakly efficient input vectors)[4] and showed that the Debreu-Farrell, Färe-Lovell, and Zieschang indexes each satisfy various combinations of the weaker and stronger axioms. None of these three indexes dominates any other in term of these axioms; hence, the choice among these efficiency indexes depends upon the investigator's opinion about the relative attractiveness of the various axioms.

In a different direction, Russell [1990] introduced axioms of *continuity* in inputs and technologies (or outputs). He argued that continuity is important because it provides some assurance that "small" errors of measurement of output and input quantities do not result in "large" errors in the calculation of the efficiency index. Only the Debreu-Farrell index passes muster with respect to continuity, and this index encounters a problem with continuity in technologies at the boundary. Russell [1990] also showed that continuity in input quantities is incompatible with the indication property.

A major advance in our understanding of efficiency measures was made by Dmitruk and Koshevoy [1991], who completely characterized the class of technologies for which there exists an efficiency index satisfying indication, monotonicity, and homogeneity. Although the characterization is indirect and does not provide a procedure for generating the technologies, there are important special cases. Most important, Dmitruk and Koshevoy indicate that their condition is satisfied if the efficient set is compact.

The primary objective of this paper is to investigate the ability of the three efficiency indexes described above to satisfy the proposed axioms on data-generated technologies.[5] In applications of efficiency measurement, a finite number of observations on the inputs and outputs of production units is used to construct a reference technology. The most common method of generating the technology is to use linear-programming or integer-programming techniques to envelop the data in the tightest fitting set satisfying certain criteria. If the technology is assumed to satisfy free-disposability and convexity, the technology is a convex polyhedral set. If the technology is assumed to satisfy free-disposability but not convexity, the generated technology has a finite number of efficient points. In each case, the efficient set is compact and the Dmitruk-Koshevoy theorem implies the ex-

[4] Most would consider the Färe-Lovell indication property to be the appropriate axiom in the spirit of Koopmans' notion of efficiency. Nevertheless, one can argue that indication of weakly efficient input vectors is appropriate for a measure of *technical* inefficiency (the consequence of being above the isoquant in input space), as distinguished from *allocative* inefficiency (the consequence of being on an economically inefficient point on the isoquant); after all, a Koopmans-inefficient point on the isoquant of a technology satisfying free disposability is allocatively efficient at zero prices of the redundant inputs.

[5] We use the phrase "data-generated technologies" to refer to technologies implicit in the mathematical-programming approach to efficiency measurement (surveyed in Färe, Lovell, and Grosskopf [1995]). These technologies are entirely data driven up to assumptions about convexity and returns to scale, in contrast to the stochastic (econometric) approach to frontier analysis, which is highly parameterized (see Kumbhakar and Lovell [2003] for a through description of these methods). The results on general technologies (in this paper and elsewhere), however, are applicable to the technologies implicit in either approach.

istence of an efficiency index that satisfies the indication, monotonicity, and homogeneity axioms.

The use of convex polyhedral sets to construct reference technologies was pioneered by Farrell [1957] and extended by Charnes, Cooper, and Rhodes [1978]. Under the assumption of constant returns to scale, the reference technology is the "smallest," or "tightest fitting," convex (free disposal) cone that envelops the data; under the assumption of nonincreasing returns, the reference technology is the tightest fitting convex set that envelops the data (the convex, free-disposal hull of the observed input-output combinations and the origin).[6] Charnes, Cooper, and Rhodes call this approach "data envelopment analysis" (DEA).

The approach pioneered by Deprins, Simar, and Tulkens [1984] and promoted by Tulkens [1993] eschews convexity and envelops the data in the tightest fitting free-disposal set—the set of points that (weakly) vector dominate at least one observed point. This is referred to as the free-disposal-hull (FDH) approach. Although it also entails envelopment of the data, we stick to convention by reserving the appellation DEA for the convex-programming approach.

We have three main results. First, restricting technologies to the DEA class does improve the properties of two of the efficiency measures: both the Färe-Lovell and the Zieschang index are continuous in input quantities on the DEA class but do not satisfy this property for general technologies. Second, restricting technologies to the DEA class does not enable any of the indexes to satisfy all of the proposed axioms. Even if we restrict the axioms to those proposed by Färe and Lovell [1978] (indication, monotonicity, and homogeneity), none of the efficiency indexes under study satisfies these axioms for this restricted class of technologies. Third, restricting technologies to the FDH class does not improve the properties of any of the efficiency measures.

Our results have three main implications. First, a researcher selecting one of the existing efficiency indexes must decide which properties are most important: since none of the indexes satisfies all the desirable properties, trade-offs remain. Second, the trade-offs depend on the class of technologies to which the indexes are to be applied. Third, for DEA and FDH technologies, there exist indexes superior to those currently known, since the Dmitruk and Koshevoy results guarantee the existence of an index that satisfies the Färe-Lovell axioms. Whether there exist indexes that satisfy the continuity axioms as well for this restricted class of technologies is a topic for further study.[7]

6 Another possibility is to take the convex, free-disposal hull of the data excluding the origin, in which case the technology is characterized by variable returns to scale. The resultant reference technology is not convex, but level sets are. See Färe, Grosskopf, and Lovell [1995] for a thorough discussion of these various constructions.

7 Another (perhaps more fundamental) axiom is independence of units of measurement (commensurability), introduced by Russell [1987]; as all three indexes evaluated in this paper satisfy this property for all technologies, we do not consider it.

The paper unfolds as follows. Section II describes the general technologies, the DEA technologies, and the FDH technologies. Section III describes the three efficiency indexes studied in this paper, while Section IV describes the axioms. Section V summarizes the known results and contributes some new results on the axioms satisfied by the indexes on general technologies. Sections VI and VII prove our results on the satisfaction of the Färe-Lovell and continuity axioms for efficiency measurement on DEA and FDH technologies, respectively. Section VIII concludes by synthesizing the results from from the literature and this paper.

II. Technologies.

The theoretical literature on technical efficiency measurement has focused on a general class of technologies satisfying only very weak regularity conditions. The input vector $x \in \mathbf{R}^n_+$ is constrained to lie in the input-requirement set L (the set of input vectors that can produce a stipulated vector of outputs).[8] Let $\mathcal{L}$ be the collection of non-empty, closed input-requirement sets that exclude the origin of $\mathbf{R}^n_+$ and satisfy the free-disposability condition, $L = L + \mathbf{R}^n_+$.[9] To simplify the language in the results that follow, we refer to "all technologies" when we mean "all input-requirement sets in $\mathcal{L}$."

