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A Class of Two-Group Polarization Measures

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I. Introduction

Polarization is a phenomenon that has attracted an increasing amount of attention recently, both in Economics and in other social sciences. While it appears to be widely acknowledged that, in the context of income distributions, polarization has to do with the "disappearing middle class" (Wolfson, 1994, p. 353), a precise definition of the term has remained elusive. There are similarities to the notion of income inequality because certain mean-preserving spreads are typically assumed to increase both inequality and polarization. But there is more to polarization than that. In order to formalize the concept, society needs to be partitioned into groups with strong group identification and clear differentiation between groups.

Love and Wolfson (1976) introduced the concept of polarization into the study of income inequality in order to focus on the issue of a disappearing middle class. In later papers, Wolfson (1994, 1997) formalized the concept and introduced a specific measure of polarization, crediting some earlier unpublished joint work with James Foster. This notion of polarization divided people into two groups: those with incomes below and those with incomes above the median income. The measure increases with clustering of incomes within groups and with the difference of the mean income between groups. Wang and Tsui (2000) have described these concepts with two axioms called Increased Bipolarity and Increased Spread.

Esteban and Ray (1994) proposed a measure of polarization based on the notions of within group identification and intergroup alienation. They introduced a series of axioms that have a similar intent to the Foster-Wolfson axioms but have some important differences. They emphasized that polarization is related to the ability of a society to cooperatively make and implement group decisions. Esteban and Ray (1999) present a model of conflict in which the degree of polarization is related to the equilibrium level of conflict. Schultz (1996) uses a political economy model of public good provision to show that polarized preferences of political parties result in inefficient equilibria that do not reveal private information.

Existing measures of polarization have been applied empirically in many countries. The polarization of income distributions and its causes have been studied in Spain by Gradín (2000, 2002), in Italy by D'Ambrosio (2001), and in China by Zhang and Kanbur (2001). Duclos, Esteban and Ray (2004) present polarization estimates for the income distributions of 21 countries taken from the Luxembourg Income Study. Seshanna and Decornez (2003) study polarization for the distribution of income across countries in the world. Ravallion and Chen (1997) estimate Foster-Wolfson polarization indices for 67 developing and transitional economies.

We adopt an ordinal approach and, in addition to a basic anonymity requirement, introduce two axioms that are intended to capture the spirit of polarization as described by Wolfson (1994), Esteban and Ray (1994), and Wang and Tsui (2000).

One of these is stated in terms of Pigou-Dalton transfers of income. An income transfer from a person with a higher income to a person with a lower income that maintains their pairwise ranking is a *simple Pigou-Dalton transfer.* The first of the Foster-Wolfson axioms requires that a simple Pigou-Dalton transfer involving two individuals on the same side of the median does not decrease polarization, provided that the median is the same before and after the transfer.

The second axiom is a dominance property. Suppose two income distributions x and y with the same median are such that all individuals below the median are no richer in y than in x and all individuals above the median are no poorer in y than in x. The axiom requires that y is at least as polarized as x.

Note that both of these axioms are rather weak because their conclusions are conditional on the medians of the distributions to be compared being the same. This raises the questions whether more demanding axioms could be employed in order to rank more pairs of distributions and whether more general notions of group identification than that induced by the median could be defined. We will provide examples illustrating that this cannot be done easily and only at the expense of having to give up much of the intuitive appeal of the weaker axioms.

We prove three main results. First, we characterize the polarization quasi-orderings presented by Wolfson (1994) and refined by Wang and Tsui (2000) in terms of the generalized Lorenz quasi-ordering. We then move on to polarization measures which are real-valued representations of polarization orderings. In our second theorem, we identify all polarization measures satisfying suitably formulated versions of the axioms used in our first result. Third, we characterize all polarization measures that satisfy our basic axioms and an independence condition.

Our first result is a complete characterization of the class of polarization quasi-orderings that satisfy anonymity and the two axioms outlined above. This class is given by all supersets of the relation that is obtained by the conjunction of the generalized Lorenz quasi-orderings applied to the distributions below and above the median, conditional on the median being the same in two distributions to be ranked.

After identifying this class of polarization quasi-orderings, we examine polarization measures. We show that our axioms impose natural curvature and monotonicity properties on a polarization measure. Adding an independence property that requires the subgroup of those with incomes below (above) the median to be separable from its complement allows us to characterize an intuitively appealing class of polarization measures whose members can be interpreted as combinations of inequality-averse aggregators defined for the two groups.

Our characterization results are formulated for the specific case of two groups where the criterion used for the partitions is the median of the income distribution. It is natural to ask whether our setting can be generalized. We examine the possibilities of extending the basic axioms by allowing for a larger number of groups or for movements of individuals across the dividing line. Unfortunately, it turns out that appealing versions of the axioms are too weak to obtain the clear and transparent structure obtained for the basic model, and stronger axioms that would allow us to extend our results are far from intuitive.

Clearly, the method chosen to partition the population into groups is crucial for the plausibility of the resulting framework for measuring polarization. We examine the most obvious candidates for defining these groups and illustrate by means of some simple examples that all of them—including the median-based partition—lead to environments where the two fundamental axioms do not appear to have strong intuitive appeal. We conclude that the current approach to polarization based on intra-group coherence and inter-group differences may not be the best way of capturing many social phenomena involving notions of polarization.

II. Preliminaries

Suppose there are $n \geq 2$ individuals in a society. For simplicity, we assume that the population size n is even to avoid ambiguities when partitioning the population on the basis of the median income. Denote the income vector of an economy by $x = (x_1, \ldots, x_n) \in \mathbf{R}^n_{++}$ with x_i the positive income of person $i \in N = \{1, \ldots, n\}$. We denote an income vector with $x_i \leq x_{i+1}$ for all $i \in \{1, \ldots, n-1\}$ by $x_\uparrow = (x_{[1]}, \ldots, x_{[n]})$. The median income for the income vector x is denoted by $m(x)$ and the mean income is denoted by $\mu(x)$. For any positive integer r, we use $\mathbf{1}_r$ to denote the vector consisting of r ones.

An income vector y can be obtained from an income vector x through a *simple Pigou-Dalton transfer* if there exists an amount $\delta > 0$ and two individuals i and j in N such that $y_i = x_i + \delta < x_j - \delta = y_j$. Define the set $\mathcal{SPD}$ to be the set of all income pairs (y, x) such that y can be obtained from x by a simple Pigou-Dalton transfer.

The notion of a Pigou-Dalton transfer is closely related to the *Lorenz criterion.* Define the Lorenz function $L\colon \mathbf{R}^n \times N \to \mathbf{R}$ by

$$L(x, k) = \sum_{i=1}^{k} x_{[i]} \tag{2.1}$$

for all $k \in N$. The Lorenz quasi-ordering $\succeq_L$ applies to distributions with equal means only and is defined as follows. For all $x, y \in \mathbf{R}^n$ such that $\mu(x) = \mu(y)$,

$$y \succeq_L x \;\Leftrightarrow\; L(y, k) \geq L(x, k) \;\; \forall k \in N. \tag{2.2}$$

A basic property of the Lorenz quasi-ordering is that, for any pair of income vectors with the same mean, $y \succeq_L x$ if and only if y can be obtained from x by a sequence each element of which is a permutation or a simple Pigou-Dalton transfer.

