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Abstract

In this paper we use a probabilistic approach to risk factor selection in the arbitrage pricing theory model. The methodology uses a bayesian framework to simultaneously select the pervasive risk factors and estimate the model. This will enable correct inference and testing of the implications of the APT model. Furthermore, we are able to make inference on any function of the parameters, in particular the pricing errors. We can also carry out tests of efficiency of the APT using the posterior odds ratio and bayesian confidence intervals. We investigate the macroeconomic risk factors of Chen, Roll, and Ross (1986) and the firm characteristic factors of Fama and French (1992,1993). Using monthly portfolio returns grouped by size and book to market, we find that the economic variables have zero risk premia although some appear to have non zero posterior probability. The "Market" factor is not priced. An APT model with factors mimicking size (SMB), book to market equity (HML), value-weighted portfolio and Standard and Poor, is supported by a conditionally independent prior and offers a significant decrease in the pricing error over a two-factor APT with SMB and HML. The posterior probability and cumulative distributions functions of the average risk premia and the pricing errors are compared to the normal distribution. The results show that under certain conditions the distortions are very small.

JEL Classification: C1, C22, C52
Keywords: Variable selection, Posterior density, Bayes factors, MCMC, APT models.

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1 Introduction

In this paper, we address model selection in the context of a linear factor model with potentially measured and latent factors. The study proposes an exact statistical framework for the estimation and inference in a factor model. We use a bayesian framework to implicitly incorporate model uncertainty into the estimation of the parameters and model inference.

Since the inception of the arbitrage pricing theory (APT) by Ross (1976, 1977), there is an increased interest in the use of linear factor models in the study of Asset pricing. There is a growing evidence that high returns are driven by a multi-factor model rather than the one factor capital pricing model (CAPM)[1]. The APT has the attractive feature of minimal assumptions about the nature of the economy. However, this tractability comes at the cost of certain ambiguities such as an approximate pricing relation and an unknown set of pervasive factors. In order to test the implications of the APT, one must specify the number and the identity of the factors.

There are two main streams in the literature of factor selection in the APT. The first view toward model determination uses latent (unobservable) factors as sources of common variations. These common factors are estimated from sample covariance matrices using statistical techniques like factor analysis and principal components. Bai and Ng (2000) develops an econometric approach to consistently determine the dimension of the model for large panels. Bai (2001) addresses the asymptotic properties of the estimated model. This literature addresses the asymptotic properties of the distributions and therefore is based on the model selection being consistent and therefore treated as deterministic.

The second alternative view suggests the use of observed economic variables as factors. There is no doubt that asset prices are intimately linked to macroeconomic activity and that the influences go in both directions. However, little is done to formalize the search for the set of significant influential variables. Chen, Roll and Ross (1986) (CRR) asserts "A rather embarrassing gap exists between the theoretically exclusive importance of systematic "state variables" and our complete ignorance of their identity". CRR attempts to explore this identification terrain by combining a set of "likely" macroeconomic and financial candidates for pervasive risk in asset returns. The selection procedure consists of a series of t-test for the significance of average risk premia corresponding to each of the variables allowed into the regression. The authors identify five common risk factors that are significantly priced in the stock market. Using a different approach, based on firm characteristics as a proxy for the firms' sensitivity to systematic risk in the economy, Fama and French (1993) shows that the variation in returns on stocks and bonds can be explained by five size and book-to-market based factors. However, these studies raise two fundamental critiques. First, the number of factors is often assumed and arbitrarily prespecified. Second, the set of potential pervasive factors is subjectively reduced to a few number of candidates and only a few specifications are tested. Hence, no statistical justification is provided to justify the selected set of variables. Ouysse (2006) proposes a formal econometric procedure to consistently select the set of pervasive factors in panels with large dimensions. However, the study does not address the post-selection properties of the model estimates.

The APT implies nonlinear restrictions on the model parameters, which make it very difficult and complex to derive the asymptotic distribution of the restricted estimates, let alone, the distribution of post-selection restricted estimates. The bayesian approach enables exact inference by making it possible to implicitly incorporate model uncertainty and to derive post-selection distributions of any functions of the parameters.

Geweke and Zhou (1996) uses a bayesian framework to analyze the APT. The authors propose the use of the pricing error to test the implication the APT that the expected returns are approximately linear function of the risk premium on systematic factors. The authors use latent factors and do not perform a selection of the appropriate number of factors. They borrow the results from the asymptotic principal component analysis of Connor and Korajczyk (1986,1993) and propose the use of 1 to 4 factors.

The present study extends [[16]] to allow for both latent and measured factors. In particular, this method makes it possible to derive the exact post-selection posterior distributions for the measures of the pricing error and for the measure of the systematic risk and risk premia.

[1]For example, Fama and French (1996) indicates that market anomalies largely disappear in a three-factor model. Ferson and Korajczyk (1995) finds that a five factors model capture a large fraction of asset returns predictability.

2 Methodology

2.1 Asset Pricing model

In the finance literature, the debate on what drives excess returns continues. A large number of studies use factor models to identify the common sources of systematic risk in expected returns. The factors used are classified into latent factors estimated through statistical methods as principal components and factor analysis and observable factors based on the sensitivity of stocks returns to economic and financial news. The use of observable variables to explain excess returns is particularly appealing. The estimated factor loadings have a meaningful interpretation. The estimated pricing relationship can be used to stimulate the financial markets through the pervasive economic and financial variables. The ability to predict excess returns with tangible factors can be useful for portfolio management and stock market investment decisions.

Unlike the Capital Asset Pricing model, the APT allows for multiple risk factors to enter the return generating process for asset returns. In a rational asset pricing model with multiple beta, expected returns of securities are related to their sensitivities to changes in the state of the economy.

Let Y_{it} be the return on asset i at time t. Assume that asset returns follow an approximate[2] k_0-factor model[3],

$$Y_{it} = \alpha_i + \sum_{j=2}^{k_0} b_{ij}\mathbf{x}_{tj} + \varepsilon_{ti} \tag{1}$$

The intercept α_i is the expected return on asset i, $\alpha_i = E[Y_{it}]$. The risk factors are common across the assets. Therefore, the asset risk can be divided into a common diversifiable risk due to the exposure to the k_0 common risk factors $\mathbf{x}_t$, and an idiosyncratic non-diversifiable risk due to the idiosyncratic factor ε_{it}. The betas b_{ij}, or factor loadings of the j^{th} factor for asset i, represent the amount of exposure to each risk factor. There are T time periods and N assets.

In our analysis, it is convenient to work with the pooled form of the model. Let $y = vec(Y)$,$\beta = vec(B)$, and $\varepsilon = vec(v)$,

$$\underset{NT\times 1}{y} = \underset{NT\times Nk}{(I_N \otimes X)}\ \underset{Nk\times 1}{\beta} + \underset{NT\times 1}{\varepsilon} \tag{2}$$

The idiosyncratic factors are assumed to be uncorrelated with the factors, X. They are also assumed to have a normal distribution with mean zero and covariance matrix $\Sigma \otimes I_T$[4].

The absence of riskless arbitrage opportunities implies well-known restrictions on (1), namely

$$\begin{aligned} \alpha_i &\approx \delta_0 + \beta_{i1}\delta_1 + ... + \beta_{ik_0}\delta_{k_0} \\ i &= 1,..,N \end{aligned} \tag{3}$$

where λ_1[5] is the riskless return, provided at least one risk-averse investor holds a portfolio without any residual risk and λ_j is the risk premium on j^{th} factor. Equation (3) represents an approximate[6] linear relationship between the expected asset returns and their risk exposures.It implies that the risk premium on an asset, $(\beta_{i2}\lambda_2 + ... + \beta_{ik_0}\lambda_{k_0})$, by its factor loadings. Exact arbitrage pricing obtains when (3) holds with equality.

Geweke and Zhou (1996) proposes a measure of pricing error given by the average of squared deviations from the restriction across assets. The pricing error is measured by

$$\begin{aligned} Q_N^2 &= \sum_{i=1}^{N} \left(\alpha_i - \delta_0 - \beta_{i1}\delta_1 - ... - \beta_{ik_0}\delta_{k_0}\right)^2 /N \\ Q_N &\to 0 \text{ as } N \to \infty \end{aligned}$$

[2]An approximate factor structure, first introduced by Chamberlain and Rothschild (1983), relaxes the static factor model assumption of diagonal idiosyncratic covariance matrix to allow for a limited amount of cross-section dependence.

[3]The APT is based on the pricing relation for a countably infinite vector of returns to a countably infinite set of traded assets.

[4]If there is serial correlation, the covariance matrix will be of the more general form, $E(\varepsilon\varepsilon') = \Sigma \otimes \Gamma$.

[5]The riskless rate will be measured by the 30-day Treasury-bill rate that is known at the beginning of each period month.

[6]Connor (1984) replaces the approximation with equality under the assumption of competitive equilibrium.

The authors argue that for Connor's equilibrium APT, Q is equal to zero. For Ross's asymptotic APT, the pricing error will converge to zero as the number of assets approaches infinity. Although for fixed N, the pricing error is not necessarily small, the authors suggest the use of the pricing error to examine some of the testable implications of the APT.