Some properties of efficiency indexes hold on the interior of $\mathbf{R}^n_+$ but not at the boundary. On occasion, therefore, we consider the subclass of input-requirement sets in which $L \subset \mathbf{R}^n_{++}$; denote this class of (non-empty, closed) input requirement sets (excluding the origin) by $\mathcal{L}^o$.

An input vector $x \in L$ is efficient (in the sense of Koopmans [1951]) if $x > \bar{x}$ implies $\bar{x} \notin L$; it is weakly efficient if $x \gg \bar{x}$ implies $\bar{x} \notin L$.[10] Under the free-disposability assumption, the set of weakly efficient input vectors is equivalent to the isoquant, defined by

$$\mathrm{Isoq}(L) = \{\ x \in L \mid \lambda x \notin L \ \forall\ \lambda \in [0,1)\ \}. \tag{2.1}$$

The set of efficient points of L, which we denote by $\mathrm{Eff}(L)$, is a subset of $\mathrm{Isoq}(L)$.

---

8 A complete characterization of the technology would be a correspondence mapping output vectors into subsets of input space. Since, however (in the tradition of axiomatic analysis of efficiency measurement), we consider only input-based measures of efficiency for fixed output vectors, it is not necessary to formally incorporate output into our analysis, at least for the analysis of the Färe-Lovell axiomatic structure. When we analyze continuity of efficiency indexes, however, we implicitly allow output to vary by considering sequences of input requirement sets in $\mathcal{L}$.

9 Nonemptiness, closedness, and exclusion of the origin are necessary to guarantee that our efficiency indexes are well defined, but the free disposability assumption could be dispensed with (theoretically). The only change that would be needed in what follows would be to redefine the Debreu-Farrell index on the free-disposal hull of L rather than on L itself (as in Russell [1987]).

10 Vector notation: $\bar{x} \geq x$ if $\bar{x}_i \geq x_i$ for all i; $\bar{x} > x$ if $\bar{x}_i \geq x_i$ for all i and $\bar{x} \neq x$; and $\bar{x} \gg x$ if $\bar{x}_i > x_i$ for all i.

Data envelopment analysis (DEA), the most common mathematical-programming, data-based method of measuring efficiency, constructs input-requirement sets that are convex, free-disposal polyhedrons—that is, intersections of a finite number of closed half spaces with semi-positive normals. Let $\mathcal{P}$ denote the set of convex polyhedral technologies in $\mathcal{L}$ and let $\mathcal{P}^o$ denote the set of convex polyhedral technologies in $\mathcal{L}^o$. Figure 2 below contains an example of a convex polyhedral input-requirement set.

The free-disposal-hull (FDH) method of measuring efficiency adds no explicit technological restrictions to the general case, but this data-driven method implicitly restricts the technologies to be unions of empirically dominated input sets—that is, finite unions of affine transformations of non-negative orthants. See, *e.g.*, Tulkens [1993] for a description of these methods. We refer to these sets as "FDH technologies." Let $\mathcal{F}$ denote the set of free-disposal-hull technologies in $\mathcal{L}$ and let $\mathcal{F}^o$ denote the set of free-disposal-hull technologies in $\mathcal{L}^o$. An example of an FDH input-requirement set is shown in Figure 6 below.

III. Efficiency Indexes.

An (input) efficiency index is a mapping, $E : \Xi \to (0, 1]$, with image $E(x, L)$, where $\Xi = \{\langle x, L\rangle \in L \times \mathcal{L} \mid x \in L\}$; it is intended to measure the inefficiency of an input vector $x \in L$ (given a technology and an output vector). The general idea underlying existing efficiency indexes is to measure the maximal "distance" an input vector may be contracted while remaining feasible. The alternative indexes differ in the method of contraction and the notion of distance.

The Debreu [1951] - Farrell [1957] index, defined by

$$E_{DF}(x, L) = \min\{\, \lambda \mid \lambda x \in L\}, \tag{3.1}$$

measures the maximal radial contraction of the input vector consistent with production feasibility.

The Färe-Lovell [1978] index is based on coordinatewise contractions of the input vector. The index is defined by

$$E_{FL}(x, L) = \min_{\kappa} \left\{ \sum_i \kappa_i \Big/ \sum_i \delta_i(x_i) \;\Big|\; Kx \in L \;\wedge\; \kappa_i \in [0, 1]\; \forall i \right\} \tag{3.2}$$

where $\delta(x) = 1$ if $x_i > 0$, $\delta(x_i) = 0$ if $x_i = 0$, and K is the diagonal matrix with $\langle \kappa_1, \ldots, \kappa_n\rangle$ on the diagonal.[11] This index measures the maximal average of coordinatewise contractions.

[11] The correction entailing the indicator function δ is needed for the case in which one coordinate value of an efficient bundle vanishes, in which case the corresponding shrinkage factor could be set at zero, thus rendering the efficiency index less than one. If the input requirement set is contained in the interior of $\mathbf{R}^n_+$, the denominator in the objective function is just n.

The Zieschang [1984] index combines the radial contraction of the Debreu-Farrell index with the coordinatewise contraction of the Färe-Lovell index. It is defined by

$$E_Z(x, L) = E_{DF}(x, L)E_{FL}\big(E_{DF}(x, L)x, L\big). \tag{3.3}$$

This index measures the multiple of the maximal radial contraction to the isoquant and the average of coordinatewise contractions along the isoquant.[12]

IV. Axioms.

The three axioms proposed by Färe and Lovell [1978][13] are as follows:

Indication of Efficient Input Bundles (I): For all $x \in L$, $E(x, L) = 1 \iff x \in \text{Eff}(L)$.

Monotonicity (M): For all $\langle x, \bar{x}\rangle \in L \times L$, $x > \bar{x} \implies E(x, L) < E(\bar{x}, L)$.

Homogeneity (H): For all $x \in L$, $E(\kappa x, L) = \kappa^{-1}E(x, L) \quad \forall\, \kappa > 0$.

Russell [1985, 1987] proposed alternatives to the indication and monotonicity axioms:

Indication of Weakly Efficient Input Bundles (IW): For all $x \in L$, $E(x, L) = 1$ if and only if $x \in Isoq(L)$ (*i.e.*, x is "weakly efficient").

Weak Monotonicity (WM): For all $\langle x, \bar{x}\rangle \in L \times L$, $x \geq \bar{x} \implies E(x, L) \leq E(\bar{x}, L)$.

Russell [1990] extended the Färe-Lovell axiomatic structure by adding three continuity axioms.

Continuity in x (C–x): E is continuous in x.

Continuity in L (C–L): E is continuous in L.