Shorrocks (1983) introduced the *generalized* Lorenz quasi-ordering $\succeq_{GL}$ to extend the Lorenz criterion to pairs of income distributions whose means are not necessarily the same. It is defined as follows. For all $x, y \in \mathbf{R}^n$,

$$y \succeq_{GL} x \;\Leftrightarrow\; L(y,k) \geq L(x,k) \;\forall k \in N. \tag{2.3}$$

The difference between the Lorenz criterion and its generalized counterpart is that the former applies to distributions with identical means only, whereas the latter is capable of ranking vectors with different means as well. Both are based on the Lorenz function L. The generalized Lorenz criterion is equivalent to second-order stochastic dominance. Clearly, the Lorenz quasi-ordering and the generalized Lorenz quasi-ordering agree on all pairs of income vectors with the same mean. If the means of two income vectors differ, then the generalized Lorenz ranking will depend on the Lorenz ranking and a comparison of the means.

III. Axioms for Polarization Quasi-Orderings

A polarization quasi-ordering is a reflexive and transitive binary relation $\succeq_P$ defined on income distributions. Thus, $y \succeq_P x$ means that income distribution $y \in \mathbf{R}^n_{++}$ is at least as polarized as income vector $x \in \mathbf{R}^n_{++}$. If $\succeq_P$ is representable by a real-valued function $P\colon \mathbf{R}^n_{++} \to \mathbf{R}$, we refer to P as a polarization measure.

We begin with an anonymity axiom that ensures that a polarization quasi-ordering only depends on incomes and ignores the identities of the individuals.

Anonymity (A): For all $x, y \in \mathbf{R}^n_{++}$, if y is a permutation of x, then $y \succeq_P x$ and $x \succeq_P y$.

The foundation for polarization is a partition of society into identifiable groups. We follow Wolfson (1994) in dividing the population into a low-income group with incomes below the median income and a high-income group with incomes above the median. Letting $x^L = (x_{[1]}, \ldots, x_{[n/2]}) \in \mathbf{R}^{n/2}_{++}$ denote the rank-ordered income vector of the low-income individuals and $x^H = (x_{[n/2+1]}, \ldots, x_{[n]}) \in \mathbf{R}^{n/2}_{++}$ the rank-ordered vector composed of the high-income individuals, the partition of the income vector x induced by the median into the low-income group and the high-income group is (x^L, x^H).

We characterize polarization quasi-orderings in terms of within-group Pigou-Dalton transfers and between-group spread. The axioms we employ are analogous to the axioms of Increased Bipolarity and Increased Spread, respectively, of Wang and Tsui (2000).

The *Within Group Clustering* axiom (WGC) requires that polarization does not decrease as a consequence of a simple Pigou-Dalton transfer between a pair of individuals who are in the same group. A progressive transfer between two people on the same side of the median will cause a mean-preserving contraction of the distribution of income for the group. In these cases, incomes are being clustered around the group mean income. The axiom requires that such a change in the distribution will not decrease polarization. The idea is that a clustering of incomes either above or below the median will increase the similarity and hence the cohesion of the group; see Figure 1.

Within-Group Clustering (WGC): For all $x, y \in \mathbf{R}^n_{++}$ such that $m(x) = m(y)$,

(i) if $(y^L, x^L) \in \mathcal{SPD}$ and $y^H = x^H$, then $y \succeq_P x$;

(ii) if $(y^H, x^H) \in \mathcal{SPD}$ and $y^L = x^L$, then $y \succeq_P x$.

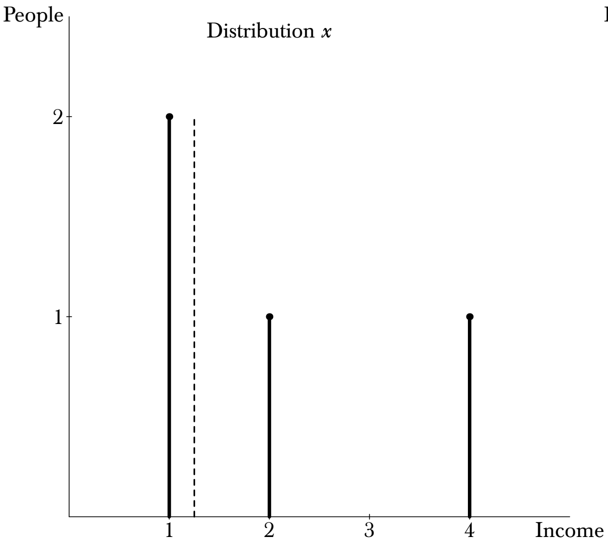


Figure 1x.

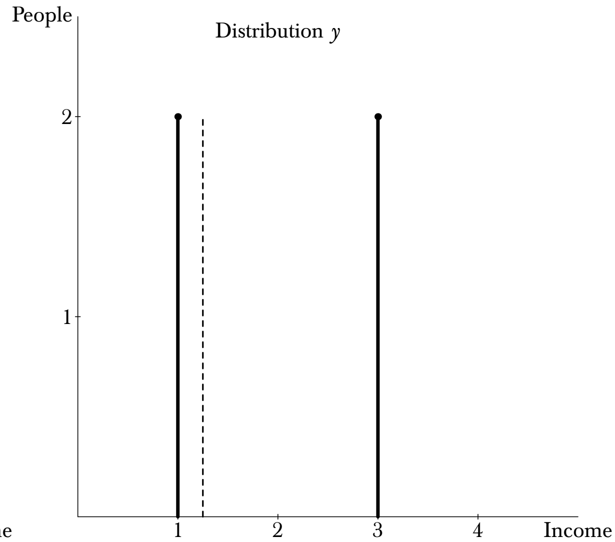
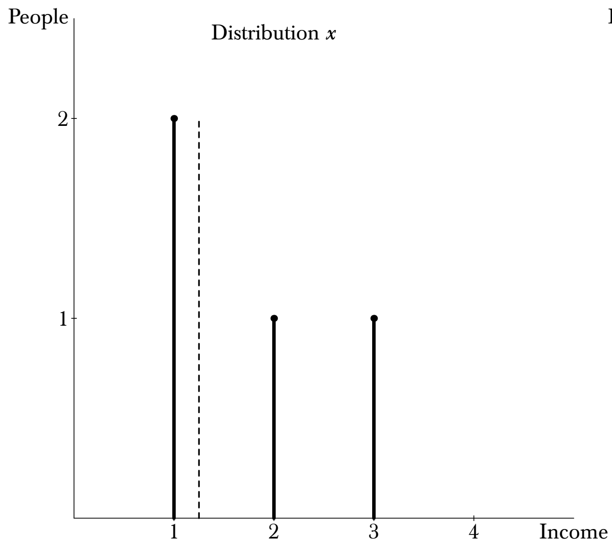


Figure 1y.

The *Between-Group Spread* axiom (BGS) requires that a polarization measure does not decrease if those below the median become (weakly) worse off and those above the median become (weakly) better off. This move (weakly) decreases the low-group mean and (weakly) increases the high-group mean. Since the two group means move further apart, polarization does not decrease.

Between-Group Spread (BGS): For all $x, y \in \mathbf{R}^n_{++}$ such that $m(x) = m(y)$, if $y^L \leq x^L$ and $y^H \geq x^H$, then $y \succeq_P x$.