Another measure of the pricing error which takes into account the cross section correlation structure in the idiosyncratic term is given by,

$$\begin{aligned}
\widetilde{Q}_N^2 &= \frac{\left(\alpha - \widetilde{B}'\delta\right)' \Omega \left(\alpha - \widetilde{B}'\delta\right)}{N} \\
\widetilde{B} &= (\imath, B);\ \delta = (\delta_0, ..., \delta_{k_0})' \\
\widetilde{Q}_N &\rightarrow 0 \text{ as } N \rightarrow \infty
\end{aligned}$$

Stated differently, the pricing theory imposes a testable cross-equation restriction on the parameters of a multivariate regression of asset excess returns on the factors[7] It implies zero intercepts in a regression of asset excess returns on the factors. A test of miss-pricing is a test for non-zero intercept.

2.2 Review of the Classical Inference

There are mainly two approaches to estimating and testing the APT. First, traditional factor analysis uses a likelihood ratio to test the restrictions implied by the APT. This involves computing the maximum likelihood estimates under the nonlinear restrictions in (3.) This is a very difficult task in practice and the model inference is non standard. Indeed, [1] shows that the asymptotic distributions of the model parameters estimates are very complex in factor analysis, and the constrained estimates should be even more complex. This complexity makes it difficult to derive the asymptotic distribution of the likelihood ratio tests.

The second approaches surmount the complexity of the estimation under nonlinear restrictions by a two-pass approach. In the first pass, either the factor loading or the factors are estimated. In order to estimate the factors betas (factor loading) of assets, excess returns are regressed against the common factors using the time series from $t-60$ to $t-1$ to get the conditional betas $\widehat{\beta}_{ik,t-1}$.The estimates $\left\{\widehat{\beta}_{ik}\right\}_{i=1,..,N}^{k=1,..,K}$ are then used in the second pass.

Treating these estimates as the true variables, the APT restrictions in (3) become linear constraints on the coefficients of the multivariate regression. In fact, these restrictions imply zero intercepts and can be tested using the standard methods. To estimate the risk premia, a cross sectional regression model is utilized for each time point to get the time series of each risk premia. For each month t of the next 12 months, perform cross section regressions: $Y_{it} = \delta_{0t} + \sum_{k=1}^{K} \beta_{ik,t-1}\delta_{kt} + \varepsilon_{it}$ with $i = 1, .., N$ and get an estimate of the sum of risk premium $\widehat{\delta}_{kt}$ for month t associated to variable k, $t = 1, .., 12$. The two-pass steps are repeated for each time period in the sample.

Estimation technique: (i) Regress excess returns on the economic variables using the time series from $t-60$ to $t-1$ to get the conditional betas $\widehat{\beta}_{ik,t-1}$. (ii) For each month t of the next 12 months, perform cross section regressions: $R_{it} = \delta_{0t} + \sum_{k=1}^{K} \beta_{ik,t-1}\delta_{kt} + \varepsilon_{it}$ with $i = 1, .., N$ and get an estimate of the sum of risk premium $\widehat{\delta}_{kt}$ for month t associated to variable k, $t = 1, .., 12$. (iii) Steps (i) and (ii) are repeated for each year in the sample period. The time series means of the series of of risk premium

[7] The equilibrium version of the APT implies that

$$E(y_t) = r_{Ft}\mathbf{e} + \widetilde{B}\lambda_t \tag{4}$$

where r_{Ft} represents the return on a riskless asset, $\mathbf{e}$ is a vector of ones, λ_t is a k_0-vector of factor risk premiums. Combining equations (**??**) and (4) gives,

$$\widetilde{Y} = \widetilde{B}\widetilde{X} + v$$

where the $N \times T$ matrix of excess returns is given by $\widetilde{Y} = Y - I_N\ r'_F$ and $\widetilde{X}$ is the $T \times k_0$ realizations of $(\mathbf{x}_t + \lambda_t)$. The pricing theory imposes a testable cross-equation restriction on the parameters of a multivariate regression of asset excess returns on the factors. Let μ be the vector of intercepts in a regression of asset excess returns on the factors. The pricing theory implies that μ should be identically zero.

estimates associated to each variable are then tested for their significance. A factor with statistically significant risk premia is priced by investors in the market.

This method is based on the estimates from the first pass being consistent. However, in small samples, this procedure suffers from errors in variables problem. The uncertainty about the first pass estimates can lead to misleading inference.

Variable selection adds an extra source of uncertainty to the model and an extra dimension to the complexity of the first method and to the unreliability of the inference in the second method. Indeed, the empirical literature so far used ad hoc methods to select the variables to enter the APT model and failed to address the issue of post variable selection inference. Ouysse (2006)[17] developed an information criterion to produce consistent estimates of the set of pervasive common factors in a large Panel with observable factors . However, there has been no attempt to address the distributional properties of the estimated set of factors to enable proper inference on the post-selection model parameters. The issue of incorporating model uncertainty is still an open research question in this literature.

3 Bayesian Inference

The factors in (**??**) are unknown but are assumed to be elements of a finite set of potential variables. Let K be the total number of potential variables represented by the columns of the matrix X. Further, let X^0 be the set of pervasive factors in the true data generating process. Define the Bernoulli random variable γ_j as: $\gamma_j = \begin{cases} 1 \; if \;\; X_j \subseteq X^0 \\ 0 \qquad otherwise \end{cases}$. Therefore, $\gamma = \{\gamma_j\}_{j=1}^{K}$, is a selector vector over the column of X. Let q_γ be the "Binomial "random variable representing the number of variables in the selected model, therefore

$$q_\gamma = \sum_{i=1,..,K} \gamma_i$$

The objective of this paper is to perform a factor selection and test the adequacy of the APT as a pricing model for the assets. The bayesian approach makes it possible to implicitly incorporate the uncertainty about the risk factors and to estimate simultaneously in one step the betas and the risk premia which circumvents the shortcomings of the two-pass procedure. Furthermore, we are able to make inference on any function of the parameters, in particular the pricing errors. We can also carry out tests of efficiency of the APT using the posterior odds ratio and bayesian confidence intervals. We use a full bayesian specification to evaluate the posterior distributions of the parameters of the model. We will consider both diffuse and informative priors and we will use a Markov Chain Monte Carlo (MCMC) with a Gibbs sampler for the case of measured economic factors and a reversible a reversible Jump MCMC for the case of measured and latent risk factors. We will use a *Gibbs sampler* to draw from the posterior distributions of γ, β, λ and Σ.

3.1 Linear Factor model with measured factors

Based on the idea that asset prices react sensitively to economic news, [9] used economic forces to proxy for the systematic influences in stock returns. Using intuition and empirical investigation, the authors combined macroeconomic variables and financial markets variables to capture the systematic risk in asset returns and suggests a five factor APT model. To assess whether the risk associated to a given variable is rewarded in the market, the authors test the significance of the estimated risk premia using a $t-statistic$ using 20 equally weighted portfolios constructed on the basis of firm size as dependent variables. Their results show evidence of five factors. CRR concludes that the spread between long and short interest rate (UTS), expected (EI)and unexpected inflation (UEI), monthly industrial production (MP) and the spread between high- and low-grade bonds (URP) are significantly priced. However, neither the market portfolios (EWNY, VWNY) nor consumption (CG) are priced separately.

Fama and French [12] argued that size and book to market equity are related to economic fundamentals. They suggested the use of firm characteristics, such as size (ME) and book to market equity

(BE/ME), to construct factors portfolios proxy for sensitivity to common risk factors in returns. The authors used slopes and R^2 values to test whether these mimicking portfolios capture shared variation in stock and bond returns. Their results show that the portfolios constructed to mimic risk factors related to ME and BE/ME capture strong variations in stock returns. Using 25 stock portfolios as dependent variables , their results show evidence that a three factor model, using Market, SMB and HML as risk factors, captures the common variations in the cross section of stock returns.

3.1.1 The general Normal-Wishart prior

The diffuse prior was first introduced into bayesian multivariate analysis by Geisser and Cornfield (1963). It is a prior of 'minimum prior information'. However, if one has prior information on β, an informative prior should be used. Indeed, Monte Carlo integration makes it possible to work with many choices of priors. In this section we will derive the full conditionals under a general form of Normal-Wishart prior. The amount of prior information and its importance relative to the sample information will be determined by the covariance matrix of the prior density of β.