Joint continuity (C–$\langle x, L\rangle$): E is jointly continuous in x and L.

As noted earlier, Russell [1990] argued (page 256) that continuity is a compelling property, "for it provides assurance that 'small' errors of measurement (of, *e.g.*, input or output quantities) result only in 'small' errors of efficiency measurement." If the technology is constructed from data on input-output vectors, the argument for continuity in the technology is even more compelling.

[12] A more recent candidate for measuring efficiency is the (input based) directional distance function, adapted from the benefit function of Luenberger [1992] to the measurement of efficiency by Chung, Färe, and Grosskopf [1997]. This concept, however, is qualitatively different from the three traditional efficiency indexes: it is parameterized by the directional vector and maps into the real line instead of the (0,1] interval. As a result, it is not amenable to straightforward application of the Färe-Lovell axioms. A separate axiomatic analysis of the directional distance function as a measure of efficiency is under way.

[13] More precisely, Färe and Lovell proposed a fourth axiom: that the input bundle is "compared to" an efficient bundle. Russell [1985], however, argued that the compared-to axiom was ill-defined (not a formal mathematical construct). He formalized the concept but then showed that the compared-to axiom is implied by the other three axioms.

V. Results for General Technologies.

The search for an efficiency index satisfying the Färe-Lovell axioms was halted by the following impossibility result of Bol [1986]:[14]

Fact 1: There does not exist an efficiency index satisfying (H), (M), and (I) for all technologies $L \in \mathcal{L}$.[15]

Further results by Russell [1990] demonstrate the incompatibility of certain continuity conditions with some of the Färe-Lovell conditions.

Fact 2:

- There does not exist an efficiency index satisfying (I) and (C–x) for all technologies $L \in \mathcal{L}$.
- There does not exist an efficiency index satisfying (I) and (C–L) for all technologies $L \in \mathcal{L}$.

The known results on the compatibility of the three indexes with all of the axioms are encapsulated in the following:

Fact 3 (Färe and Lovell [1978], Färe, Lovell, and Zieschang [1983], Zieschang [1984], and Russell [1985, 1987, 1990]):

- E_{DF} satisfies (IW), (WM), and (H) for all $L \in \mathcal{L}$ and (C–$\langle x, L\rangle$) for all $L \in \mathcal{L}^o$ and fails to satisfy (I), (M) and (C–$\langle x, L\rangle$) for all $L \in \mathcal{L}$;
- E_{FL} satisfies (I) and (WM) and fails to satisfy (M), (H), (C–x), and (C–L) for all $L \in \mathcal{L}$; and
- E_Z satisfies (I) and (H) and fails to satisfy (WM), (C–x), and (C–L) for all $L \in \mathcal{L}$.

The (Russell [1985]) counterexample showing that E_{FL} violates (M) relies critically on the input requirement set intersecting the boundary of $\mathbf{R}^n_+$, as demonstrated by the following result:

Theorem 1: *E_{FL} satisfies (M) on $\mathcal{L}^o$.*

[14] In the facts and results that follow, we refer to, *e.g.*, "all $L \in \mathcal{L}$" or "all $L \in \mathcal{P}$" when we more formally mean "all $\langle x, L\rangle \in \Xi$" or "all $\langle x, L\rangle \in \Xi_P$," where $\Xi_P = \{\langle x, L\rangle \mid L \in \mathcal{P} \ \wedge \ x \in L\}$.

[15] Bol's three-dimensional example purporting to show that convexity is not relevant contains a minor error: his input-requirement set,

$$L = \left\{ \langle x_1, x_2, x_3\rangle \mid x_1 = 1 \wedge x_2 = 2 \wedge x_3 = e^{-(x_1-1)(x_2-1)} \right\}, \tag{5.1}$$

is not convex. A minor change to

$$L = \left\{ \langle x_1, x_2, x_3\rangle \mid x_1 = 1 \wedge x_2 = 2 \wedge x_3 = e^{-(x_1-1)^{1/2}(x_2-1)^{1/2}} \right\}, \tag{5.2}$$

however, results in an input-requirement set that establishes his point.

Proof: With L restricted to the interior of $\mathbf{R}^n_+$, the Färe-Lovell index can be written as

$$E_{FL}(x,L) = \min_y \Big\{ \sum_i \frac{y_i}{x_i} \;\Big|\; y \leq x \;\wedge\; y \in L \Big\}. \tag{5.3}$$

Suppose that

$$\overset{*}{y} = \operatorname*{argmin}_y \Big\{ \sum_i \frac{y_i}{x_i} \;\Big|\; y \leq x \;\wedge\; y \in L \Big\} \tag{5.4}$$

and $\bar{x} > x$. Since $\overset{*}{y} < x$ and $x < \bar{x}$, $\overset{*}{y}$ is a feasible solution in the problem,

$$E_{FL}(\bar{x},L) = \min_y \Big\{ \sum_i \frac{y_i}{\bar{x}_i} \;\Big|\; y \leq \bar{x} \;\wedge\; y \in L \Big\}, \tag{5.5}$$

and

$$\sum_i \frac{\overset{*}{y}_i}{\bar{x}_i} < \sum_i \frac{\overset{*}{y}_i}{x_i}. \tag{5.6}$$

Thus, $E_{FL}(\bar{x},L) < E_{FL}(x,L)$. ■

The positive result on continuity in x for the Debreu-Farrell index (Russell [1990]) can be extended to the boundary:

Theorem 2: *The E_{DF} satisfies (C–x) on $\mathcal{L}$.*

Proof: The Debreu-Farrell index can be written as

$$E_{DF}(x,L) = \min \big\{ \lambda \mid \lambda \in \Lambda(x,L) \big\}, \tag{5.7}$$

where

$$\Lambda(x,L) = \big\{ \lambda \in \mathbf{R}^n_+ \mid \lambda x \in L \big\}. \tag{5.8}$$

By the maximum theorem, E_{DF} is continuous in x if Λ is continuous (upper and lower hemi-continuous) in x given L.

We first show that Λ is upper hemi-continuous. Consider a sequence $\{x^\nu\} \subset L$ satisfying $x^\nu \to x^o$ and $x^\nu \in L$ for all ν and an associated sequence $\{\lambda^\nu\}$ satisfying $\lambda^\nu \to \lambda^o$ and $\lambda^\nu \in \Lambda(x^\nu, L)$ for all ν. Thus, $\lambda^\nu x^\nu \in L$ for all ν and $\lambda^\nu x^\nu \to \lambda^o x^o$. By closedness of L, $\lambda^o x^o \in L$. Hence, $\lambda^o \in \Lambda(x^o, L)$.