Figure 2 displays an income change as in the axiom statement. BGS implies that this change does not decrease polarization.



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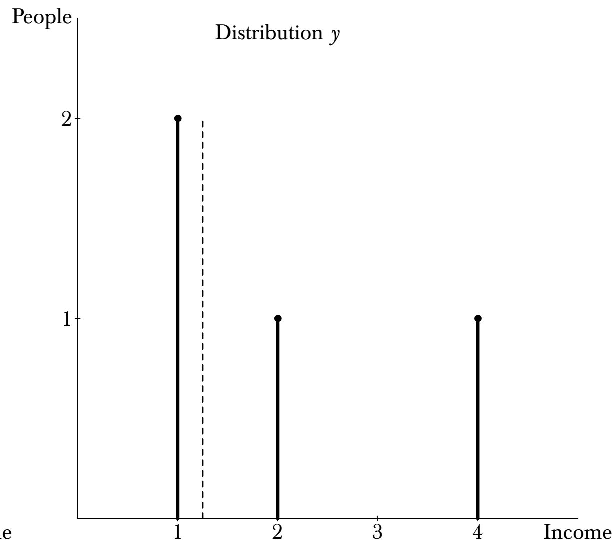

Figure 2y.

IV. Existing Polarization Measures

Wolfson (1994), crediting unpublished work in collaboration with James Foster, proposed a polarization measure P_{FW} that is defined as follows. For all $x \in \mathbf{R}^n_{++}$,

$$P_{FW}(x) = \frac{2\mu(x)}{m(x)} \left[\frac{\big(\mu(x^H) - \mu(x^L)\big)}{\mu(x)} - G(x) \right] \tag{4.1}$$

where $G\colon \mathbf{R}^n_{++} \to \mathbf{R}$ is the Gini inequality index. This measure has an interesting geometrical connection to the Lorenz curve that is explored in Wolfson (1994).

Esteban and Ray (1994) assume that a polarization measure has an additive representation based on intragroup identification and intergroup alienation. Their assumption regarding the functional form of the index is a strong requirement that severely restricts the set of admissible polarization measures. With their axioms and this assumption, they derive a class of polarization measures that are closely linked to the Gini index when applied to groups rather than individuals. To define their measures, we require more notation. For $x \in \mathbf{R}^n_{++}$, let $K(x) \in N$ be the number of distinct incomes in distribution x. Furthermore, let $\xi(x) = (\xi_1(x), \ldots, \xi_{K(x)}(x))$ be the vector of these distinct incomes and let $\pi(x) = (\pi_1(x), \ldots, \pi_{K(x)}(x))$ be the vector of population shares with the corresponding income. That is, $\pi_k(x)$ is the proportion of individuals in N with income $\xi_k(x)$. Esteban and Ray's (1994) class of measures P^{α}_{ER} is defined by

$$P^{\alpha}_{ER}(x) = \frac{1}{2\mu(x)} \sum_{k=1}^{K(x)} \sum_{\ell=1}^{K(x)} \pi_k(x)^{1+\alpha} \pi_\ell(x) |\xi_k(x) - \xi_\ell(x)| \tag{4.2}$$

for all $x \in \mathbf{R}^n_{++}$, where $\alpha \in (0, 1.6]$ is a parameter. The index approaches the Gini index defined in terms of income groups as α approaches zero. For larger values of α, polarization differs from an inequality measure as required by the axioms.

Esteban, Gradín, and Ray (2007) introduced a modification of the earlier Esteban and Ray (1994) measure to incorporate an error in grouping people into the required discrete groups. Their measure is a generalization of both the Esteban and Ray (1994) measure and the Foster-Wolfson measure. The additive structure employed by Esteban and Ray (1994) is preserved.

Duclos, Esteban, and Ray (2004) measured polarization for continuous rather than discrete distributions and proposed a suitable framework for a sampling theory. These measures can be viewed as continuous analogues of those introduced in the discrete setting by Esteban and Ray. As in Esteban and Ray (1994), polarization is assumed to be proportional to the "sum" of effective antagonisms.

Wang and Tsui (2000) proposed a generalized Foster-Wolfson measure of polarization that satisfies their axioms of Increased Bipolarity and Increased Spread. In addition to imposing these axioms, they restricted themselves to polarization measures that are additive in rank-ordered individual incomes. Specifically, they required that a polarization measure P can be written as

$$P(x) = \frac{1}{n}\sum_{i=1}^{n} a_i x_{[i]} \tag{4.3}$$

for all $x \in \mathbf{R}^n_{++}$, with constants $a_i \in \mathbf{R}$ for all $i \in N$. They demonstrated that the only polarization measures with this representation that satisfy their axioms and a normalization are given by

$$P_{WT}(x) = \sum_{i=1}^{n} a_i \left|\frac{m(x) - x_{[i]}}{m(x)}\right| \tag{4.4}$$

for all $x \in \mathbf{R}^n_{++}$, where the coefficients are such that $0 < -a_i < -a_{i+1}$ for all $i \in \{1, \ldots, n/2 - 1\}$ and $a_i > a_{i+1} > 0$ for all $i \in \{n/2 + 1, \ldots, n - 1\}$. Furthermore, they demonstrated that the Foster-Wolfson measure is a special case of this formulation.

There are alternatives to the additive structure shared by the above-described measures. For instance, Chakravarty and Majumder (2001) discuss polarization measures that are linked to specific classes of inequality measures and that do not satisfy the stringent additivity requirements as formulated above.

V. A Class of Polarization Quasi-Orderings

We now identify the class of polarization quasi-orderings that satisfy our three axioms. They can be expressed in terms of the generalized Lorenz criterion, restricted to comparisons of vectors with the same median. Specifically, we prove that all polarization quasi-orderings satisfying the axioms are supersets of a quasi-ordering $\succeq_0$ that is defined in terms of the generalized Lorenz quasi-ordering.

Theorem 1: *A polarization quasi-ordering $\succeq_P$ satisfies A, WGC, and BGS if and only if $\succeq_0 \subseteq \succeq_P$ where*

$$y \succeq_0 x \;\Leftrightarrow\; m(x) = m(y) \;\text{ and }\; -y^L \succeq_{GL} -x^L \;\text{ and }\; y^H \succeq_{GL} x^H. \tag{5.1}$$

for all $x, y \in \mathbf{R}^n_{++}$.

Proof: If. Suppose $\succeq_0 \subseteq \succeq_P$.

That A is satisfied by $\succeq_P$ follows immediately by definition.

To prove that $\succeq_P$ satisfies WGC, let $x, y \in \mathbf{R}^n_{++}$ be such that $m(x) = m(y)$. To establish part (i), suppose $(y^L, x^L) \in \mathcal{SPD}$ and $y^H = x^H$. This implies

$$y^L \succeq_L x^L \tag{5.2}$$

and, by definition of the Lorenz criterion,

$$-y^L \succeq_L -x^L. \tag{5.3}$$

Because $\mu(y^L) = \mu(x^L)$, we obtain

$$-y^L \succeq_{GL} -x^L. \tag{5.4}$$

Moreover,

$$y^H \succeq_{GL} x^H \tag{5.5}$$

because $\succeq_{GL}$ is reflexive. Hence, $y \succeq_0 x$ and, because $\succeq_0 \subseteq \succeq_P$ by assumption, we obtain $y \succeq_P x$. Part (ii) is proven analogously.