Lemma 3.1 *Consider the Normal-Wishart prior for* β*and*Σ, $\beta|\Sigma \sim N\left(\beta_0, \Sigma \otimes \underline{H}_\beta\right)$ *and* $\Sigma^{-1} \sim W_N\left(m, \Phi^{-1}\right)$ *wishart distribution with location* Φ*and scale parameter* $m > N+1$[8]*.Under uniform priors on* γ*, the full conditionals are given by*

1. $\beta|y, \Omega, \gamma \sim N\left(\widetilde{\beta}_\gamma, \Sigma \otimes D^{-1}\right)$ *where* $D_\gamma = \left(X'_\gamma X_\gamma + \underline{H}_\beta^{-1}\right)$

$$\begin{aligned} \widetilde{\beta}_\gamma &= \left(I_N \otimes D_\gamma^{-1}\underline{H}_\beta^{-1}\right)\beta_0 + \left(I_N \otimes D_\gamma^{-1}X'_\gamma X_\gamma\right)\widehat{\beta}_{GLS} \\ \widehat{\beta}_{GLS} &= \left[I_N \otimes \left(X'_\gamma X_\gamma\right)^{-1} X'_\gamma\right] y \end{aligned}$$

2. $\Omega|y, \gamma \sim W_N(m+T, (S(\gamma)+\Phi)^{-1})$ *where*

$$\begin{aligned} S(\gamma) &= (Y - B_0 W)'\left(I_T - X_\gamma D_\gamma^{-1} X'_\gamma\right)(Y - B_0 W) + B'_0 L B_0 \\ W_\gamma &= \left[I_T - X D^{-1} X'\right]^{-1} X D_\gamma^{-1} \underline{H}_\beta^{-1} \\ L &= \Sigma^{-1} \otimes V_\gamma \\ V_\gamma &= \underline{H}_\beta^{-1} - \underline{H}_\beta^{-1} D^{-1} \underline{H}_\beta^{-1} - \underline{H}_\beta^{-1} D_\gamma^{-1} X' W_\gamma \\ \beta_0 &= vec(B_0) \end{aligned}$$

3. $p\left(y|\gamma, X\right) \propto \left|\underline{H}_\beta\right|^{-\frac{N}{2}} \left|X'X + \underline{H}_\beta^{-1}\right|^{-\frac{N}{2}} |\Phi|^{\frac{m}{2}} |\Phi + S|^{-\frac{(T+m)}{2}}$

In order to implement the selection process, the hyperparameters β_0 and $\underline{H}_\beta$.determining the on β need to be specified. Assuming that no subjective information about these parameters is available, their values will be set in order to minimize their influence. Two values of the prior mean are considered in this application. The default prior $\beta_0 = 0$ which reflects indifference between positive and negative values and, a somewhat more informative prior which centers the prior distribution around the generalized least squares estimator $\beta_0 = \widehat{\beta}_{GLS}$.

The covariance matrix $\underline{H}_\beta$ determines the amount of information in the prior and will influence the likelihood covariance structure. In the literature, the specification is simplified to $\underline{H}_\beta = c\underline{V}_\beta$. The preset form $\underline{V}_\beta$, can be chosen to either replicate the correlation structure of the likelihood by setting $\underline{V}_\beta = \left(X'X\right)^{-1}$, this is also the g-prior recommended by Zellner (1986); or to weaken the covariance in the likelihood by setting, $\underline{V}_\beta = I_K$, which implies that the components of β are conditionally independent. The scalar c is a tuning parameter controlling the amount of prior information. The larger the value of c, the more diffuse (more flat) is the prior over the region of plausible values of β. The value of c should be large enough to reduce the prior influence. However, excessively large values can generate a form of the Bartlett-Lindley paradox by putting increasing probability on the

[8]Given this prior, the first moment of Σ is $E(\Sigma) = \frac{\Phi}{m-N-1}$ and therefore the scale parameter $m > N+1$.

null model as $c \to \infty$. In the literature, different values of c were recommended depending on the application at hand. In this paper we will consider three choices of $c \in \{4, max\{T, K^2\}, \widehat{c}^{EB}\}$[9], where

$$\begin{aligned} \widehat{c}^{EB} &= maxF_\gamma - 1, 0 \\ F_\gamma &= \tfrac{R_\gamma^2/k_\gamma}{(1-R_\gamma^2)/(n-1-k_\gamma)} \end{aligned}$$

A local empirical Bayes estimate $\widehat{c}^{EB}$ for c is required for each model. Therefore, this approach generates values of c that are data dependent, (See, [18] for a thorough discussion). As pointed out in Chipman, George and McCulloch (2001), there is an asymptotic correspondence between these choices of c and the classical information criteria AIC, BIC and RIC respectively when $\underline{V}_\beta = (X'X)^{-1}$. The case of $c = n$ corresponds to the so called *unit information priors* which corresponds to choosing priors with the same amount of information about β as that contained in one observation. This prior leads to Bayes factors with BIC behavior. A risk information criterion (RIC) corresponds to a choice of $c = K^2$ as shown in Foster and George (1994)([14]).
Given that $D = \left(X'X + \underline{H}_\beta^{-1}\right)$, one can see that under $\underline{V}_\beta = I_K$ the posterior correlations will be less than those of the design correlation. The posterior correlations are however identical to those of the design matrix under the prior $\underline{V}_\beta = (X'X)^{-1}$.
The conditionally independent prior for β,

$$\beta|\Sigma \sim N\left(\widehat{\beta}_{GLS}, \Sigma \otimes cI_K\right) \tag{5}$$

is equivalent to the prior $B \sim N(\Sigma, cI_K)$, a matrix variate representation used by Brown *et al. (1998),* where cI_K is the covariance matrix of $B|\Sigma$. Therefore, the columns of B in **??** are independent under this prior.
Given this prior, the posterior densities for the parameters and the covariance matrix are given by

$$\beta|\Sigma, y, \gamma, X \sim N\left(\widehat{\beta}_\gamma, \left[\Omega \otimes \left(\frac{1}{c}I_{q_\gamma} + X_\gamma' X_\gamma\right)\right]^{-1}\right) \tag{6}$$

where

$$\widehat{\beta}_\gamma = \left[I_N \otimes \left(X_\gamma' X_\gamma\right)^{-1} X_\gamma'\right] y \tag{7}$$

The posterior density of the inverse covariance matrix, conditional on the sample information,

$$\Omega|r, \gamma, X \sim W_N\left(m + T, (S(\gamma) + \Phi)^{-1}\right) \tag{8}$$

where the location matrix depends on the sample data through the sum of squers residuals $S(\gamma) = Y'(I - X_\gamma(X_\gamma' X_\gamma)^{-1} X_\gamma')Y.$
Given the same uniform prior on the indicator variable, the posterior density is adjusted in light of the informative prior in (5). The posterior density for γ, conditional on the data, is

$$p(\gamma|y, X) \propto c^{-(Nq_\gamma/2)} \left|X_\gamma' X_\gamma + \frac{1}{c}I_{q_\gamma}\right|^{-\frac{N}{2}} |\Phi|^{\frac{m}{2}} |S(\gamma) + \Phi|^{-\frac{\delta}{2}} \tag{9}$$

Now lets consider a different informative prior for the parameters of the model,

$$\beta|\Sigma \sim N\left(0, \Sigma \otimes c(X'X)^{-1}\right) \tag{10}$$

The parameter c measures the amount of information in the prior relative to the sample. Setting $c = 50$ gives the prior the same importance as 2% of the sample. Given this prior and given the Wishart prior the full conditional densities for the unknown parameters of the model are given by,

$$\beta|\Sigma, y, \gamma, X \sim N\left(\widetilde{\beta}_\gamma, \frac{c}{1+c}\Sigma \otimes (X'X)^{-1}\right) \tag{11}$$

[9]In a simulation study of the effect of the choice of c on the posterior probability of the true model, Fernandez, Ley and Steel (2001) found that the effect depends on the true model and noise level and they recommend the use of $c = \max\{T, K^2\}$.

where $\widetilde{\beta_\gamma} = \frac{c}{1+c}\left((X'_\gamma X_\gamma)^{-1}X'_\gamma \otimes I_N\right)y$. The full posterior density for the covariance matrix is a wishart,

$$\Omega|y,\gamma,X \sim W_N(m+T,\left(\widetilde{S}(\gamma)+\Phi\right)^{-1}) \tag{12}$$

where $\widetilde{S}(\gamma) = Y'\left[I - \frac{c}{1+c}X_\gamma(X'_\gamma X_\gamma)^{-1}X'_\gamma\right]Y$.