To show lower hemi-continuity of Λ, consider a sequence $\{x^\nu\}$ satisfying $x^\nu \to x^o$ and $\lambda^o \in \Lambda(x^o, L)$. We need to show that there exists a sequence $\{\lambda^\nu\}$ satisfying $\lambda^\nu \in \Lambda(x^\nu, L)$ for all ν and $\lambda^\nu \to \lambda^o$. Let $y^o = \lambda^o x^o$ and define, for all ν,

$$\lambda^\nu = \min \big\{ \lambda \mid \lambda x^\nu \in y^o + \mathbf{R}^n_+ \big\}. \tag{5.9}$$

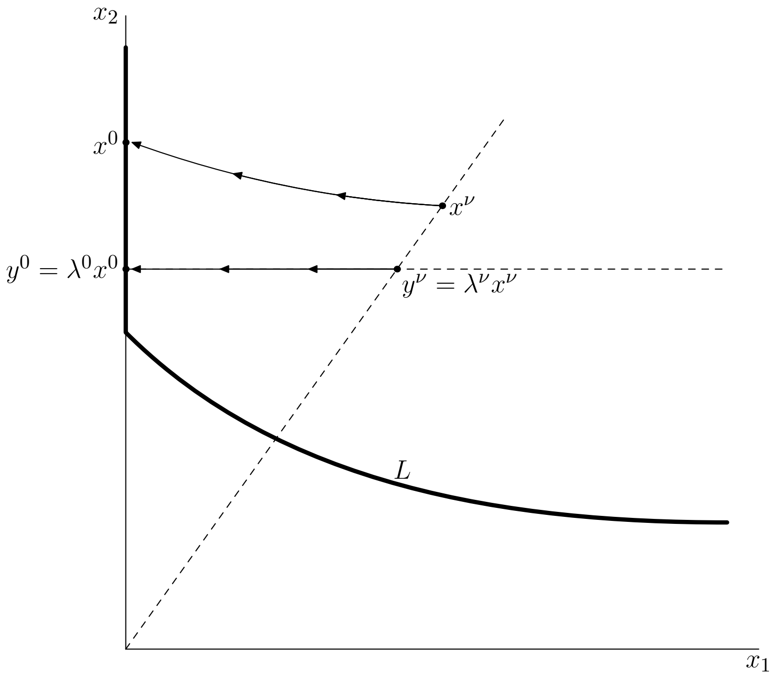

Figure 1.

Because $y^o \in L$ and L satisfies free disposability, $\lambda^\nu \in \Lambda(x^\nu, L)$ for all ν. The sequence $\{y^\nu\}$ defined by $y^\nu = \lambda^\nu x^\nu$ for all ν clearly converges to y^o (see Figure 1), so that $\lambda^\nu \to \lambda^o$. ■

Fact 3 and Theorems 1–2 indicate that none of the three indexes dominates any other, in terms of the axioms introduced in Section IV, on general technologies (for all $L \in \mathcal{L}$). The results thus underscore the trade-offs among the three efficiency indexes. The choice between E_{DF} and E_{FL} reflects the trade-off between homogeneity and continuity of E_{DF} and the strong-efficiency form of the indication condition (and strict monotonicity on the interior of input space) of E_{FL}. The choice between E_{DF} and E_Z reflects the trade-off between weak monotonicity and continuity of E_{DF} and the strong-efficiency form of the indication condition satisfied by E_Z. Choosing between E_{FL} and E_Z reflects the trade-off between monotonicity and homogeneity.

VI. Results for Convex Polyhedral Technologies.

Our next theorem examines the possibility of obtaining stronger results when the technologies are restricted to those generated by DEA methods of measuring efficiency and, *pari passu*, generating reference technologies. In the theorem, we only state results that are not immediately implied by the results for general technologies (Fact 3 and Theorems 1–2).

Theorem 3:

- *E_{DF} fails to satisfy (C–L) for all $L \in \mathcal{P}$ and fails to satisfy (I) and (M) for all*

$L \in \mathcal{P}^o$.[16]

- *E_{FL} satisfies (C–x) and fails to satisfy (M) for all $L \in \mathcal{P}$; E_{FL} fails to satisfy (H) and (C–L) for all $L \in \mathcal{P}^o$.*
- *E_Z satisfies (C–x) and fails to satisfy (WM) and (C–L) for all $L \in \mathcal{P}^o$.*

Proof: The counterexample showing that E_{DF} index fails to satisfy continuity in L on $\mathcal{P}$ is illustrated in Figure 2.[17] In this example, $L^\nu \to L^o$ as the cusp $z^\nu \to z^o$ and $E_{DF}(x, L^\nu) = 1$ for all ν, but $E_{DF}(x, L^o) < 1$.

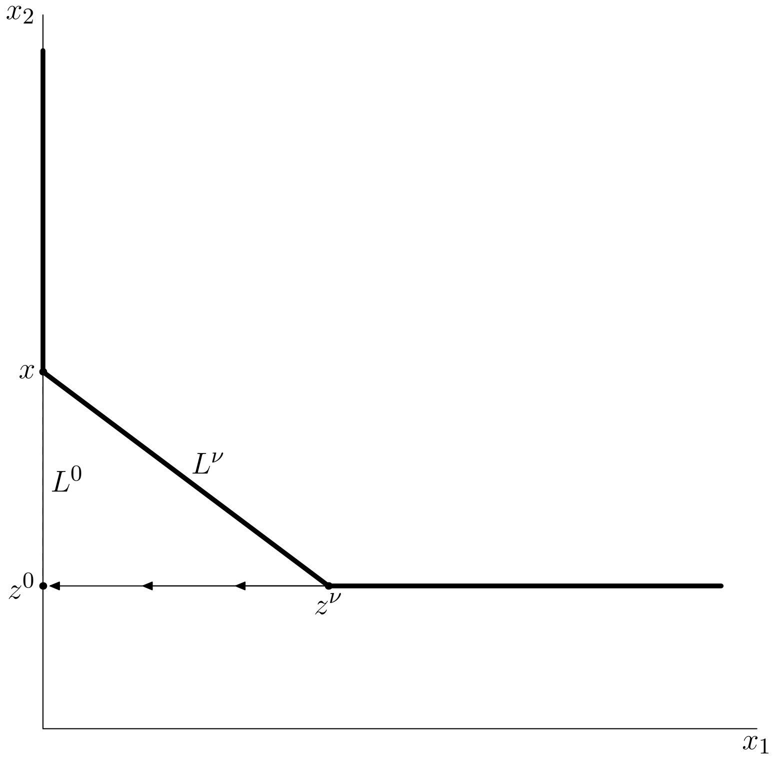

Figure 2.