To establish BGS, suppose $x, y \in \mathbf{R}^n_{++}$ are such that $m(x) = m(y)$, $y^L \leq x^L$, and $y^H \geq x^H$. This implies

$$-y^L \geq -x^L \tag{5.6}$$

and, thus,

$$-y^L \succeq_{GL} -x^L \;\text{ and }\; y^H \succeq_{GL} x^H. \tag{5.7}$$

By definition, $y \succeq_0 x$ and hence $y \succeq_P x$.

Only if. Suppose $\succeq_P$ satisfies A, WGC, and BGS, and let $x, y \in \mathbf{R}^n_{++}$ be such that $y \succeq_0 x$. We have to prove that $y \succeq_P x$. By definition of $\succeq_0$, we have $m(x) = m(y)$, $-y^L \succeq_{GL} -x^L$, and $y^H \succeq_{GL} x^H$. Define $z_1^L = x_1^L + \sum_{i=1}^{n/2} y_i^L - \sum_{i=1}^{n/2} x_i^L$ and $z_i^L = x_i^L$ for all $i \in \{2, \ldots, n/2\}$. Because $-y^L \succeq_{GL} -x^L$, it follows that

$$\sum_{i=1}^{n/2} -y_i^L - \sum_{i=1}^{n/2} -x_i^L \geq 0 \tag{5.8}$$

and, thus,

$$z_1^L = x_1^L + \sum_{i=1}^{n/2} y_i^L - \sum_{i=1}^{n/2} x_i^L \leq x_1^L \tag{5.9}$$

so that

$$z^L \leq x^L. \tag{5.10}$$

Furthermore,

$$\sum_{i=k}^{n/2} -z_i^L = \sum_{i=k}^{n/2} -x_i^L \leq \sum_{i=k}^{n/2} -y_i^L \quad \forall k \in \{2, \dots, n/2\} \tag{5.11}$$

and

$$\sum_{i=1}^{n/2} -z_i^L = -x_1^L + \sum_{i=1}^{n/2} -y_i^L + \sum_{i=1}^{n/2} x_i^L + \sum_{i=2}^{n/2} -x_i^L = \sum_{i=1}^{n/2} -y_i^L. \tag{5.12}$$

Because the rank-order of the components of $-z^L$ and $-y^L$ is the reverse of the rank-order of those of z^L and y^L, it follows that

$$-y^L \succeq_L -z^L \tag{5.13}$$

and thus

$$y^L \succeq_L z^L. \tag{5.14}$$

Analogously, define $z_{n/2}^H = x_{n/2}^H + \sum_{i=1}^{n/2} y_i^H - \sum_{i=1}^{n/2} x_i^H$ and $z_i^H = x_i^H$ for all $i \in \{1, \dots, n/2 - 1\}$. Because $y^H \succeq_{GL} x^H$, it follows that

$$\sum_{i=1}^{n/2} y_i^H - \sum_{i=1}^{n/2} x_i^H \geq 0 \tag{5.15}$$

and, thus,

$$z_{n/2}^H = x_{n/2}^H + \sum_{i=1}^{n/2} y_i^H - \sum_{i=1}^{n/2} x_i^H \geq x_{n/2}^H \tag{5.16}$$

so that

$$z^H \geq x^H. \tag{5.17}$$

Furthermore,

$$\sum_{i=1}^{k} z_i^H = \sum_{i=1}^{k} x_i^H \leq \sum_{i=1}^{k} y_i^H \quad \forall k \in \{1, \dots, n/2 - 1\} \tag{5.18}$$

and

$$\sum_{i=1}^{n/2} z_i^H = \sum_{i=1}^{n/2-1} x_i^H + x_{n/2}^H + \sum_{i=1}^{n/2} y_i^H - \sum_{i=2}^{n/2} x_i^H = \sum_{i=1}^{n/2} y_i^H \tag{5.19}$$

so that

$$y^H \succeq_L z^H. \tag{5.20}$$

By (5.10), (5.17), and BGS, $z \succeq_P x$. By (5.14), (5.20), repeated application of A or WGC, and transitivity,

$$y = (y^L, y^H) \succeq_P (z^L, y^H) \text{ and } (z^L, y^H) \succeq_P (z^L, z^H) = z \tag{5.21}$$

and the transitivity of $\succeq_P$ implies $y \succeq_P x$. ■

Wolfson (1994) defined the Foster-Wolfson polarization quasi-ordering in terms of income distribution functions with common medians. Wang and Tsui (2000) presented the following definition in terms of the income vectors. For any two income distributions $x, y \in \mathbf{R}^n_{++}$ such that $m(x) = m(y) = M$,

$$\begin{aligned} y \succeq_{FW} x \;\Leftrightarrow\; & \sum_{i=k}^{n/2}(M - x_i) \;\leq\; \sum_{i=k}^{n/2}(M - y_i) \;\; \forall k \in \{1, \ldots, n/2\} \\ \text{and } & \sum_{i=n/2+1}^{k} (x_i - M) \;\leq\; \sum_{i=n/2+1}^{k} (y_i - M) \;\; \forall k \in \{n/2+1, \ldots, n\}. \end{aligned} \tag{5.22}$$

Simple algebra establishes that $\succeq_0 = \succeq_{FW}$ when the quasi-orderings are restricted to comparisons involving the same median and, thus, an alternative way of characterizing the quasi-orderings of Theorem 1 is to describe them as supersets of the Foster-Wolfson quasi-ordering. Thus, we obtain

Corollary 1.1: $\succeq_0 = \succeq_{FW}$.

VI. A Class of Polarization Measures

Suppose that $\succeq_P$ is representable by a polarization measure $P: \mathbf{R}^n_{++} \to \mathbf{R}$ (which implies, in particular, that $\succeq_P$ is an ordering and not merely a quasi-ordering). We now examine the consequences of our axioms in terms of monotonicity and curvature properties of P. The relevant curvature property is *S-concavity;* see Marshall and Olkin (1979). Clearly, A, WGC, and BGS can be expressed as properties of P by replacing every occurrence of the term "$y \succeq_P x$" with "$P(y) \geq P(x)$." For any $M \in \mathbf{R}_{++}$, let $D_M = \{x \in \mathbf{R}^n_{++} \mid m(x) = M\}$. Furthermore, let $P_M: D_M \to \mathbf{R}$ be the restriction of P to D_M. We obtain

Theorem 2: *A polarization measure P satisfies A, WGC, and BGS if and only if, for all $M \in \mathbf{R}_{++}$, P_M is non-increasing and S-concave in x^L and non-decreasing and S-concave in x^H.*

Proof: If. Suppose P_M is non-increasing and S-concave in x^L and non-decreasing and S-concave in x^H for all $M \in \mathbf{R}_{++}$.

As is well-known, A is implied by S-concavity.

To show that part (i) of WGC is satisfied, suppose $x, y \in D_M$ are such that $(y^L, x^L) \in \mathcal{SPD}$ and $y^H = x^H$. This implies

$$P(y) = P_M(y^L, y^H) \geq P_M(x^L, y^H) = P_M(x^L, x^H) = P(x) \tag{6.1}$$

because P_M is S-concave in x^L. Part (ii) of WGC is proven analogously.