Finally, given the same uniform prior on the indicator variable,the posterior density for γ, conditional on the data is as follows

$$p(\gamma|y,X) \propto (1+c)^{-\frac{Nq_\gamma}{2}}\,|\Phi|^{\frac{m}{2}}\left|\Phi+\widetilde{S}(\gamma)\right|^{-\frac{(T+m)}{2}} \tag{13}$$

3.1.2 Random draw from the posterior density $\gamma|y,X$

In the Gibbs sampler each element, γ_k in the indicator variable is generated at a fixed or random order from the full conditional distributions. Each γ_k is a Bernoulli random variable with

$$p_k = P(\gamma_k = 1|\gamma_{/k},y,X) = \frac{L}{1+L}$$

where

$$\begin{aligned}\gamma_{/k} &= \{\gamma_1,..\gamma_{k-1},\gamma_{k+1},..,\gamma_K\}\\ L &= \frac{p\left[\gamma=(\gamma_1,..\gamma_{k-1},1,\gamma_{k+1},..,\gamma_K)|y,X\right]}{p\left[\gamma=(\gamma_1,..\gamma_{k-1},0,\gamma_{k+1},..,\gamma_K)|y,X\right]}\end{aligned}$$

For the *Normal-Wishart g-prior* case, the random mechanism general informative prior (See Appendix),

$$logL = -\frac{N}{2}\log\frac{\left|\underline{H}_\beta(1)\right|}{\left|\underline{H}_\beta(0)\right|} - \frac{N}{2}\log\frac{\left|X'_{\gamma(1)}X_{\gamma(1)}+\underline{H}_\beta(1)^{-1}\right|}{\left|X'_{\gamma(0)}X_{\gamma(0)}+\underline{H}_\beta(0)^{-1}\right|} - \frac{(T+m)}{2}\log\frac{|\Phi+S(1)|}{|\Phi+S(0)|} \tag{14}$$

this term represents the ratio of the marginal likelihood of the data conditional on the model. In the case of the data dependent g-prior,

$$L = (1+c)^{-\frac{N}{2}}\left(\frac{\left|\Phi+\widetilde{S}_{\gamma(1)}\right|}{\left|\Phi+\widetilde{S}_{\gamma(0)}\right|}\right)^{-\frac{(T+m)}{2}}$$

$$\log\widetilde{L} = -\frac{N}{2}\log(1+c) - \frac{(T+m)}{2}\log\frac{\left|\widetilde{S}_{\gamma(1)}+\Phi\right|}{\left|\widetilde{S}_{\gamma(0)}+\Phi\right|} \tag{15}$$

4 Empirical Results

4.1 Measured Economic and Firm Characteristics Variables

Based on the idea that asset prices react sensitively to economic news, Chen, Roll and Ross (1986) (CRR) uses economic forces to proxy for the systematic influences in stock returns. Using intuition and empirical investigation, CRR combines macroeconomic variables and financial markets variables to capture the systematic risk in asset returns and suggests a five factor APT model. Using a set of 20 equally weighted portfolios[10] constructed on the basis of firm size as dependent variables, the authors apply Fama and MacBeth (1973) two-step estimation procedure to estimate the average risk

[10]The authors use portfolios instead of individual stocks in order to control for the errors-in-variables problems that arises from the use of the two steps estimation technique.

Measured Variables		Measured Variables	
Januaruy Effect Dummy	JAN		
Consumption	CG	Market portfolio	MARKET
Term Structure	UTS	Small Minus Big	SMB
Risk Premium	URP	High Minus Low	HML
Expected Inflation	EI	Momentum Factor	MOM
Unexpected Inflation	UEI	Value-Weighted return	VWRET
Change in EI	DEI	Value-Weighted return Ex dividend	VWRETE
Prod Growth (M)	MP	Equally-Weighted return	EWRET
Prod Growth (A)	YP	Equally-Weighted return Ex dividend	EWRETE
Unemployment	UNEP		
Producer Price Index	PPI		

premia associated to the variables included in each regression. To assess whether the risk associated to a given variable is rewarded in the market, they test the significance of the estimated risk premia using a $t-statistic$. Results show evidence of five factors: CRR concludes that the spread between long and short interest rate (UTS), expected (EI)and unexpected inflation (UEI), monthly industrial production (MP) and the spread between high- and low-grade bonds (URP) are significantly priced. However, neither the market portfolio (EWNY, VWNY) nor consumption (CG) are priced separately. Furthermore, they found no evidence that oil prices (OG) are rewarded separately.

Fama and French (1992) documents that size and book to market equity are related to economic fundamentals. This motivates the use of these firm characteristics to construct factors portfolios. Fama and French (1993) suggests that variables that are related to average returns, such as size (ME) and book to market equity (BE/ME) must proxy for sensitivity to common risk factors in returns. The authors use slopes and R^2 values to test whether these mimicking portfolios capture shared variation in stock and bond returns. Their results show that the portfolios constructed to mimic risk factors related to ME and BE/ME capture strong variations in stock returns. Using 25 stock portfolios as dependent variables , their results show evidence that a three factor model, using Market, SMB and HML as risk factors, captures the common variations in the cross section of stock returns.

In this application we consider the monthly value weighted returns for the intersections of 10 ME portfolios and 10 BE/ME portfolios from Fama and French. The portfolios are constructed at the end of June. ME is market cap at the end of June. BE/ME is book equity at the last fiscal year end of the prior calendar year divided by ME at the end of December of the prior year. The sample period considered in the variable selection is $1960:10-2000:12$. We also report results on the subperiod $1980:01-2000:12$. These two periods are used estimation. In a later section, we will assess the performance of the most promising models over the period $2000:01-2005:12$ to explain the cross section of expected returns..

Given the Normal-Wishart priors described in section 3.1.2, we use a Gibbs sampler to draw an $ergodic$ Markov chain sequence for γ, β and Ω. These parameters are drawn from their full conditional distributions as described in *Lemma 1*. The posterior distribution of the average pricing errors are then straightforward obtained from these samples of the model parameters. The following are the quantities of interest computed from the MCMC sequences,

a. The posterior mean of the indicator variable γ, denoted $\overline{\gamma}|y$. The elements in which $\overline{\gamma}|y$ will represent the posterior probability of each variable being in the true data generating process.

b. Iterates for the risk premia, δ_i for each variable i with nonzero posterior probability. In addition of reporting their posterior mean,$\overline{\delta}|y$, and standard deviation, $Std(\delta|y)$, we plot their histogram and a kernel density estimator of the posterior distribution. To assess significance of the risk premia, we also compute the bayesian confidence intervals for 90, 95 percent confidence level.

c. Since variable selection will depend on the loss function considered, we report the model estimates for a selection based on the highest posterior model probability $p(y|\gamma)$ as well as the results for a selection based on minimizing the pricing errors. As a notation, we use $\delta_{\max p(y|\gamma)}$ and $\delta_{\min Q}$ respectively.

d. Iterates for the pricing errors, Q^2 and $\widetilde{Q}^2$, their histogram and kernel density estimate of their posterior distribution as well as the bayesian confidence intervals.

e. It is of interest to examine the expected returns, the systematic risks, and the unsystematic risks in the APT model. We provide bayesian point estimates of the expected asset returns computed as the posterior means of the α_i, i.e.$\alpha_i^{pm} = \frac{1}{M}\sum_{j=1}^{M}\alpha_i^{[j]}$. The posterior mean, Σ^{pm}, of the Σ iterates represents the bayesian estimate for the idiosyncratic risks. The bayesian estimate for the total risk is given as the sum of the estimate for the systematic risk $B'_{pm}X'_{\gamma}X_{\gamma}B_{pm}$ and the estimate for the non-diversifiable Σ^{pm}. We report the proportion of systematic risk to the total risk denoted $D1$ and given the ratio of the diagonal elements of the systematic risk to the diagonal elements of the total risk, $D1 = \frac{diag(B'_{pm}X'XB_{pm})}{diag\left(B'_{pm}X'XB_{pm}+\Sigma_{pm}\right)}$. We also report the proportion of bayesian total risk to the sample covariance of observed returns, $D2 = \frac{diag\left(B'_{pm}X'XB_{pm}+\Sigma_{pm}\right)}{diag(V(Y))}$, where B_{pm} is posterior mean of the iterates of B, which is a bayesian estimate for the factors betas B.

- To assess the convergence of the Gibbs sampler, we plot the autocorrelation function for the risk premia and the pricing errors iterates.

e Finally, we use kernel density to estimate the posterior probability distribution of the risk premia and pricing errors.

We set the warming period to 100000 iterations and the sampling period to 50000. The initial values for the indicator variable are one for the constant term and zero for all other variables. The results were unchanged with different starting values. The initial value for the idiosyncratic covariance matrix is $0.2 \times I_N$. The location matrix for the Wishart distribution is $\Phi = I_N$ and the scale parameter $m = N + 2$. The following points summarize the main results:

1. The most favored model using the *g-prior* $\underline{H}_\beta = c\,(X'X)^{-1}$ with $c = \max\{T, K^2\}$ is the APT with the two factors $\{SMB, HML\}$ *Table 1* shows that the two factors have a posterior probability of one to be in the DGP. The point estimate for the risk premium for SMB is 0.01% while the risk premium associated to the factor HML is about 1.9%. None of the measured economic variables nor the financial variables were pervasive. *Table 2* represents the results for the conditionally independent prior $\underline{H}_\beta = \mathbf{cI}_K$. The most favored factors are $\{VWRET, SP500, SMB, HML\}$ all with 100% posterior probability to be in the DGP. The money growth, GB has only a 0.05% posterior odds to be part of the pervasive set of factors, while the monthly production growth, MP, appears to have a 0.05% probability. Both GB and MP have zero risk premia. The posterior means of the risk premia for SMB and HML are negative and are equal to -0.0022 and -0.0014 respectively. The $VWRET$ has a negative risk premia of -0.006, which is slightly smaller than point estimate for the risk premia on the $SP500$ which is equal to -0.0042.