A similar counterexample shows that the E_{FL} and E_Z indexes fail to satisfy continuity in L on $\mathcal{P}^o$. In Figure 3, $L^\nu \to L^o$ as the cusp $z^\nu \to z^o$ and $E_{FL}(x, L^\nu) = E_Z(x, L^\nu) = 1$ for all ν, but $E_{FL}(x, L^o) < 1$ and $E_Z(x, L^o) < 1$.

Violation of (I) and (M) on $\mathcal{P}^o$ by E_{DF} is immediately apparent by taking any convex polyhedral set and two points with positive slack on the same facet. To see that E_{FL} violates (M) on $\mathcal{P}$, consider a Leontief input requirement set with facet x^o satisfying $x_i^o = 0$ and $x_j^o > 0$ for all $j \neq i$, an input bundle x satisfying $x_i > 0$ and $x_j = x_j^o$ for all

[16] And fails, a fortiori, to satisfy (I) and (M) on $\mathcal{P}$.

[17] This counterexample is reproduced from Russell [1990], who attributed it to Rolf Färe. Throughout the paper, we provide counterexamples in 2-space (except for the violation of (WM) by E_Z, which does not hold in 2-space). Each could be explicitly extended to n-space, but only at the cost of tedious calculations that do not enhance the clarity of the counterexample.

Figure 3.

$j \neq i$, and another input bundle $\bar{x}$ satisfying $\bar{x}_i > x_i$ and $\bar{x}_j = x_j$ for all $j \neq i$. Since $\bar{x} > x$ and $E_{FL}(x, L) = (n-1)/n = E_{FL}(\bar{x}, L)$, E_{FL} violates (M).

To show that E_{FL} does not satisfy (H) on $\mathcal{P}^o$, consider the two-dimensional, closed, convex, polyhedral input-requirement set in Figure 4. $E_{FL}(\lambda\bar{x}, L) < \lambda^{-1} E_{FL}(\bar{x}, L) = \lambda^{-1}$ if $1 < (\alpha+1)/2 < \lambda^{-1}$ (*e.g.*, $\lambda = 1.5$ and $\alpha = .25$).

To show that E_{FL} satisfies (C–x), first re-write this index as follows:

$$\begin{aligned} E_{FL}(x, L) &= \min_y \left\{ \frac{\sum_{i \in I^o} y_i/x_i}{|I^o|} \;\middle|\; y \leq x \;\wedge\; y \in L \right\} \\ &= \min_y \left\{ \frac{\sum_{i \in I^o} y_i/x_i}{|I^o|} \;\middle|\; y \in D(x, L) \cap L =: \Gamma(x, L) \right\}, \end{aligned} \tag{6.1}$$

where $D(x, L) = \{y \mid y \leq x\}$, I^o is the set of coordinates for which $x_i \neq 0$, and $|I^o|$ is the cardinality of this set. To employ the maximum theorem, we will show that the mapping Γ is continuous in x. To prove upper hemi-continuity of Γ in x, consider a sequence $\{x^\nu\}$ converging to x^o and a sequence $\{y^\nu\}$ converging to y^o and satisfying $y^\nu \in \Gamma(x^\nu, L)$ for all ν. Suppose that $y^o \notin \Gamma(x^o, L)$. As L is closed, it must be that $y^o \notin D(x^o, L)$. Consequently, for some i and some ν', $y_i^\nu - x_i^o > \epsilon$ for all $\nu > \nu'$. As $x^\nu \to x^o$, there exists a ν'' such that $x^\nu \in N_{\epsilon/2}(x^o)$ for all $\nu > \nu''$. This implies that $y_i^\nu > x_i^\nu$, and hence $y^\nu \notin D(x^\nu, L)$, for all $\nu > \max\{\nu', \nu''\}$, a contradiction. To prove lower hemi-continuity

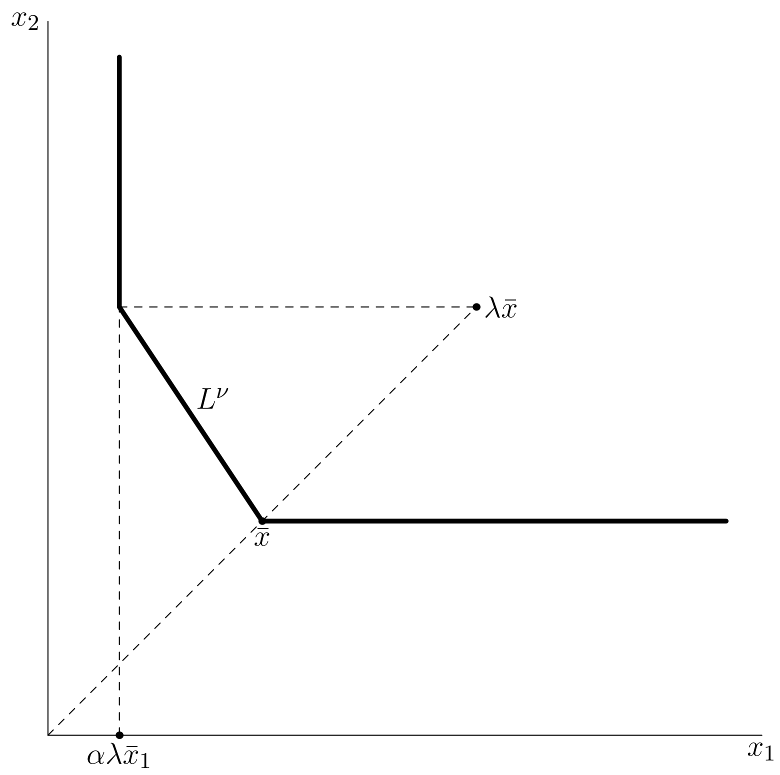

Figure 4.

of Γ in x, consider a sequence $\{x^\nu\}$ converging to x^o and $y^o \in \Gamma(x^o, L)$. For all ν take any $z^\nu \in D(x^\nu, L)$ and define

$$\alpha^\nu = \min\big\{\alpha \mid \alpha z^\nu + (1-\alpha)y^o \leq x^\nu \ \wedge \ \alpha \in [0,1]\big\}. \tag{6.2}$$

The sequence $\{y^\nu\}$ defined by $y^\nu = \alpha^\nu z^\nu + (1-\alpha^\nu)y^o$ for all ν clearly converges to y^o (see Figure 5). Moreover, convexity of L, together with $y^o \in L$ and $z^\nu \in L$ for all ν, implies that $y^\nu \in \Gamma(x^\nu, L)$ for all ν. Thus, Γ is continuous and, by the maximum theorem, E_{FL} is continuous in x.