Now suppose $x, y \in \mathbf{R}^n_{++}$ are such that $m(x) = m(y) = M$, $y^L \leq x^L$, and $y^H \geq x^H$. Because P_M is non-increasing in x^L and non-decreasing in x^H, it follows immediately that

$$P(y) = P_M(y) \geq P_M(x) = P(x) \tag{6.2}$$

and BGS is satisfied.

Only if. Suppose P satisfies A, WGC, and BGS.

We first establish that P_M must be S-concave in x^L for all $M \in \mathbf{R}_{++}$. Consider any $x = (x^L, x^H) \in D_M$ and any bistochastic $(n/2) \times (n/2)$ matrix B. By the properties of a bistochastic matrix (see, for instance, Marshall and Olkin, 1979), Bx^L can be reached from x^L through a finite sequence of simple Pigou-Dalton transfers or permutations. Therefore, there exist $K \in \mathbf{N}$ and $z^0, \ldots, z^K \in \mathbf{R}^{n/2}_{++}$ such that $Bx^L = z^0$, $(z^{k-1}, z^k) \in \mathcal{SPD}$ or z^k is a permutation of z^{k-1} for all $k \in \{1, \ldots, K\}$, and $z^K = x^L$. Moreover, by definition, $m(z^k, x^H) = M$ for all $k \in \{1, \ldots, K\}$. By repeated application of A or part (i) of WGC,

$$P_M(Bx^L, x^H) = P(Bx^L, x^H) = P(z^0, x^H) \geq \ldots \geq P(z^K, x^H) = P(x^L, x^H) = P_M(x^L, x^H) \tag{6.3}$$

and, thus, P_M is S-concave in x^L. That P_M is S-concave in x^H follows from replacing part (i) with part (ii) of WGC in the above argument.

That P_M is non-increasing in x^L and non-decreasing in x^H follows immediately from BGS. ■

We now add an independence property to our list of axioms in order to obtain polarization as a function that can be expressed in terms of two functions—one applied to x^L and one applied to x^H. For any fixed value of the median, the independence axiom requires that x^L is strictly separable from its complement x^H in P and, conversely, x^H is strictly separable from x^L in P; see Blackorby, Primont, and Russell (1978, Chapter 3).

Independence (IND): For all $x, y \in \mathbf{R}^n_{++}$ such that $m(x) = m(y)$,

(i) $P(y^L, y^H) \geq P(x^L, y^H) \;\Leftrightarrow\; P(y^L, x^H) \geq P(x^L, x^H)$;

(ii) $P(y^L, y^H) \geq P(y^L, x^H) \;\Leftrightarrow\; P(x^L, y^H) \geq P(x^L, x^H)$.

To illustrate that the axiom merely is a slight strengthening of a restricted version that is already implied by our other properties, we note that the variant of IND suitably defined for a polarization quasi-ordering is implied by A, WGC, and BGS on the decisive set for the quasi-ordering $\succeq_0$. To see that this is the case, define

$$\Omega = \{\, (x, y) \mid y \succeq_0 x \text{ or } x \succeq_0 y \,\}\,. \tag{6.4}$$

This immediately yields the following corollary.

Corollary 2.1: *A polarization quasi-ordering $\succeq_P$ satisfying A, WGC, and BGS satisfies IND on Ω.*

While the independence axiom is satisfied on this subset of $\mathbf{R}^n_{++} \times \mathbf{R}^n_{++}$, the result cannot be extended to the complete space. The independence property is implied by the independence of the minimal quasi-order $\succeq_0$ but need not hold for any completion of $\succeq_0$. Therefore, any polarization measure need not satisfy the independence axiom on the full domain. Our independence axiom extends the independence property to the full domain and enables us to obtain simpler functional representations.

Theorem 3: *A polarization measure P satisfies A, WGC, BGS, and IND if and only if, for all $M \in \mathbf{R}_{++}$, there exist an increasing function $\Phi_M\colon \mathbf{R}^2 \to \mathbf{R}$, a non-increasing and S-concave function $\phi^L_M\colon \mathbf{R}^{n/2}_{++} \to \mathbf{R}$, and a non-decreasing and S-concave function $\phi^H_M\colon \mathbf{R}^{n/2}_{++} \to \mathbf{R}$ such that, for all $M \in \mathbf{R}_{++}$ and for all $x \in D_M$,*

$$P_M(x^L, x^H) = \Phi_M\left(\phi^L_M(x^L), \phi^H_M(x^H)\right). \tag{6.5}$$

Proof: If. Let $M \in \mathbf{R}_{++}$ and suppose P_M is given by (6.5). That A, WGC, and BGS are satisfied follows as in Theorem 2. Part (i) of IND is established by noting that (6.5) implies

$$\begin{aligned} P(y^L, y^H) \geq P(x^L, y^H) &\Leftrightarrow P_M(y^L, y^H) \geq P_M(x^L, y^H) \\ &\Leftrightarrow \Phi_M\left(\phi^L_M(y^L), \phi^H_M(y^H)\right) \geq \Phi_M\left(\phi^L_M(x^L), \phi^H_M(y^H)\right) \\ &\Leftrightarrow \phi^L_M(y^L) \geq \phi^L_M(x^L) \end{aligned} \tag{6.6}$$

because Φ_M is increasing and, analogously,

$$\begin{aligned} P(y^L, x^H) \geq P(x^L, x^H) &\Leftrightarrow P_M(y^L, x^H) \geq P_M(x^L, x^H) \\ &\Leftrightarrow \Phi_M\left(\phi_M^L(y^L), \phi_M^H(x^H)\right) \geq \Phi_M\left(\phi_M^L(x^L), \phi_M^H(x^H)\right) \\ &\Leftrightarrow \phi_M^L(y^L) \geq \phi_M^L(x^L). \end{aligned} \tag{6.7}$$

Combining the two equivalences establishes (i). The proof of (ii) is analogous.

Only if. Suppose P satisfies A, WGC, BGS, and IND. Let $M \in \mathbf{R}_{++}$. By Theorem 2, P_M is non-increasing and S-concave in x^L and non-decreasing and S-concave in x^H. Fix an arbitrary $\bar{y}^H > M$ and define $\phi_M^L(y^L) = P_M(y^L, \bar{y}^H)$ for all y^L. ϕ_M^L is non-increasing and S-concave because P_M is non-increasing and S-concave in x^L. Analogously, fix $\bar{y}^L < M$ and let $\phi_M^H(y^H) = P_M(\bar{y}^L, y^H)$ for all y^H. ϕ_M^L is non-increasing and S-concave because P_M is non-increasing and S-concave in x^L, and ϕ_M^H is non-decreasing and S-concave because P_M is non-decreasing and S-concave in x^H. Now define $\Phi_M(\phi_M^L(y^L), \phi_M^H(y^H)) = P_M(y^L, y^H)$. To see that Φ_M^L is increasing in its first argument, use the definitions of ϕ_M^L and Φ_M and IND to obtain

$$\begin{aligned} \phi_M^L(y^L) > \phi_M^L(x^L) &\Leftrightarrow P_M(y^L, \bar{y}^H) > P_M(x^L, \bar{y}^H) \\ &\Leftrightarrow P_M(y^L, y^H) > P_M(x^L, y^H) \\ &\Leftrightarrow \Phi_M\left(\phi_M^L(y^L), \phi_M^H(y^H)\right) > \Phi_M\left(\phi_M^L(x^L), \phi_M^H(y^H)\right). \end{aligned} \tag{6.8}$$

That Φ_M is increasing in its second argument is shown analogously. ■

The function ϕ_M^H can be interpreted as an inequality-sensitive welfare indicator for the high-income group and, analogously, ϕ_M^L has the properties of an inequality-sensitive reverse welfare indicator defined for the subdistribution below the median. Thus, Theorem 3 provides an intuitive representation of polarization as an aggregate of these two components.