2. In order to compare the two favored models using the two different priors we first look at the pricing errors. *Table 3* is a summary of the posterior means, standard deviation and confidence intervals for the pricing errors for the two types of priors. With the conditionally independent prior, the mean pricing error for the most favored model with the factors $\{VWRET, SP500, SMB, HML\}$ is 0.0056% for $\widetilde{Q}^2$ (resp. 0.0347% for Q^2) compared to 0.0071% (resp. 0.055% for Q^2) for the most favored model under the g-prior with factors $\{SMB, HML\}$. The *4-factor* APT shrinks the pricing error by about 21% (resp. 36% for Q^2). To assess the economic importance of the pricing errors, we will follow Geweke & Zhou (1996) argument and compare the magnitude of the mean of Q with the monthly expected returns. For the sample period 1960-2000, the means range from 0.7015 to 1.6584 percent per month. One can regard the above means of the pricing errors as economically negligible. To further assess the pricing error we provide the 90% (95%) Bayesian confidence interval, which state that that there is 90% (95%) probability that the pricing errors in the interval. The smaller the confidence interval,

the more heavily concentrated the posterior density of the average pricing error near its mean and the more informative are the data on the pricing error.

Pricing Errors for the two types of g-priors. Case of combined set of measured factors, K = 22,N = 45 and c = T.

	cI_K		$c(X'X)^{-1}$	
Panel A: SamplePeriod 1960 : 01 − 2000 : 12, T = 480				
	Q	$\widetilde{Q}$	Q	$\widetilde{Q}$
$E(Q\|y)$	0.0347	0.0056	0.0550	0.0071
Std	0.0012	0.0005	0.0017	0.0005
90% CI	[0.0324; 0.0360]	[0.0049; 0.0064]	[0.0516; 0.0566]	[0.0062; 0.0080]
95% CI	[0.0317; 0.0361]	[0.0047; 0.0065]	[0.0505; 0.0567]	[0.0060; 0.0082]
Panel B: Sample Period 1980 : 01 − 2000 : 12, T = 240				
	Q	$\widetilde{Q}$	Q	$\widetilde{Q}$
$E(Q\|y)$	0.0292	0.0103	0.0770	0.0154
Std	0.0012	0.0012	0.0040	0.0020
90% CI	[0.0693; 0.0816]	[0.0123; 0.0187]	[0.0267; 0.0305]	[0.0083; 0.0123]
95% CI	[0.0672; 0.0818]	[0.0117; 0.0194]	[0.0260; 0.0306]	[0.0079; 0.0127]

3. The average proportion $D1$ for the 4-factor model is 99.96% and an average of 93.34% for the proportion $D2$. The two-factor model has an average proportion of 99.74% for $D1$ and 49.02% for $D2$. *Figure 1* and *Figure 2* show that the *4-factor* has both a higher $D1$ and $D2$ compared to the two-factor model for all asset returns in the sample. The higher the proportion D1, the smaller is the idiosyncratic risk relative to the systematic risk. Both models have proportions above 99.5%. However, the gap between the sample variances of the asset returns and the estimated total risk is far greater in the *two-factor* compared to the *4-factor APT*.

4. In terms of matching the average returns, *Figure 1* shows that the design matrix prior appears to produce estimates that are very similar to the classical approach. The independent prior. produces estimates for the expected returns that are quantitatively very low compared to the data dependent prior and the sample means.

Table 1: Estimates for Size Decile Portfolio Returns Factor Sensitivities and posterior probabilities for the period 1960 : 01-1989 : 12, $c_{pm} = 1.6989 \cdot 10^3$, $T = 360$,$N = 10$ and $k_{max} = 14$. The posterior mean $E(\delta_i|y) = 1$, for $i = \{Market - R_f, SMB\}$. $RTMSE = 0.0068$

	β_0	$\beta_{Market-R_f}$	β_{SMB}
decile1	0.0102	0.0320	0.0489
decile2	0.0083	0.0361	0.0443
decile3	0.0079	0.0382	0.0399
decile4	0.0081	0.0415	0.0359
decile5	0.0066	0.0421	0.0331
decile6	0.0066	0.0439	0.0276
decile7	0.0069	0.0450	0.0226
decile8	0.0054	0.0444	0.0185
decile9	0.0048	0.0459	0.0138
decile10	0.0030	0.0463	-0.0021
$\widehat{\delta_i}$	0.0083	-0.1006	0.0786
$CI_{95\%}$	(0.0011 , 0.0173)	(−0.2545 , 0.0471)	(0.0123 , 0.1345)

Table 2: Estimates for Expected Size Decile Portfolio Returns, and returns to factor risk exposure ($\delta_j\beta_{ij}$) for the period $1960:01$-$1989:12$, $c_{pm} = 1.6989 \cdot 10^3$, $T = 360$,$N = 10$ and $k_{max} = 14$. The posterior mean $E(\delta_i|y) = 1$, for $i = \{Market - R_f, SMB\}$. $RTMSE = 0.0068$

	$\overline{y}$	$\widehat{\alpha}$	$\delta_{Market}\beta_{i,Market}$	$\delta_{SMB}\beta_{i,SMB}$	$\widehat{\delta}_0 + \widehat{B}\widehat{\delta}_{[2:k]}$
decile1	0.0177	0.0102	0.0038	0.0102	0.0090
decile2	0.0141	0.0083	0.0035	0.0083	0.0083
decile3	0.0123	0.0079	0.0031	0.0079	0.0077
decile4	0.0120	0.0081	0.0028	0.0081	0.0071
decile5	0.0114	0.0066	0.0036	0.0066	0.0068
decile6	0.0111	0.0066	0.0022	0.0066	0.0062
decile7	0.0109	0.0069	0.0018	0.00697	0.0057
decile8	0.0110	0.0054	0.0015	0.00547	0.0054
decile9	0.0106	0.0048	0.0011	0.0048	0.0049
decile10	0.0089	0.0030	-0.0002	0.0030	0.0036

Table 3: Ratios of Diagonal elements of the Sample covariance matrix of the Decile returns $diag(\frac{Y'Y}{N})$ and the diagonal elements, $diag(\frac{\widehat{B}'X'X\widehat{B}}{N})$,and $diag(\widehat{\Sigma})$ for the period $1960:01$-$1989:12$, $c_{pm} = 1.6989 \cdot 10^3$, $T = 360$,$N = 10$,and $k_{max} = 14$. The posterior mean $E(\delta_i|y) = 1$, for $i = \{Market - R_f, SMB\}$. $RTMSE = 0.0068$

$Returns$	$diag(\frac{Y'Y}{N})$	$diag(\widehat{B}'X'X\widehat{B} + \widehat{\Sigma})$	$\frac{diag(\frac{\widehat{B}'X'X\widehat{B}}{N})}{diag(\frac{Y'Y}{N})}$	$\frac{diag(\widehat{\Sigma})}{diag(\frac{Y'Y}{N})}$
decile1	0.2206	0.0320	0.7577	0.4558
decile2	0.1786	0.0107	0.9047	0.1277
decile3	0.1632	0.0209	0.9270	0.3706
decile4	0.1515	0.0069	0.9835	0.0659
decile5	0.1409	0.0208	1.0014	0.4332
decile6	0.1309	0.0132	0.9912	0.2657
decile7	0.1224	0.0254	0.9734	0.6560
decile8	0.1087	0.0046	0.9755	0.0576
decile9	0.0954	0.0188	1.0511	0.6097
decile10	0.0683	0.0287	1.0972	1.4073

Table 4: Estimates for Posterior probability of γ_i for Economic Factors when $Fama's$ three Factors are ignored. The columns shows the estimates of the risk Premia and their Confidence interval. Sample period, $1960:01$-$1989:12$, $c_{pm} = 2.5921$, $T = 360$,$N = 10$ and $k_{max} = 11$

$Factors$	γ_{pm}	$\widehat{\delta_j}$	$99\%CI(\widehat{\delta_j})$
$Zero - \beta$	1	0.0022	(0,0.0079)
UTS	0.0735	-0.0062	$(-0.2851$,0.0292)
$PSAVE$	0.2206	-0.0075	$(-0.3409$,0.2060)
EI	0.0441	-0.0011	$(-0.2163$,0.0766)
UEI	0.0588	0.0030	$(-0.0852$,0.1604)

Figure 1: Out of Sample forecasts for the Size Decile Portfolio Returns for the period 1990 : 01-1991 : 12, based on 1960 : 01-1989 : 12 In-sample data, $c_{pm} = 1.6989 \cdot 10^3$, $T = 360$, $N = 10$,and $k_{max} = 14$. The posterior mean $E(\delta_i|y) = 1$, for $i = \{Market - R_f, SMB\}$. $RTMSE = 0.0068$