To show that the Zieschang index satisfies (C–x), note that it can be re-written as

$$\begin{aligned} E_Z(x,L) &= E_{DF}(x,L) \min_y \left\{ \frac{\sum_{i\in I^o} y_i/E_{DF}(x,L)\ x_i}{|I^o|} \ \middle|\ y \leq E_{DF}(x,L)x \ \wedge \ y \in L \right\} \\ &= \min_y \left\{ \frac{\sum_{i\in I^o} y_i/x_i}{|I^o|} \ \middle|\ y \in D(x,L) \cap L =: \Gamma(x,L) \right\}, \end{aligned} \tag{6.3}$$

where $D(x,L) = \{y \mid y \leq E_{DF}(x,L)x\}$. To prove upper hemi-continuity of Γ, consider a sequence $\{x^\nu\}$ converging to x^o and a sequence $\{y^\nu\}$ converging to y^o and satisfying $y^\nu \in \Gamma(x^\nu, L)$ for all ν. From Theorem 2 above, E_{DF} is continuous in x; hence, $x^\nu \to x^o$ implies that $E_{DF}(x^\nu, L)\ x^\nu \to E_{DF}(x^o, L)\ x^o$. Suppose that $y^o \notin \Gamma(x^o, L)$. As L is closed, it must be that $y^o \notin D(x^o, L)$. Consequently, for some i and some ν',

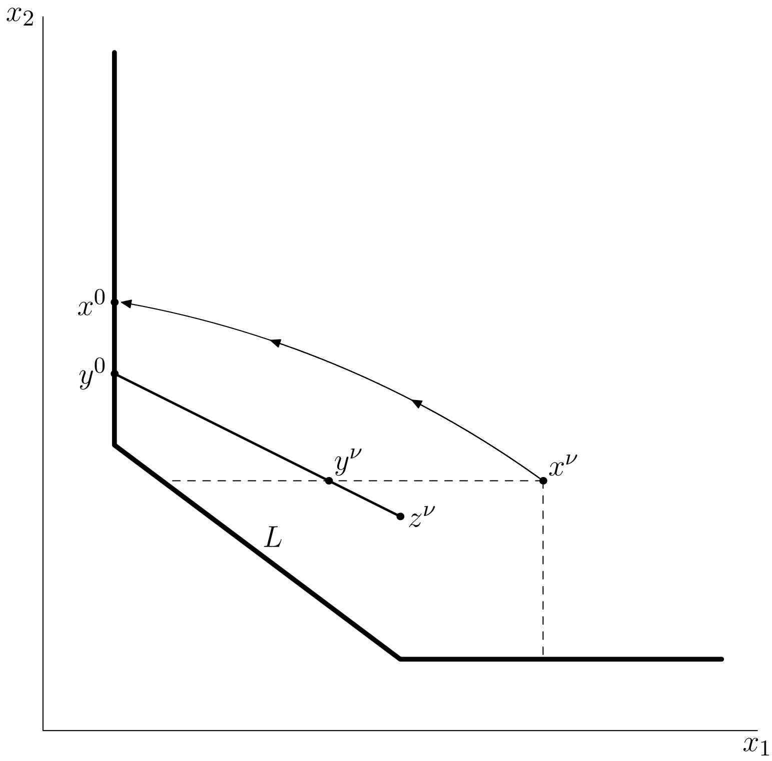

Figure 5.

$y_i^\nu - E_{DF}(x^\nu, L)\ x_i^o > \epsilon$ for all $\nu > \nu'$. As $E_{DF}(x^\nu, L)\ x^\nu \to E_{DF}(x^o, L)\ x^o$, there exists a ν'' such that $E_{DF}(x^\nu, L)\ x^\nu \in N_{\epsilon/2}(x^o)$ for all $\nu > \nu''$. This implies that $y_i^\nu > E_{DF}(x^\nu, L)\ x_i^\nu$, and hence $y^\nu \notin D(x^\nu, L)$, for all $\nu > \max\{\nu', \nu''\}$, a contradiction. Apart from the alternative definition of the mapping D, and hence the mapping Γ, the proof of lower hemi-continuity of Γ is exactly the same as that in the proof of continuity of E_{FL} above. Thus, E_Z is continuous in x.

Proof that E_Z violates (WM) on on $\mathcal{P}^o$, requires an explicit three-dimensional counterexample, since E_Z appears to satisfy (WM) in two-space. To facilitate understanding of the reason for the non-monotonicity, we begin with an illustrative counter-example in $\mathcal{P}$. Consider a convex polyhedral technology L with three efficient vertices given by $z^1 = \langle 2, 10, 3\rangle$, $z^2 = \langle 1, 30, 0\rangle$, and $z^3 = \langle 1, 0, 6\rangle$. Let $x = \langle 2, 12, 12\rangle$ and note that $E_{DF}(x, L) = 1/2$ and $E_{DF}(x, L)x = \langle 1, 6, 6\rangle$, which is an inefficient point in the flat containing z^2 and z^3. Let $x' = \langle 2.4, 12, 12\rangle$, so that $x' > x$, $E_{DF}(x', L) = 5/12$, and $E_{DF}(x', L)x' = \langle 1, 5, 5\rangle$, which, as a convex combination of z^2 and z^3, is efficient. The Färe-Lovell index contracts $\langle 1, 6, 6\rangle$ to $\langle 1, 0, 6\rangle$ so that $E_{FL}(E_{DF}(x, L)x, L) = 2/3$. Since $E_{DF}(x', L)x'$ is efficient, $E_{FL}(E_{DF}(x', L)x', L) = 1$. We then have

$$\begin{aligned} E_Z(x, L) &= E_{DF}(x, L)E_{FL}(E_{DF}(x, L)x, L) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3} \\ E_Z(x', L) &= E_{DF}(x', L)E_{FL}(E_{DF}(x', L)x', L) = \frac{5}{12} \times 1 = \frac{5}{12}, \end{aligned} \tag{6.4}$$

so that $E_Z(x, L) < E_Z(x', L)$, a contradiction of weak monotonicity.