VII. Multiple Groups and Variable Population

A first natural extension of our framework is to allow for more than two groups in assessing polarization. The extent to which the between-group spread axiom can be generalized to that setting in a natural fashion is rather limited, even if the partition of the population into groups is clear-cut and uncontroversial (see the next section for a discussion of the grouping issue). While the monotonicity property expressed by between-group spread generalizes naturally to the lowest-income and highest-income groups, there is no plausible analogue for groups in between the two extremes.

Suppose $n = 6$, $x = (\mathbf{1}_2, (2-\varepsilon)\mathbf{1}_2, 3\mathbf{1}_2)$ and $y = (\mathbf{1}_2, (2+\varepsilon)\mathbf{1}_2, 3\mathbf{1}_2)$, where $\varepsilon \in (0, 1/2)$; see Figure 3. Three groups are identified according to the vertical dotted lines. The

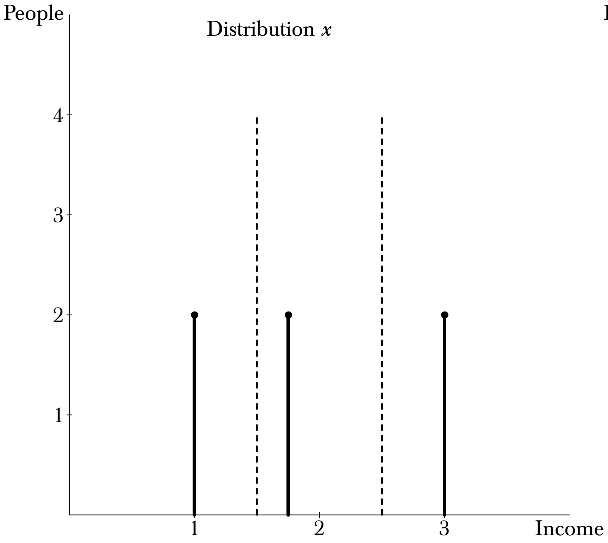

Figure 3x.

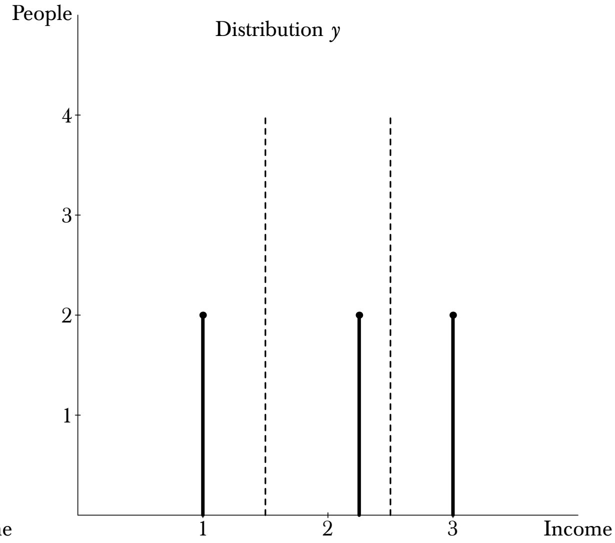

Figure 3y.

middle-group income vector has increased in the move from x to y but there is no reason to declare one of the two distributions more polarized than the other.

As a consequence of this observation, the effect of increasing or decreasing the incomes of the members of an intermediate group is not limited to the dominance considerations underlying a similar move at the extremes, and the dominance effects cannot be disentangled from distributional effects. This prevents us from extending the clear-cut separation of the two groups to a multi-group environment.

Similar problems arise if we attempt to generalize our model by making the axioms apply to situations where individuals may move from one side of the median to the other. Esteban and Ray (1994) suggest that, abstracting from other considerations, the more equal the cardinalities of the two groups are, the more polarized society is. Again, this effect cannot be disentangled from distributional and dominance effects, except in very limited circumstances that do not allow us to generalize the results of our theorems adequately. Consider the two-group case and suppose there are more individuals in the high-income group than in the low-income group. Suppose an individual in the high-income group falls below the median in the move from one distribution to another, all else unchanged. Clearly, this move is not necessarily distributionally neutral for both groups (actually, it is rather unlikely that this case occurs), which introduces a first ambiguity. Moreover, the move increases total income in the low-income group and decreases total income in the high-income group, which would tend to decrease polarization on dominance grounds. Thus, there is no reason to believe that such a move towards equalization of the sizes of the two groups increases polarization. As is the case for the multi-group extension discussed above, the impossibility of disentangling the effects of such a move across the median line prevents us from formulating a plausible axiom that would take us anywhere near a class of measures where the relative size of the groups is an additional independent determinant of polarization.

VIII. Defining Groups

While our basic axioms are plausible and appealing if there are two well-defined groups before and after the change in the income distribution considered in the respective axiom, the underlying intuition does not survive when trying to apply analogous reasoning to distributions where the division into groups is not as clear-cut. This observation applies not only to the median-based or quantile-based model considered here, but to more general notions of group definitions.

Consider first a division of the population according to some criterion that is independent of the variable studied (in our case, income)—for example, we may partition the population into males and females. In that case, the notion of polarization examined here (and in the current literature) does not seem to be suitable.

Suppose we have a population of two men and two women. In distribution x, both men have an income of 2 and the incomes of the women are 1 and 3. If, in income distribution y, the incomes of the men are unchanged and both women have an income of 2, y can be obtained from x by means of a simple Pigou-Dalton transfer among the women and, if the axiom within-group clustering is adopted in this setting, y should be at least as polarized than x. However, everyone is the same in y, and this situation is associated with a minimal degree of polarization, and the appeal of WGC appears to be highly questionable. See Figure 4 for an illustration.

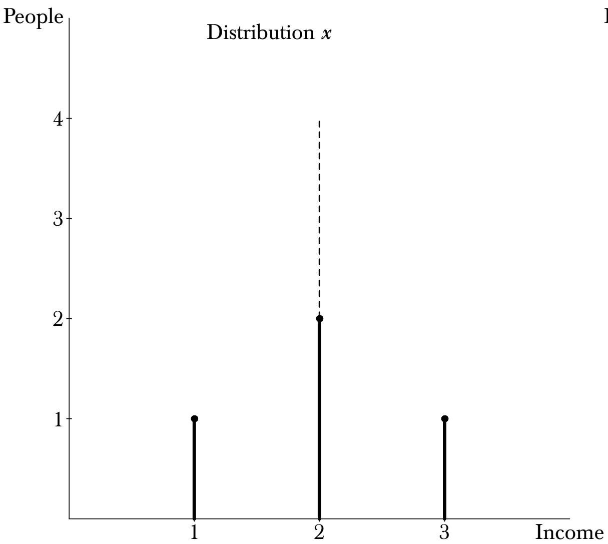

Figure 4x.

