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GENERIC DETERMINACY OF NASH EQUILIBRIUM IN NETWORK FORMATION GAMES*

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Abstract. This paper shows that the set of probability distributions over networks induced by Nash equilibria of the network formation game proposed by Myerson (1991) is finite for a generic assignment of payoffs to networks. The same result can be extended to several variations of the game found in the literature.

1. Introduction

A basic tool in applying noncooperative game theory is to have a finite set of probability distributions on outcomes derived from equilibria.[1] When utilities are defined over the relevant outcome space, it is well know that this is generically the case when we can assign a different outcome to each pure strategy profile (Harsanyi, 1973), or to each ending node of an extensive form game (Kreps and Wilson, 1982).[2]

A *game form* endows players with finite strategy sets and specifies which is the outcome that arises from each pure strategy profile.[3] It could identify, for instance, two ending nodes in an extensive game form with the same outcome. Govindan and McLennan (2001) give an example of a game form such that, in an open set of utilities over outcomes, produces infinitely many equilibrium distributions on outcomes. In view of such a negative result, we have to turn to specific classes of games to seek for positive results regarding the generic determinacy of the Nash equilibrium concept. For some examples, see Park (1997) for sender-receiver games, and De Sinopoli (2001), De Sinopoli and Iannantuoni (2005) for voting games.

This paper studies the generic determinacy of the Nash equilibrium concept when individual payoffs depend on the network connecting them. The network literature has been fruitful to describe social and economic interaction. See for instance Jackson and Wolinsky (1996), Jackson and Watts (2002), Kranton and

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[1]By *outcomes* we mean the set of physical or economic outcomes of the game (i.e. the set of different economic alternatives that can be found after the game is played) and not the set of probability distribution induced by equilibria. We will refer to the latter concept as the set of equilibrium distributions.

[2]Harsanyi (1973) actually proves that the set of Nash equilibria is finite for a generic assignment of payoffs to pure strategy profiles.

[3]More generally, it specifies a probability distribution on the set of outcomes. Game forms are formally defined in Section 2.2

Minehart (2001), or Calvo-Armengol (2004). It is, therefore, important to have theories about how such networks form. Different network formation procedures have been proposed. For a comprehensive survey of those theories the reader is referred to Jackson (2003).

The current paper is concerned with a noncooperative approach to network formation. Perhaps the first paper that follows this same line of reseach is Calvo-Armengol and İlkilic (2007). We focus on the network formation game proposed by Myerson (1991). It can be described as follows: each player simultaneously proposes a list of players with whom she wants to form a link, and a direct link between two players is formed if and only if both players agree on that. This game is simple and intuitive, however, since it takes two players to form a link, a coordination problem arises which makes the game exhibit multiplicity of equilibria. Nevertheless, we can prove that even though a network formation game may have a large number of equilibria, every probability distribution on networks induced by equilibria is generically isolated.

The network formation game is formally presented in the next section. Section 3 discusses an example. Section 4 contains the main result and its proof. To conclude, Section 5 discusses some extensions of the result to other network formation games as well as a related result for the extensive form game of network formation introduced by Aumann and Myerson (1989).

2. Preliminaries

Given a finite set A, denote as $\mathcal{P}(A)$ the power set of A, and as $\Delta(A)$ the set of probability distributions on A.

2.1. **Networks.** Given a set of players N, a *network* g is a collection of direct links. A direct link in the network g between two different players i and j is denoted by $ij \in g$. For the time being we focus on undirected networks. In an undirected network $ij \in g$ is equivalent to $ji \in g$.[4] The set of i's direct links in g is $L_i(g) = \{jk \in g : j = i \text{ or } k = i\}$.

The complete network g^N is such that $L_i(g^N) = \{ij : j \neq i\}$, for all $i \in N$. In g^N player i is directly linked to every other player. The set of all undirected networks on N is $\mathcal{G} = \mathcal{P}(g^N)$.

Each player i can be directly linked with $N - 1$ other players. The number of links in the complete network g^N is $N(N-1)/2$, dividing by 2 not to count links twice. Since $\mathcal{G}$ is the power set of g^N, it has $2^{N(N-1)/2}$ elements.