To show that E_Z violates (WM) on $\mathcal{P}^o$, let $z^1 = \langle 2, 10, 3\rangle$, $z^2 = \langle 1, 30, 1\rangle$, and $z^3 = \langle 1, 1, 6\rangle$. Following the same steps as above, we obtain $E_Z(x, L) = 0.36$ and $E_Z(x', L) = 0.42$, so that weak monotonicity is violated. ■

There are two interesting implications of Theorem 3. First, restricting the technologies to the DEA class enables the Färe-Lovell and Zieschang indexes to satisfy continuity in x. Second, one cannot escape the trade-off between indexes by restricting the technologies to the DEA class. While the choices are different, since all three indexes satisfy continuity in x on $\mathcal{P}$, there remain conflicts among indication, monotonicity, and homogeneity.

VII. Results for Free-Disposal-Hull Technologies.

In this section, we restrict the technologies to be free-disposal-hull technologies: $L \in \mathcal{F}$. In the following theorem, we again state only those results that are not immediately implied by the results for general technologies.

Theorem 4:

- *E_{DF} fails to satisfy (C–L) for all $L \in \mathcal{F}$ and fails to satisfy (I) and (M) for all $L \in \mathcal{F}^o$.*[18]
- *E_{FL} fails to satisfy (M) for all $L \in \mathcal{F}$ and fails to satisfy (H), (C–x), and (C–L) for all $L \in \mathcal{F}^o$.*
- *E_Z fails to satisfy (WM), (C–x), and (C–L) for all $L \in \mathcal{F}^o$.*

Proof: Violation of (I) and (M) by E_{DF} on $\mathcal{F}^o$ and of (M) by E_{FL} on $\mathcal{F}$ are immediately established as in the proof of Theorem 3.

Consider the FDH level set in Figure 6 in which $x^\nu \to x^o$ and $x^\mu \to x^o$ Simple calculations reveal that $E_{FL}(\lambda\bar{x}, L) < \lambda^{-1}$ if $(\alpha\lambda^{-1} + 1)/2 < \lambda^{-1} < 1$, which holds for a range of values of $\alpha < 1$ and $\lambda < 1$; *e.g.*, $\lambda = 1.5$ and $\alpha = .25$ imply that $E(\lambda\bar{x}, L) = 7/12 < 2/3 = \lambda^{-1}$. Thus, E_{FL} violates (H) on $\mathcal{F}^o$.

Next note that $E_{FL}(x^\nu, L) \to (\lambda^{-1} + 1)/2 > (\alpha + 1)/2 = E_{FL}(x^o, L)$ as $x^\nu \to x^o$ if $\lambda^{-1} > \alpha$; *e.g.*, $\alpha = .25$ and $\lambda = 2$ imply $E_{FL}(x^o, L) = 5/8 < \lim_{\nu\to\infty} E_{FL}(x^\nu, L) = 3/4$. Thus, E_{FL} violates (C–x) on $\mathcal{F}^o$. In fact, since E_Z coincides with E_{FL} on the isoquant, this construction also shows that E_Z violates (C–x) on $\mathcal{F}^o$.

To show that E_Z violates (WM) on $\mathcal{F}^o$, note that, as $x^\mu \to x^o$, $E_Z(x^\mu, L) \to 1$. But, since $E(x^o, L) < 1$, there exist x^μ (closed enough to x^o) such that $E(x^o, L) < E(x^\mu, L)$, a violation of (WM).

Discontinuity of E_{DF} in L on $\mathcal{F}$ is demonstrated in Figure 7 and discontinuity of E_{FL} and E_Z in L on $\mathcal{F}^o$ are demonstrated in Figure 8. In each case, the cusp z^ν converges to z^o, the input requirement set L^ν converges to L^o, and $E_{FL}(x, L^\nu) = E_Z(x, L^\nu) = 1$ for all ν, but $E_{FL}(x, L^o) = E_Z(x, L^o) < 1$. ■

[18] And fails, a fortiori, to satisfy (I) and (M) on $\mathcal{F}$.

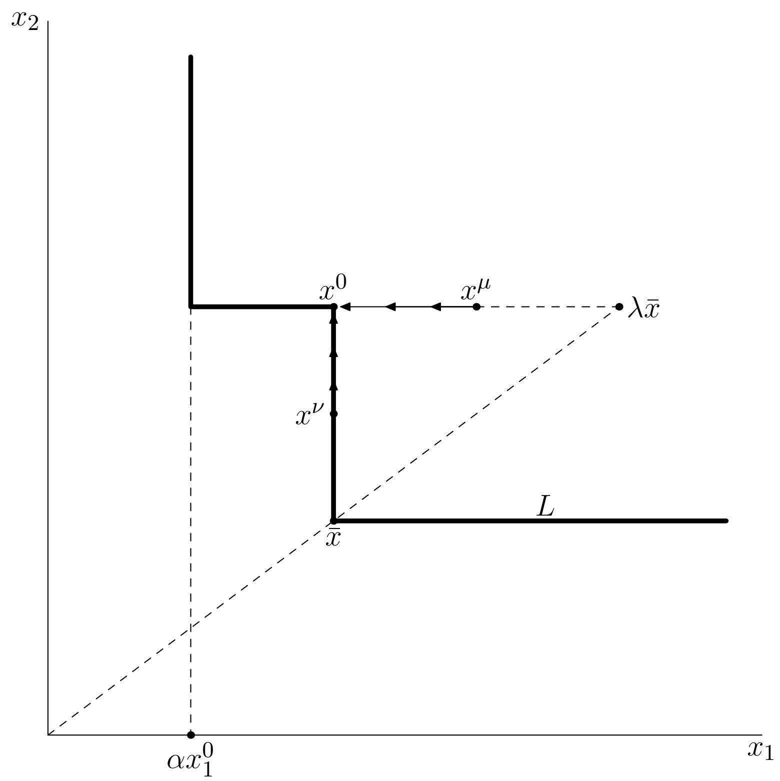

Figure 6.