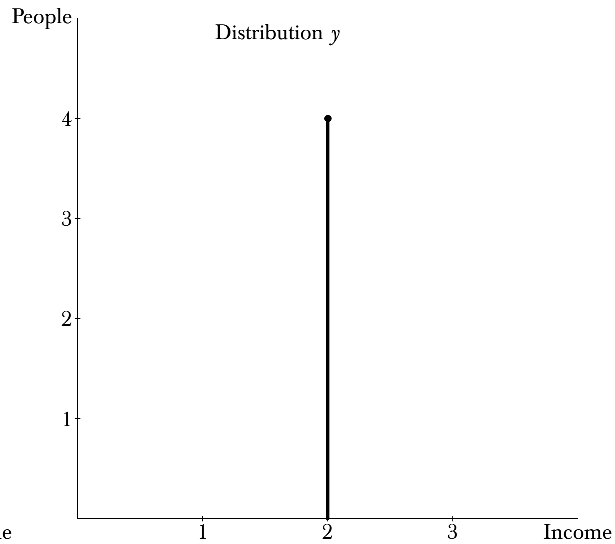

Figure 4y.

Between-group spread does not fare any better in this setting. Clearly, there is no reason to believe that polarization should be increasing in the incomes of one group and decreasing in the incomes of the other if these groups are defined independently of the distribution under consideration.

A first alternative to an independent definition of a partition is to use a fixed income level y_0 to separate the population into two groups. Suppose the low-income group is defined as the set of those with incomes below y_0 and the high-income group consists of the individuals with income y_0 or higher (the group to which a person at y_0 is assigned is arbitrary and does not affect our discussion in any way).

Suppose there are four individuals and $y_0 = 3$. In distribution x, incomes are $(1, 1, 3-\varepsilon, 3+\varepsilon)$ with $\varepsilon > 0$ small. Now consider the distribution $y = (1, 2-\varepsilon, 2, 3+\varepsilon)$. y is obtained from x via a simple Pigou-Dalton transfer among the low-income recipients, and WGC demands a weak increase in polarization. But this seems very counter-intuitive; see Figure 5.

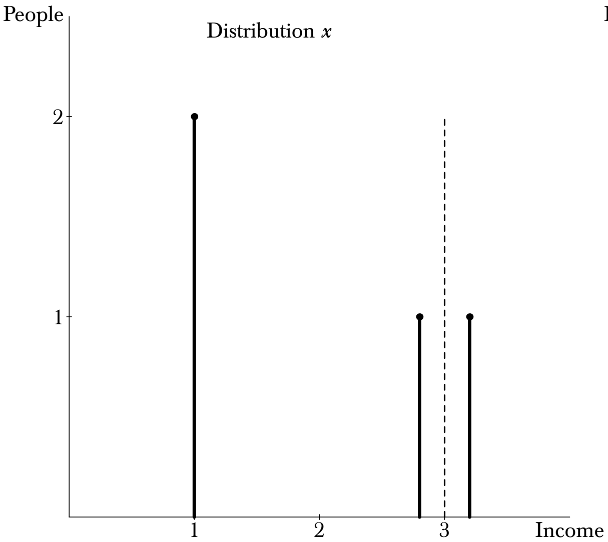

Figure 5x.

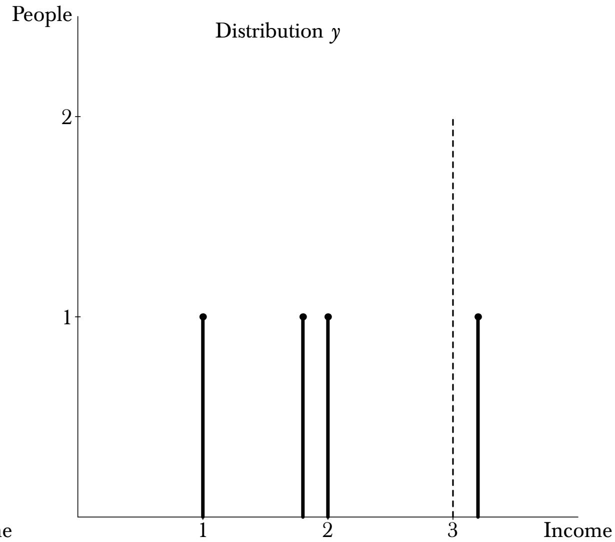

Figure 5y.

Now consider BGS. As before, suppose $n = 4$ and $y_0 = 3$. Let $x = (1, 1, 2, 3+\varepsilon)$ and $y = (1, 1, 3-\varepsilon, 3+\varepsilon)$. BGS requires that y is at most as polarized than x but, again, this does not conform to our intuition regarding the relative polarization ranking of the two distributions. See Figure 6 for an illustration of this example.

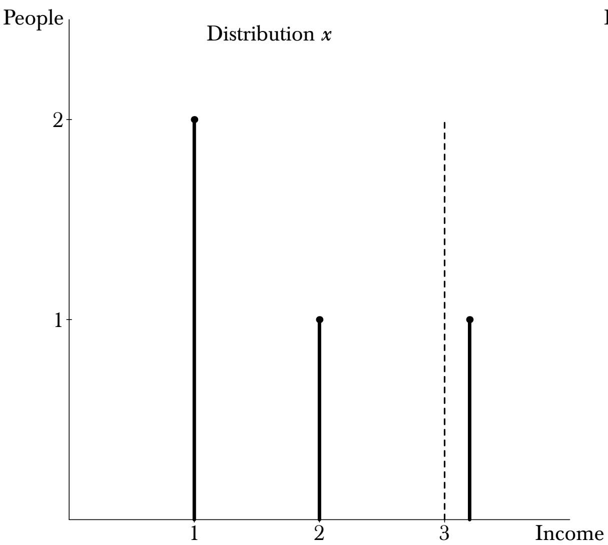

Figure 6x.

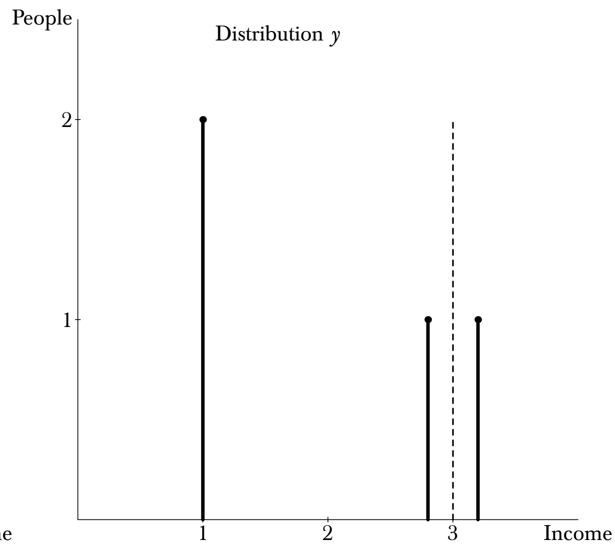

Figure 6y.