2.2. **Game forms.** A *game form* is given by a set of players $N = \{1, \ldots, n\}$, nonempty finite sets of pure strategies $S_1, \ldots, S_n$, a finite set of outcomes Ω, a function $\theta\colon S \to \Delta(\Omega)$, and utilities defined over the outcome space Ω, that is, $u_1, \ldots, u_n\colon \Omega \to \mathbb{R}$. Once we fix N, $S_1, \ldots, S_n$, Ω, and θ, a game form is given by a point in $\left(\mathbb{R}^{\Omega}\right)^N$.

Utility functions $u_1, \ldots, u_n$ over Ω induce utility functions $v_1, \ldots, v_n$ over S according to $u_1 \circ \theta, \ldots, u_n \circ \theta$. Hence, every game form has associated its finite normal form game.

[4]In a directed network, if i and j are two different agents, the link ij is different from the link ji. This two links can be regarded as different if, for instance, they explain which is the direction of information, or which is the player who is sponsoring the link.

2.3. **The Network Formation Game.** The following network formation game is due to Myerson (1991). The set of players is N. All players in N simultaneously announce the set of direct links they wish to form. Formally, the set of player i's pure strategies is $S_i = \mathcal{P}(N \setminus \{i\})$. Therefore, a strategy $s_i \in S_i$ is a subset of $N \setminus \{i\}$ and is interpreted as the set of players other than i with whom player i wishes to form a link. Mutual consent is needed to create a direct link, i.e., if s is played, ij is created if and only if $j \in s_i$ and $i \in s_j$.

We can adapt the previous general description of game forms to the present context in order to specify the game form that structures the network formation game. Let the set of players and the collection of pure strategy sets be as above. The set of outcomes is the set of undirected networks, i.e., $\Omega = \mathcal{G}$. The function θ is a deterministic outcome function, formally, $\theta \colon S \to \mathcal{G}$. Given a pure strategy profile, θ specifies which network is formed respecting the rule of mutual consent to create direct links. Utilities are functions $u_1, \ldots, u_n \colon \mathcal{G} \to \mathbb{R}$. Once the set of players N is given, the pure strategy sets are automatically created and the network formation game is defined by a point in $\left(\mathbb{R}^{\mathcal{G}}\right)^N$.

If players other than i play according to $s_{-i} \in S_{-i}$,[5] the utility to player i from playing strategy s_i is equal to $v_i(s_i, s_{-i}) = u_i(\theta(s_i, s_{-i}))$.

Let $\Sigma_i = \Delta(S_i)$ be the set of mixed strategies of player i. Furthermore, let $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$. While a pure strategy profile s results in the network $\theta(s)$ with certainty, a mixed strategy profile σ generates a probability distribution on $\mathcal{G}$, where the probability that $g \in \mathcal{G}$ forms equals

$$\mathbf{P}(g \mid \sigma) = \sum_{s \in \theta^{-1}(g)} \left(\prod_{i \in N} \sigma_i(s_i) \right).$$

If players other than i play according to σ_{-i} in Σ_{-i},[6] the utility to player i from playing the mixed strategy σ_i is equal to $V_i(\sigma_i, \sigma_{-i}) = \sum_{g \in \mathcal{G}} \mathbf{P}(g \mid (\sigma_i, \sigma_{-i})) u_i(g)$.

Definition 1 (Nash Equilibrium)**.** The strategy profile $\sigma \in \Sigma$ is a Nash equilibrium of the network formation game if $V_i(\sigma_i, \sigma_{-i}) \geq V_i(\sigma_i', \sigma_{-i})$ for all σ_i' in Σ_i, and for all i in N.

2.4. **Generic Finiteness of Equilibrium Distributions in Game Forms.** Let us first give the definition of a generic set.

Definition 2. For any $m \geq 0$, we say that $G \subset \mathbb{R}^m$ is a generic set, or generic, if $\mathbb{R}^m \setminus \operatorname{int}(G)$ has Lebesgue measure 0.