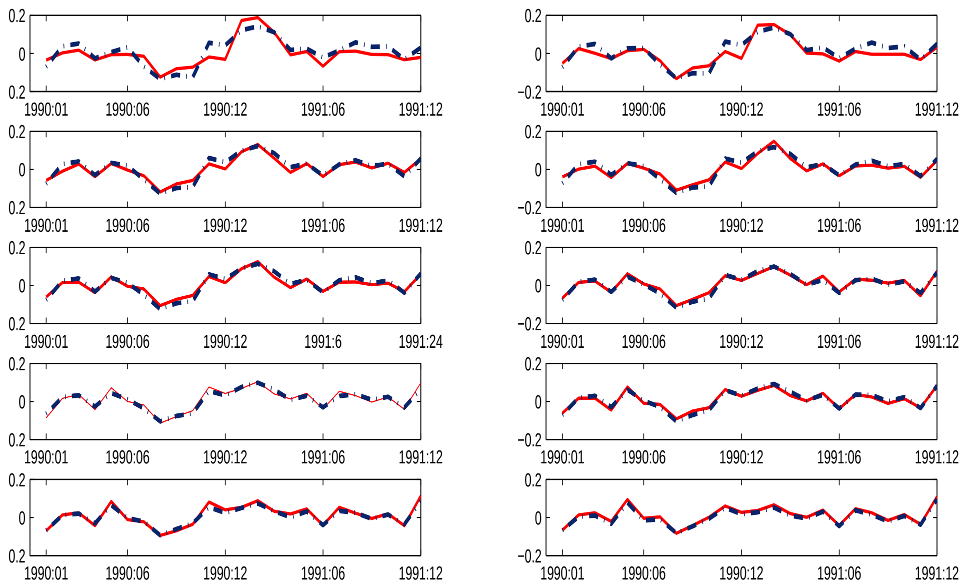


Table 5: Estimates for Size Decile Portfolio Returns Factor Sensitivities and posterior probabilities for the period 1960 : 01-1989 : 12, $c_{pm} = 2.5921$, $T = 360$,$N = 10$ and $k_{max} = 11$.

Returns	β_0	β_{UTS}	β_{PSAVE}	β_{EI}	β_{UEI}
decile1	0.0067	-0.0004	0.0003	-0.0002	0.0002
decile2	0.0043	-0.0002	-0.0001	-0.0002	-0.0001
decile3	0.0038	-0.0001	0.0002	-0.0001	0.0006
decile4	0.0047	-0.0002	0.0004	-0.0004	0.0001
decile5	0.0028	-0.0002	0.0008	-0.0005	-0.0001
decile6	0.0029	-0.0006	0.0004	-0.0005	0.0004
decile7	0.0039	0.0000	0.0001	-0.0000	0.0001
decile8	0.0032	0.0005	0.0006	-0.0012	-0.0008
decile9	0.0046	0.0001	0.0000	-0.0005	-0.0001
decile10	0.0012	-0.0000	0.0001	-0.0009	0.0006

5 Conclusion

In this article, we propose a fully bayesian framework for selecting the risk factors and examining their risk premia and the pricing restrictions implied by the APT. This a one step approach which integrates the uncertainty behind model selection and the estimation of the different functions of the parameters. In contrast to existing studies, we do not fix *a priori* the number of measured variables allowed to enter the pricing relationship. The number of measured variables and statistical factors is endogenously determined . This process is performed simultaneously with the estimation of the factor betas and their risk premia. Hence, this method avoids the errors in variables problem encountered in the main stream two-pass approach of Fama-MacBeth. Because, the bayesian approach evaluates the exact posterior distribution of the estimated parameters and any other function of the parameters, we

Table 6: Estimates for Expected Size Decile Portfolio Returns, and returns to factor risk exposure $(\delta_j\beta_{ij})$ for the period 1960 : 01-1989 : 12, $c_{pm} = 2.5921$, $T = 360$,$N = 10$ and $k_{max} = 11$. Note: The returns to individual factor risk $\delta_j\beta_i$ should be $(\times 10^{-5})$.

	$\overline{y}$	$\widehat{\alpha}$	$\widehat{\delta}_0 + \widehat{B}\widehat{\delta}_{[2:k]}$	$\delta_{UTS}\beta_i$	$\delta_{PSAVE}\beta_i$	$\delta_{EI}\beta_i$	$\delta_{UEI}\beta_i$
decile1	0.0177	0.0067	0.0022	0.2635	-0.1921	0.0197	0.0641
decile1	0.0141	0.0043	0.0022	0.1119	0.0962	0.0224	-0.0183
decile1	0.0123	0.0038	0.0022	0.0900	-0.1744	0.0145	0.1698
decile1	0.0120	0.0047	0.0022	0.1519	-0.2699	0.0399	0.0279
decile1	0.0114	0.0028	0.0022	0.1518	-0.6327	0.0540	-0.0351
decile1	0.0111	0.0029	0.0022	0.3704	-0.2649	0.0531	0.1064
decile1	0.0109	0.0039	0.0022	-0.0204	-0.1087	0.0011	0.0279
decile1	0.0110	0.0032	0.0022	-0.3299	-0.4782	0.1333	-0.2556
decile1	0.0106	0.0046	0.0022	-0.0771	-0.0374	0.0493	-0.0298
decile1	0.0089	0.0012	0.0022	0.0264	-0.0713	0.0987	0.1729

Table 7: Ratios of Diagonal elements of the Sample covariance matrix of the Decile returns $diag(\frac{Y'Y}{N})$ and the diagonal elements, $diag(\frac{\widehat{B}'X'X\widehat{B}}{N})$,and $diag(\widehat{\Sigma})$ for the period 1960 : 01-1989 : 12, $c_{pm} = 2.5921$, $T = 360$,$N = 10$,and $k_{max} = 11$.

Returns	$diag(\frac{Y'Y}{N})$	$diag(\widehat{B}'X'X\widehat{B} + \widehat{\Sigma})$	$\frac{diag(\frac{\widehat{B}'X'X\widehat{B}}{N})}{diag(\frac{Y'Y}{N})}$	$\frac{diag(\widehat{\Sigma})}{diag(\frac{Y'Y}{N})}$
decile1	0.2206	0.0030	0.0001	0.4963
decile2	0.1786	0.0027	0.0000	0.5507
decile3	0.1632	0.0021	0.0001	0.4705
decile4	0.1515	0.0028	0.0001	0.6808
decile5	0.1409	0.0029	0.0002	0.7417
decile6	0.1309	0.0019	0.0004	0.5305
decile7	0.1224	0.0010	0.0000	0.2969
decile8	0.1087	0.0021	0.0002	0.6865
decile9	0.0954	0.0028	0.0000	1.0846
decile10	0.0683	0.0023	0.0011	1.2405

Table 8: Estimates for Posterior probability of γ_i for Factors Pervasive for the Industry Portfolios Returns. The columns shows the estimates of the risk Premia and their Confidence interval. Sample period, 1960 : 01-1989 : 12, $c_{pm} = 0.0072$, $T = 360$,$N = 12$ and $k_{max} = 11$

$Factors$	γ_{pm}	$\widehat{\delta}_j$	$95\%CI(\widehat{\delta}_j)$
$Zero-\beta$	1	0.0045	(0.0043,0.0046)
JAN	0.5556	-0.1347	$(-0.8087, 0.0000)$
CG	0.8333	0.0352	(0.0000,0.1249)
UTS	0.5	-0.0100	$(-0.0604, 0.0000)$
URP	0.3333	-0.0201	$(-0.1209, 0.0000)$
$UNEM-1$	0.3333	0	NA
EI	0.5	0.2195	(0.0000,1.317)
$Mark-Rf$	0.6666	0.0218	(0.0000,0.1309)
SMB	0.5	-0.1239	$(-0.7437, 0.0000)$
HML	0.5	-0.0007	$(-0.0043, 0.0000)$

Table 9: Estimates for Industry Portfolio Returns Factor Sensitivities for the period 1960 : 01-1989 : 12, $c_{pm} = 2.5921$, $T = 360$,$N = 12$ and $k_{max} = 11$.

$Returns$	β_{JAN}	β_{CG}	β_{UTS}	β_{URP}	β_{EI}	$\beta_{Mark-Rf}$	β_{SMB}	β_{HML}
$NonDur$	-0.0001	-0.0000	0.0000	0.0001	0.0003	-0.0002	-0.0000	0.0001
$Durabl$	0.0000	-0.0001	0.0003	0.0002	-0.0000	0.0000	-0.0001	0.0001
$Manufc$	-0.0001	-0.0002	-0.0001	0.0002	-0.0001	-0.0001	-0.0001	-0.0000
$Energy$	-0.0000	0.0001	-0.0003	0.0001	0.0001	-0.0002	0.0001	-0.0002
$Chemis$	-0.0000	-0.0000	-0.0001	0.0000	0.0001	-0.0002	-0.0001	-0.0000
$BusEqu$	-0.0001	0.0008	-0.0001	0.0000	-0.0002	-0.0003	0.0001	-0.0002
$TeLcm$	-0.0001	-0.0001	0.0002	0.0002	0.0001	-0.0001	0.0001	-0.0000
$UtiLis$	-0.0000	0.0002	0.0001	0.0001	0.0000	0.0001	0.0001	0.0001
$Shops$	0.0001	0.0001	0.0001	0.0002	-0.0001	0.0003	0.0001	-0.0000
$Health$	0.0000	-0.0013	0.0002	-0.0003	0.0001	0.0003	-0.0001	0.0000
$Money$	-0.0001	0.0001	0.0000	0.0003	0.0001	0.0001	-0.0000	0.0000
$Other$	0.0000	-0.0003	0.0000	-0.0001	0.0000	-0.0001	0.0000	-0.0001

Table 10: Expected Industry Portfolio Returns, and returns to factor risk exposure (in %)($\delta_j\beta_{ij}$) for the period 1960 : 01-1989 : 12, $c_{pm} = 0.0072$, $T = 360$,$N = 12$ and $k_{max} = 11$.