The final method we consider is the median-based criterion (and its generalization to arbitrary quantiles). Even in this case, examples that call into doubt the intuitive appeal of the two axioms are readily found.

Suppose $n = 24$, $x = (\mathbf{1}_{12}, 1+\varepsilon, 7\mathbf{1}_{11})$ and $y = (\mathbf{1}_{12}, (6+\varepsilon/6)\mathbf{1}_6, 7\mathbf{1}_6)$, where $\varepsilon > 0$ is small. We have $y^L = x^L$ and $(y^H, x^H) \in \mathcal{SPD}$ but it is by no means clear that $y \succeq_P x$—which is required by WGC—is a reasonable requirement. See Figure 7 for an illustration.

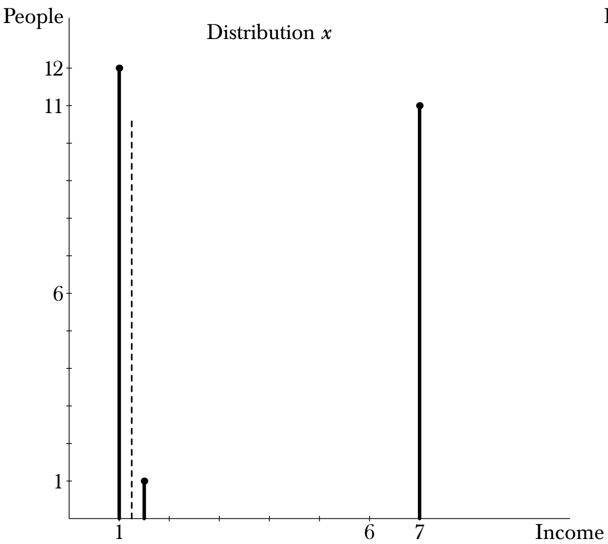

Figure 7x.

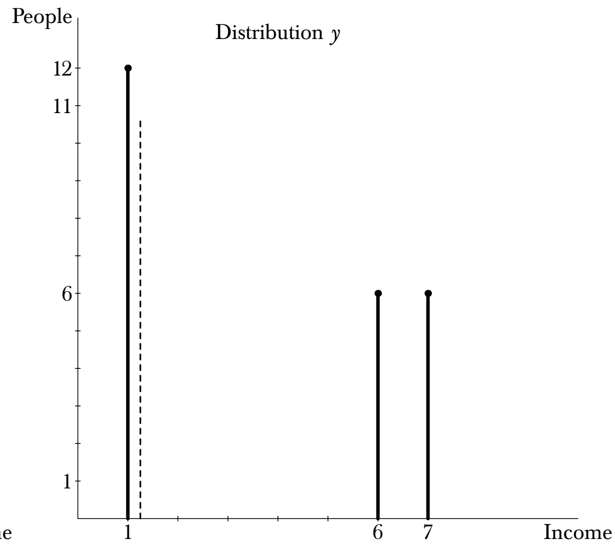

Figure 7y.

As a final example, let $n = 4$, $x = (1, 1, 1+\varepsilon, 3)$ and $y = (1, 1, 2, 3)$ with a small $\varepsilon > 0$. It follows that $y^L \leq x^L$ and $y^H \geq x^H$ but it is not clear that y should be considered at least as polarized as x, as required by BGS; see Figure 8.

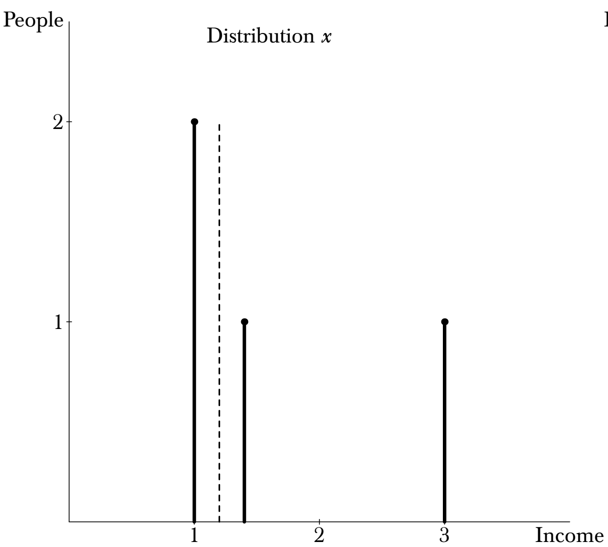

Figure 8x.

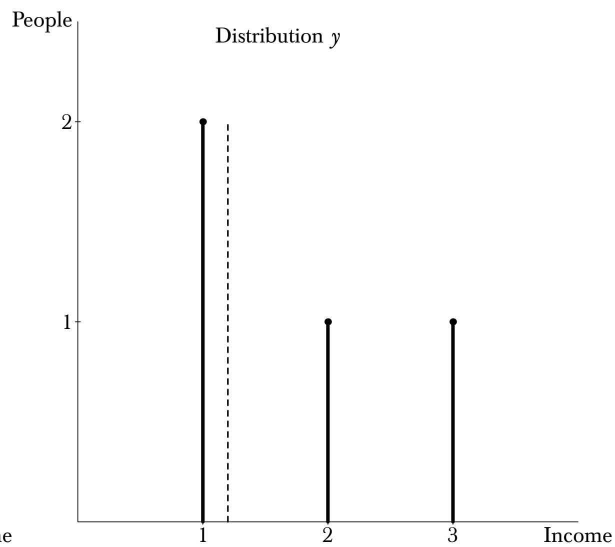

Figure 8y.

The last two examples are formulated using the median as the basis of partitioning the population but it is immediate that they generalize easily to arbitrary quantiles. While the three general methods of defining groups discussed in this section do not exhaust all possibilities, it appears that the method employed in the examples can be used to generate analogous examples for arbitrary ways of assigning partitions to distributions. Note that the quantiles category encompasses all situations where the set of individuals above (below) the dividing line is the same in two distributions to be compared, and situations where people may move from one side to another have been dealt with in the previous section. We do not present a formal way of defining a pair of examples for each possible way of defining a partition because, clearly, the examples would have to depend on the grouping method employed.

IX. Concluding Remarks

Our theorems illustrate that the Foster-Wolfson approach can be given an intuitive interpretation in terms of viewing polarization as an aggregate of (inverse) welfare indicators of the two groups under consideration. However, the previous two sections have shown that it seems unlikely to extend these results beyond the limited circumstances of the model discussed here. Moreover, even the two-group case can be called into question if the median dividing line (or any quantile-based partitioning of a distribution) fails to capture the intuitive notion of distinct groups in a distribution. The approach of Esteban and Ray (1994) and Duclos, Esteban, and Ray (2004) is subject to analogous observations. As mentioned in the discussion following Axiom 1 in Duclos, Esteban, and Ray (2004), this axiom requires that two opposing effects of a specific change in an income distribution are always traded off in favor of one direction. Moreover, note that the additive structure employed in both of these papers already encompasses a prescription regarding the trade-offs that appear as consequences of certain transfer and dominance principles. Thus, we conclude that uncontroversial axioms do not appear to generalize to distributions where we do not have clear and unambiguous ways of definition a population partition that remains the same after a transformation as those employed in the axiom statements.

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