Govindan and McLennan (2001) give an example of a game form that, in an open set of utilities over outcomes, produces infinitely many equilibrium distributions on the outcome space.[7] Nevertheless, they also provide a number of positive results. Consider the general specification of game forms given in Section 2.2. The following theorem is a slight modification of Theorem 5.3 in Govindan and McLennan (2001).

Theorem 1. *If θ is such that at all completely mixed strategy tuples and for each agent i the set of distributions on Ω that agent i can induce by changing her strategy*

[5] $S_{-i} = \prod_{j \neq i} S_j$.

[6] $\Sigma_{-i} = \prod_{j \neq i} \Sigma_j$.

[7] Their counterexample needs at least three players. In a recent paper, Kukushkin et al. (2007) provide a counterexample for the two player case.

is $(|S_i|-1)$*-dimensional, then for generic utilities there are finitely many completely mixed equilibria.*

To maintain the paper self-contained, the proof of Theorem 1 is offered in the appendix.

3. An Example

Consider a 3 person network formation game. The corresponding game form is depicted in Figure 1. Player 1 is the row player, player 2 the column player, and player 3 the matrix player. The symbol g^0 denotes the empty network, g^N denotes the complete network, g^{ij} denotes the network that only contains link ij, and g^i denotes the network where player i is connected to every other player and such that there are no further links.[8]

	$\{\emptyset\}$	$\{1\}$	$\{3\}$	$\{1,3\}$	$\{\emptyset\}$	$\{1\}$	$\{3\}$	$\{1,3\}$
$\{\emptyset\}$	g^0	g^0	g^0	g^0	g^0	g^0	g^0	g^0
$\{2\}$	g^0	g^{12}	g^0	g^{12}	g^0	g^{12}	g^0	g^{12}
$\{3\}$	g^0	g^0	g^0	g^0	g^{13}	g^{13}	g^{13}	g^{13}
$\{2,3\}$	g^0	g^{12}	g^0	g^{12}	g^{13}	g^1	g^{13}	g^1
			$\{\emptyset\}$				$\{1\}$	
$\{\emptyset\}$	g^0	g^0	g^{23}	g^{23}	g^0	g^0	g^{23}	g^{23}
$\{2\}$	g^0	g^{12}	g^{23}	g^2	g^0	g^{12}	g^{23}	g^2
$\{3\}$	g^0	g^0	g^{23}	g^{23}	g^{13}	g^{13}	g^3	g^3
$\{2,3\}$	g^0	g^{12}	g^{23}	g^2	g^{13}	g^1	g^3	g^N
			$\{2\}$				$\{1,2\}$	

Figure 1. The game form of a network formation game with three players.

Suppose that the utility function of player $i=1,2$ is $u_i(g)=|L_i(g)|$, i.e. player $i=1,2$ derives an utility from network g equal to the number of direct links that she maintains in g. Suppose also that player 3 has the same utility as players 1 and 2, except that she derives an utility equal to 2 from network g^2. Specifically,

$$\begin{aligned}
u_i(g^0) = &\quad 0 \text{ for all } i,\\
u_i(g^{jk}) = &\begin{cases} 1 \text{ if } i=k \text{ or } i=j \\ 0 \text{ otherwise,} \end{cases}\\
u_i(g^j) = &\begin{cases} 2 \text{ if } i=j \\ 2 \text{ if } i=3 \text{ and } j=2 \\ 1 \text{ otherwise,} \end{cases}\\
g_i(g^N) = &\quad 2 \text{ for all } i.
\end{aligned}$$

[8]This network architecture is often referred to as a *star*, see Bala and Goyal (2000)

Figure 2 displays the set of Nash equilibria of this game. The subset of Nash equilibria of line (i) supports the empty network, the subsets of line (ii) support, respectively, networks g^{12}, g^{13} and g^{23}, the subsets of line (iii) support, respectively, networks g^1, g^2 and g^3.