	$\widehat{\alpha}$	$\widehat{\delta}_0 + \widehat{B}\widehat{\delta}$	$\delta_1\beta_1$	$\delta_2\beta_2$	$\delta_3\beta_3$	$\delta_4\beta_4$	$\delta_5\beta_5$	$\delta_6\beta_6$	$\delta_7\beta_7$	$\delta_8\beta_8$
Ind1	-0.0269	0.0046	0.0018	-0.0000	-0.0048	-0.0001	-0.0057	-0.0004	0.0002	-0.00
Ind2	-0.0168	0.0045	-0.0003	-0.0003	-0.0003	-0.0004	-0.0001	0.0000	0.0011	-0.00
Ind3	-0.0175	0.0045	0.0013	-0.0008	0.0001	-0.0004	-0.0013	-0.0001	0.0008	0.00
Ind4	0.0179	0.0046	0.0006	0.0004	0.0002	-0.0002	0.0026	-0.0003	-0.0009	0.00
Ind5	-0.0434	0.0046	0.0006	-0.0001	0.0001	-0.0000	0.0013	-0.0004	0.0010	0.00
Ind6	-0.0056	0.0045	0.0010	0.0028	0.0001	-0.0000	-0.0036	-0.0007	-0.0008	0.00
Ind7	-0.0140	0.0046	0.0010	-0.0002	-0.0001	-0.0004	0.0025	-0.0002	-0.0012	0.00
Ind8	0.0124	0.0045	0.0006	0.0006	-0.0000	-0.0002	0.0006	0.0001	-0.0012	-0.00
Ind9	0.0541	0.0045	-0.0017	0.0003	-0.0001	-0.0003	-0.0023	0.0006	-0.0014	0.00
Ind10	0.0075	0.0045	-0.0002	-0.0044	-0.0002	0.0006	0.0032	0.0006	0.0008	-0.00
Ind11	-0.378	0.0046	0.0012	0.0004	-0.0000	-0.0005	0.0012	0.0002	0.0004	-0.00
Ind12	0.0059	0.0045	-0.0002	-0.0009	-0.0000	0.0001	0.0001	-0.0002	-0.0004	0.00

Table 11: Ratios of Diagonal elements of the Sample covariance matrix of the Industry returns $diag(\frac{Y'Y}{N})$ and the diagonal elements, $diag(\frac{\widehat{B}'X'X\widehat{B}}{N})$,and $diag(\widehat{\Sigma})$ for the period 1960 : 01-1989 : 12, $c_{pm} = 0.0072$, $T = 360$,$N = 12$,and $k_{max} = 11$.

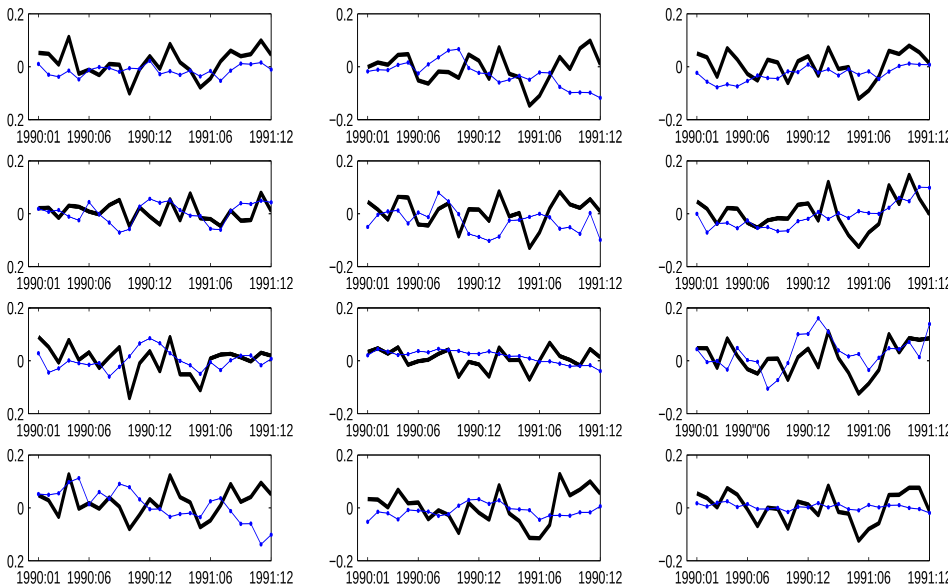
$Returns$	$diag(\frac{Y'Y}{N})$	$diag(\widehat{B}'X'X\widehat{B} + \widehat{\Sigma})$	$\frac{diag(\frac{\widehat{B}'X'X\widehat{B}}{N})}{diag(\frac{Y'Y}{N})}$	$\frac{diag(\widehat{\Sigma})}{diag(\frac{Y'Y}{N})}$
$NonDur$	0.0022	0.0029	0.0001	1.3177
$Durabl$	0.0030	0.0004	0.0001	0.1254
$Manufc$	0.0027	0.0002	0.0000	0.0816
$Energy$	0.0028	0.0022	0.0000	0.7752
$Chemis$	0.0024	0.0010	0.0000	0.4336
$BusEqu$	0.0034	0.0015	0.0002	0.4522
$UtiLis$	0.0016	0.0023	0.0001	1.4392
$Shops$	0.0031	0.0007	0.0001	0.2152
$Health$	0.0029	0.0016	0.0006	0.5539
$Money$	0.0026	0.0003	0.0001	0.1210
$Other$	0.0032	0.0012	0.0000	0.3818

are able to produce bayesian confidence intervals for the risk premia to gage if the market does price a certain factor. Inference is also done on the average pricing errors in order to evaluate the extent to which the APT restrictions deviate from the data. In an APT with only measured economic variables are allowed along with Fama and French three factors, the choice of the prior on the factor betas influences the posterior distribution of the promising factors. In the case of Zellner g-prior where the

Table 12: Estimates for the Pricing Error (in %) for the period 1960 : 01-1989 : 12, $c_{pm} = 0.0072$, $T = 360$,$N = 12$,and $k_{max} = 11$.

	$E(.)$	$std(.)$	$CI(95\%)$
Q^2	0.0014	0.0028	(0 , 0.0076)
Q	0.0191	0.0321	(0 , 0.0874)

Figure 2: Out of Sample forecasts for the Industry Portfolio Returns for the period 1990 : 01-1991 : 12, based on 1960 : 01-1989 : 12 In-sample data, $c_{pm} = 0.0072$, $T = 360$, $N = 12$,and $k_{max} = 14$. $RTMSE = 0.0503$



prior covariance matrix is a replica of the design matrix, the pervasive factors are Fama and French size and book to market risk factors SMB and HML. However, using the conditionally dependent prior, in addition to SMB and HML,some economic variables appear to be priced by the market. More specifically, inflation, unexpected inflation, return on value-weighted portfolio and return on the standard and poor portfolio.

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6 APPENDIX:

6.1 Data Appendix

6.1.1 Fama and French Portfolio Factors.

First the stock returns are ranked on size (prices times shares). The median size is then used to split the stocks into two groups, small and big (S and B). The returns are also broken into three book-to-market groups based on the bottom 30% (L), middle 40% (M) and the top 30% (H). Six portfolios (S/L, S/M, S/H, B/L, B/M, B/H) are then constructed from the intersection of the two size groups and the three BE/ME groups. Two additional portfolios are constructed from these intersections. HML (high minus low) meant to mimic the risk factor in return related to book to market equity BE/ME. It is defined as the difference between the average return on the two high-BE/ME portfolios and the average returns on the two low-BE/ME. The portfolio SMB (small minus big) meant to mimic the risk factor in return related to size. It is the difference between small and big stocks with about the same book to market equity. Finally, a value-weighted portfolio on the six size-BE/ME portfolio to proxy for market factor. To simplify notations, the set of factors used in Fama and French (1993) will be denoted F^{FF}.

6.1.2 Chen, Roll and Ross macroeconomic factors

From Roll and Ross (1986), the following set of variables is constructed:

- Consumption growth CG: growth rate in real per capita consumption constructed by dividing the series of seasonally adjusted real personal consumption (excluding durables) by the population. The series are from $FRED$ (Federal reserve bank of St. Louis).

- Term structure of interest rate UTS: the spread between the return on a long term government bond and the lagged return on one month bills. The two series are from $CRSP$ US Treasury and Inflation Indices.