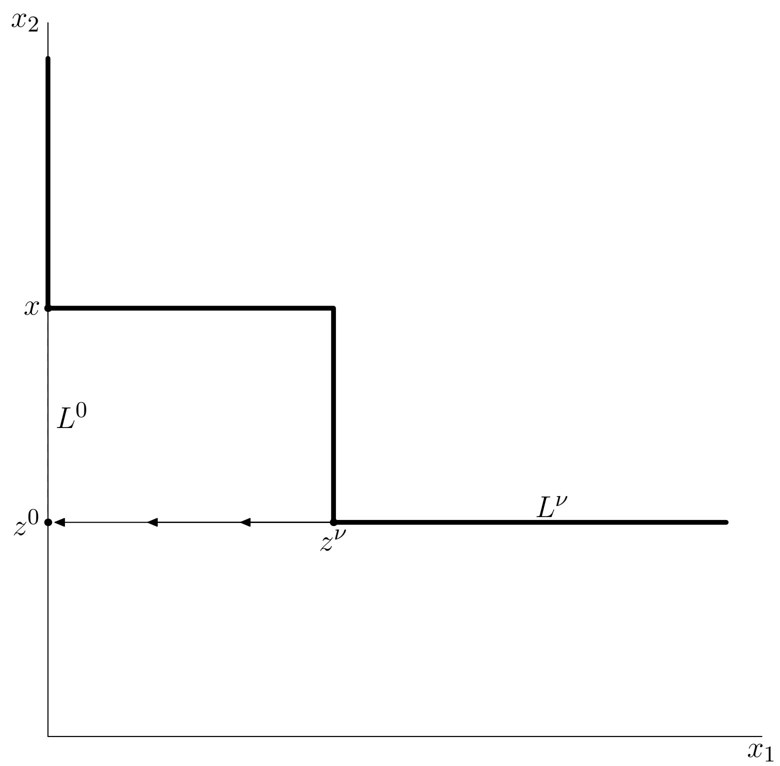

Figure 7.

The main implication of this theorem is that restricting technologies to the FDH class

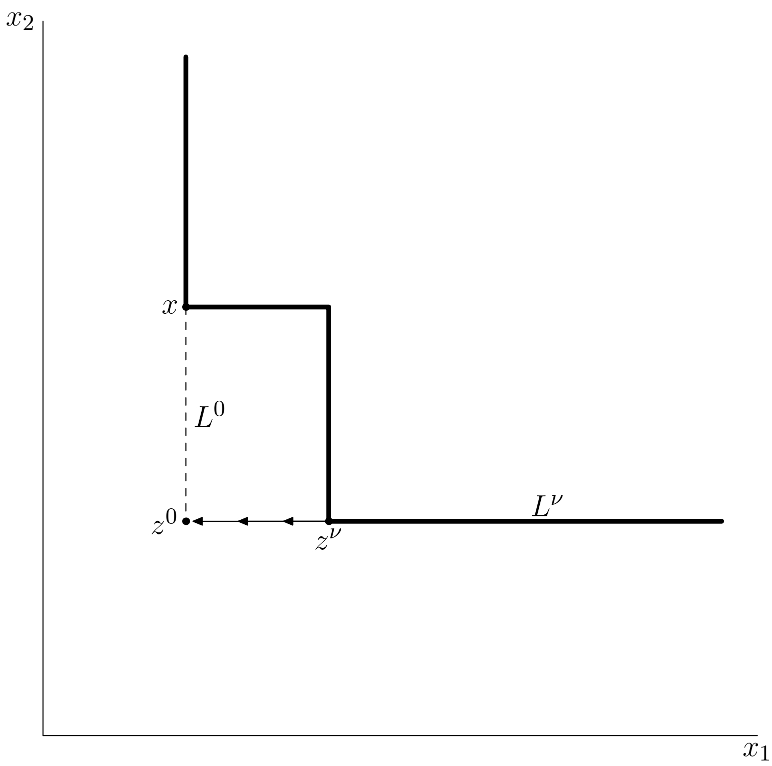

Figure 8.

does not improve their properties. This is quite surprising, since the FDH class is such a "small" subset of the general class of technologies.

VIII. Concluding Remarks.

To conclude, we collect the results from the literature and from this paper and present a summary of the properties of the indexes. Table 1 displays for each index the largest sets on which a particular property holds. For example, weak monotonicity holds for E_{DF} on $\mathcal{L}$ and, therefore, on $\mathcal{L}^o$, $\mathcal{P}$, $\mathcal{P}^o$, $\mathcal{F}$, and $\mathcal{F}^o$. A blank space indicates that a property does not hold for either $\mathcal{P}^o$ or $\mathcal{F}^o$ and, therefore, not for any of their supersets.[19] For example, E_Z is not weakly monotonic on any of these sets of technologies.

Table 1. Properties of Efficiency Indexes.

Indexes	I	IW	M	WM	H	C–x	C–L	C–$<x, L>$
E_{DF}		$\mathcal{L}$		$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}^o$	$\mathcal{L}^o$
E_{FL}	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}^o$	$\mathcal{L}$		$\mathcal{P}$		
E_Z	$\mathcal{L}$	$\mathcal{L}$			$\mathcal{L}$	$\mathcal{P}$		

[19] Of course, $\mathcal{F}$ and $\mathcal{P}$ are not nested, so that it is possible, in principle, for there to be two largest sets for which a particular index would satisfy a particular property; but this does not happen.

Quite surprisingly, the restriction to the set $\mathcal{F}$ of free-disposal-hull technologies has no effect on the results. The properties of the three indexes are the same on $\mathcal{L}$ (and $\mathcal{L}^o$) as on $\mathcal{F}$ (and $\mathcal{F}^o$). Therefore, the trade-offs among these indexes (summarized for all technologies at the end of Section V) persist even when the class of technologies is severely restricted to the FDH class.

The major effect of restricting technologies to convex polyhedral technologies generated by DEA methods is to ensure that the Färe-Lovell and Zieschang indexes are continuous in input quantities. A principal advantage of the Debreu-Farrell index on $\mathcal{L}$ is therefore eliminated when technologies are restricted to $\mathcal{P}$.

Serious trade-offs in the choice of indexes remain, however, in the convex polyhedral case. If continuity in technologies, homogeneity, and weak monotonicity are most important, one should choose the Debreu-Farrell index. If homogeneity and indication are most important, choose the Zieschang index. If indication and weak monotonicity are most important, then use of the Färe-Lovell index is appropriate.

Our view is that weak monotonicity is critical for an efficiency index and that the failure to satisfy this property on both DEA and FDH technologies should eliminate the Zieschang index from consideration. The choice between the Debreu-Farrell and Färe-Lovell indexes for DEA technologies depends on the importance of indication and monotonicity versus homogeneity and continuity in technologies.

While we have shown that the most commonly used efficiency indexes do not satisfy one or more of the desirable axioms, the Dmitruk-Koshevoy [1991] results imply that, for DEA and FDH technologies, there exists an index that satisfies indication, monotonicity, and homogeneity. Whether or not continuity in inputs and technologies can be added to this list is an open question.

The proof of Dmitruk and Koshevoy [1991] is constructive, defining a class of indexes that satisfy the Färe-Lovell conditions, but, as stated, implementation would require an infinite number of (programming) steps. This construction nevertheless might provide the basis for formulating a calculable index satisfying at least the Färe-Lovell axioms.

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