(i) $\text{NE} = \Big\{ (\{\emptyset\}, \{\emptyset\}, \{\emptyset\}) \Big\} \bigcup$

(ii) $\Big\{ (\{2\}, \{1\}, \{\emptyset\}) \Big\} \bigcup \Big\{ (\{3\}, \{\emptyset\}, \{1\}) \Big\} \bigcup \Big\{ (\{\emptyset\}, \{3\}, \{2\}) \Big\} \bigcup$

(iii) $\Big\{ (\{2,3\}, \{1\}, \{1\}) \Big\} \bigcup \Big\{ (\{2\}, \{1,3\}, \{2\}) \Big\} \bigcup \Big\{ (\{3\}, \{3\}, \{1,2\}) \Big\} \bigcup$

(iv) $\Big\{ (\{2,3\}, \{1,3\}, \lambda\{2\} + (1-\lambda)\{1,2\}) : \lambda \in [0,1] \Big\}.$

Figure 2. Set of Nash equilibria of the 3 person network formation game discussed in Section 3.

The subset of equilibria of line (iv) induces a continuum of probability distribution over the set of networks that give probability λ to network $g^2$and probability $(1-\lambda)$ to the complete network g^N for $\lambda \in [0,1]$.

Now perturb independently the utility that each player obtains from each network. The subsets of strategy profiles of lines (i) through (iii) are still equilibrium strategy profiles. In addition, there are two possibilities:

- Player 3 ranks the complete network g^N over network g^2. In this case the set of Nash equilibria is composed of lines (i) through (iii) united to
$$\Big\{ (\{2,3\}, \{1,3\}, \{1,2\}) \Big\},$$
which supports the complete network.
- Player 3 ranks network g^2 over the complet nerwork g^N. Then, no Nash equilibrium gives positive probability to the complete network. The set of Nash equilibria is composed of lines (i) through (iii) united to
$$\Big\{ (\lambda\{2\} + (1-\lambda)\{2,3\}, \{1,3\}, \{2\}) : \lambda \in [0,1) \Big\},$$
which supports network g^2.

In either case, there is a finite number of probability distributions on networks induced by equilibria.

4. The Result

Proposition. *For generic $u \in \left(\mathbb{R}^{\mathcal{G}}\right)^N$ the set of probability distributions on networks induced by Nash equilibria of the network formation game is finite.*

Proof. Given a network formation game, there are a finite number of different normal form games obtained by assigning to each player i an element of $\mathcal{P}(S_i)$ as her strategy set.

Let $T = T_1 \times \cdots \times T_n$, where $T_i \subseteq S_i$. The normal form game Γ_T is defined by the set of players N, the collection of strategy sets $\{T_i\}_{i \in N}$, and the collection of utility functions $\{v_i^T\}_{i\in N}$, where v_i^T is the restriction of v_i to T. Furthermore, let $\mathcal{G}_T = \theta(T)$.

It is enough to prove that for a generic assignment of payoffs to networks, completely mixed Nash equilibria of each of those games induce a finite set of probability distributions on $\mathcal{G}$. Notice that every equilibrium of any game can be obtained as a completely mixed equilibrium of the modified game obtained by eliminating unused strategies.

Consider the game Γ_T. If there exists a strategy $t_i \in T_i$ with $j \in t_i$ and there does not exist a strategy $t_j \in T_j$ such that $i \in t_j$, replace strategy t_i with $t'_i = t_i \setminus \{j\}$ in case t'_i is not already contained in T_i, otherwise just eliminate strategy t_i from T_i. Notice that by making this change, the set of probability distributions on $\mathcal{G}_T$ that can be obtained through mixed strategies remains unaltered. Most importantly, for every completely mixed Nash equilibrium of Γ_T, there exists a completely mixed Nash equilibrium of the modified game that induces the same probability distribution on $\mathcal{G}_T$.

Repeat the same procedure with t'_i: if there exists a $k \in t'_i$ and there does not exist a strategy t_k in T_k with $i \in t_k$ substitute t'_i for $t''_i = t'_i \setminus \{k\}$ in case t''_i is not already contained in T_k. Continue eliminating and replacing pure strategies in the same vein, for every t_i in T_i and for every i in N, until every link proposal that any player has in some on her strategies is formed with positive probability under a completely mixed strategy profile. Let $\hat{T}$ denote the resulting set of pure strategy profiles, and notice that $\mathcal{G}_{\hat{T}} = \mathcal{G}_T$.