- Risk premium URP: the spread between the return on low grade bonds (Moody's seasonally adjusted Baa corporate bond yield) and a long term government bond.

- Monthly growth of industrial production $MP(t)$: measured the change in industrial production lagged by one month.

- Annual growth of industrial production $YP(t)$.

- Oil price growth OG: the producer price index for crude petroleum. Source: Bureau of Labor Statistics.

- Expected inflation $EI(t)$: constructed using Fama and Gibbons (1984).

- Unexpected inflation $UEI(t) = I(t) - E(I(t)|t-1)$.

- Change in expected inflation $DEI(t) = E(I(t+1)|t) - E(I(t|t-1))$.

- Financial market indices: the return on value weighted $VWRETD$ and equally weighted $EWRTD$ portfolios of $NYSE$ listed stocks.

To these variables, an additional set of potential sources of variation are added:

- The unemployment rate $UNEMP$:Civilian unemployment rate percent, seasonally adjusted. Source: Bureau of Labor Statistics ($FRED$).

- The growth rate of money base GB :Currency component of money stock plus demand deposits seasonally adjusted ($FRED$).

- Private saving rate $PSAVE$:Percent, seasonally adjusted. All the variables used in CRR are collected in a set denoted by F^{RR}.

- where $\lambda = vec(\Lambda)$; thus $\underset{(N\times r)}{\Lambda} = (\lambda_1, .., \underset{(r\times 1)}{\lambda_i} .., \lambda_N)'$.

6.2 Proofs

6.2.1 Gibbs Sampler

The Gibbs sampler generates an ergodic Markov chain,

$$\gamma^{(0)}, \gamma^{(1)}, \Omega^{(1)}, \beta^{(1)}, ...\gamma^{(j)}, \Omega^{(j)}, \beta^{(j)}, ..., \gamma^{(M)}, \Omega^{(M)}, \beta^{(M)}$$

Except for $\gamma^{(0)}$ which is initialized as $\gamma^{(0)} = (1, 0, 0, ..., 0)$, the subsequent values of $\gamma^{(j)}, \Omega^{(j)}, \beta^{(j)}$ are obtained obtained by successively simulationg values according to the following iterated scheme.

1. Given $\gamma^{(j-1)}$, the next iterate $\gamma^{(1)}$ is obtained by sampling from $p(\gamma|y, X)$

$$\begin{aligned} p_k &= P(\gamma_k = 1|\gamma_{/k}, y, X) = \frac{L}{1+L} \\ L &\propto \left(\frac{\left|\underline{H}_\beta(1)\right|}{\left|\underline{H}_\beta(0)\right|}\right)^{-\frac{N}{2}} \left(\frac{\left|X'_{\gamma(1)}X_{\gamma(1)} + \underline{H}_\beta(1)^{-1}\right|}{\left|X'_{\gamma(0)}X_{\gamma(0)} + \underline{H}_\beta(0)^{-1}\right|}\right)^{-\frac{N}{2}} \left(\frac{|\Phi + S(1)|}{|\Phi + S(0)|}\right)^{-\frac{(T+m)}{2}} \end{aligned}$$

Draw $u \sim U(0, 1)$

if $u < p_k$ **then** $\gamma_k^{(j)} = 1$; $\gamma^{(j)} = \gamma_{_{(1)}}$ and $S^{(j)} = S(1)$

 (a) **else** $\gamma_k^{(j)} = 0$; $\gamma^{(j)} = \gamma_{_{(0)}}$and $S^{(j)} = S(0)$ **end else**

2. Given $\gamma^{(j)}$, draw $\Omega^{(j)}$ by sampling from the Wishart distribution $\Omega^{(j)}|y, \gamma^{(j)} \sim W_N(m + T, \left(S(\gamma^{(j)}) + \Phi\right)^{-1})$, where

$$\begin{aligned} S(\gamma^{(j)}) &= Y'\left(I_T - X_{\gamma^{(j)}}D^{-1}_{\gamma^{(j)}}X'_{\gamma^{(j)}}\right)Y \text{ if we take a prior with } \beta_0 = \mathbf{0} \\ D_{\gamma^{(j)}} &= \left(X'_{\gamma^{(j)}}X_{\gamma^{(j)}} + \underline{H}_\beta^{-1}\right) \end{aligned}$$

3. Given $\gamma^{(j)}$, $\Omega^{(j)}$, draw $\beta^{(j)}$ by random sampling from $\beta^{(j)}|y, \Omega^{(j)}, \gamma^{(j)} \sim N\left(\widetilde{\beta}_\gamma, \Sigma^{(j)} \otimes D^{-1}_{\gamma^{(j)}}\right)$ where $\Sigma^{(j)} = \left(\Omega^{(j)}\right)^{-1}$ and

$$\widetilde{\beta}_{\gamma^{(j)}} = \left(I_N \otimes D^{-1}_{\gamma^{(j)}}\underline{H}_\beta^{-1}\right)\beta_0 + \left(I_N \otimes D^{-1}_{\gamma^{(j)}}X'_{\gamma^{(j)}}X_{\gamma^{(j)}}\right)\widehat{\beta}_{GLS}$$

4. Compute the risk premia iterates $\delta^{(j)} = \left(\widetilde{B}^{(j)\prime}\widetilde{B}^{(j)}\right)^{-1}\widetilde{B}^{(j)\prime}\alpha^{(j)}$ and the pricing errors iterates

$$\begin{aligned} \widetilde{Q}_N^{2^{(j)}} &= \frac{\alpha^{(j)\prime}\left(I_N - \widetilde{B}^{(j)}\left(\widetilde{B}^{(j)\prime}\widetilde{B}^{(j)}\right)^{-1}\widetilde{B}^{(j)\prime}\right)'\Omega^{(j)}\left(I_N - \widetilde{B}^{(j)}\left(\widetilde{B}^{(j)\prime}\widetilde{B}^{(j)}\right)^{-1}\widetilde{B}^{(j)\prime}\right)\alpha^{(j)}}{N} \\ \widetilde{B}^{(j)} &= (\imath, \widetilde{B}^{(j)}) \\ Q_N^{2^{(j)}} &= \frac{\alpha^{(j)\prime}\left(I_N - \widetilde{B}^{(j)}\left(\widetilde{B}^{(j)\prime}\widetilde{B}^{(j)}\right)^{-1}\widetilde{B}^{(j)\prime}\right)\alpha^{(j)}}{N} \end{aligned}$$

6.2.2 Posterir density of $\gamma|y, X$

For the *Normal-Wishart g-prior* case, the random mechanism general informative prior (See Appendix),

$$\begin{aligned} L &= \frac{p\left[\gamma(1)|y\right]}{p\left[\gamma(0)|y\right]} \propto \frac{p(y|\gamma(1))}{p(y|\gamma(0))} \\ &\propto \left(\frac{\left|\underline{H}_\beta(1)\right|}{\left|\underline{H}_\beta(0)\right|}\right)^{-\frac{N}{2}} \left(\frac{\left|X'_{\gamma(1)}X_{\gamma(1)} + \underline{H}_\beta(1)^{-1}\right|}{\left|X'_{\gamma(0)}X_{\gamma(0)} + \underline{H}_\beta(0)^{-1}\right|}\right)^{-\frac{N}{2}} \left(\frac{|\Phi + S(1)|}{|\Phi + S(0)|}\right)^{-\frac{(T+m)}{2}} \\ logL &= -\frac{N}{2}\log\frac{\left|\underline{H}_\beta(1)\right|}{\left|\underline{H}_\beta(0)\right|} - \frac{N}{2}\log\frac{\left|X'_{\gamma(1)}X_{\gamma(1)} + \underline{H}_\beta(1)^{-1}\right|}{\left|X'_{\gamma(0)}X_{\gamma(0)} + \underline{H}_\beta(0)^{-1}\right|} - \frac{(T+m)}{2}\log\frac{|\Phi + S(1)|}{|\Phi + S(0)|} \qquad (16) \end{aligned}$$

In the informative diffuse prior case 1:

$$L \propto c^{-\frac{N}{2}} \left(\frac{\left|X'_{\gamma(1)}X_{\gamma(1)} + \frac{1}{c}I_{q_{\gamma(1)}}\right|}{\left|X'_{\gamma(0)}X_{\gamma(0)} + \frac{1}{c}I_{q_{\gamma(0)}}\right|}\right)^{-\frac{N}{2}} \left(\frac{|\Phi + S(1)|}{|\Phi + S(0)|}\right)^{-\frac{(T+m)}{2}}$$

$$\log L = -\frac{N}{2}\log c - \frac{N}{2}\log\frac{\left|X'_{\gamma(1)}X_{\gamma(1)} + \frac{1}{c}I_{q_{\gamma(1)}}\right|}{\left|X'_{\gamma(0)}X_{\gamma(0)} + \frac{1}{c}I_{q_{\gamma(0)}}\right|} - \frac{(T+m)}{2}\log\frac{\left|S_{\gamma(1)} + \Phi\right|}{\left|S_{\gamma(0)} + \Phi\right|} \qquad (17)$$