At every completely mixed strategy profile σ of $\Gamma_{\hat{T}}$, every network in $\mathcal{G}_T$ receives positive probability. At the strategy profile (t_i, σ_{-i}), only those networks $g \in \mathcal{G}_T$ such that $\{ij : j \in t_i\} \subset g$ receive positive probability, and since for every player i each of her pure strategies is different, we have that:

$$\operatorname{rank} \frac{\partial \mathbf{P}}{\partial \sigma_i}(\cdot \mid \sigma) = |\hat{T}_i| - 1.$$

Therefore, at every completely mixed strategy profile of $\Gamma_{\hat{T}}$ the set probability distributions on $\mathcal{G}_T$ that player i can induce by varying her strategy is $(|\hat{T}_i| - 1)$-dimensional. We can apply Theorem 1 to the game form given by $\hat{T}$ and $\theta_{\hat{T}}$, the restriction of θ to $\hat{T}$. This implies that for generic utilities over $\mathcal{G}_T$ there are finitely many completely mixed equilibria of $\Gamma_{\hat{T}}$, which in turn implies that the set of probability distributions on $\mathcal{G}_T$ induced by completely mixed Nash equilibria of Γ_T is generically finite.

Let $T \subseteq S$, we can write $\left(\mathbb{R}^{\mathcal{G}}\right)^N = \left(\mathbb{R}^{\mathcal{G}_T}\right)^N \times \left(\mathbb{R}^{\mathcal{G}\setminus\mathcal{G}_T}\right)^N$. Let K be a closed set of zero measure in $\left(\mathbb{R}^{\mathcal{G}_T}\right)^N$, i.e., the closure of the set of payoffs over $\mathcal{G}_T$ such that the set of completely mixed Nash equilibria of Γ_T induces infinitely many probability distributions on $\mathcal{G}_T$, then for any closed set H in $\left(\mathbb{R}^{\mathcal{G}\setminus\mathcal{G}_T}\right)^N$ the closed set $K \times H$ has zero measure in $\left(\mathbb{R}^{\mathcal{G}}\right)^N$. The same is true for any other $T' \subseteq S$. This concludes the proof. □

5. Remarks

5.1. **Absence of Mutual Consent.** Models of network formation can be found in the literature that do not require common agreement between the parties to create a direct link, see for instance Bala and Goyal (2000). Thus, suppose that mutual consent is not needed to create a direct link. Let N be the set of players, let $S_1, \ldots, S_n$ be the collection of pure strategy sets, where $S_i = \mathcal{P}(N \setminus \{i\})$ for all i in N, and let $\mathcal{G}$ be the outcome space. In the model analyzed in Section 4, a link may not be created even if a player wants it to be created. In the current model, a link may be created even if a player does not want it to be created.

In this modified network formation game, generically, the set of equilibrium distributions on $\mathcal{G}$ is also finite. Notice that we can reinterpret pure strategies $s_i \in S_i$ as the set of players other that i with whom player i does not want to form a link. The link ij is not created only if player i does not want to be linked with player j and player j does not want to be linked with player i. Define $\theta' : S \to \mathcal{G}$ according to $\theta'(s) = g^N \setminus \theta(s)$, where θ is the one defined in Section 2.3. Now, apply the proof of Section 4.

5.2. **Directed Networks.** Sometimes links ij and ji cannot be treated as equivalent for reasons coming from the nature of the phenomena being modeled. Directed networks respond to this necessity, for an example see again Bala and Goyal (2000). Denote the set of directed networks as $\mathcal{G}^d$. Suppose first that link formation does not need mutual consent. The strategy set of player i is $S_i = \mathcal{P}(N \setminus \{i\})$. A strategy $s_i \in S_i$ is interpreted as the set of players other than i with whom player i wants to start an arrowhead link pointing at herself, i.e. the set of links that player i wishes to receive.[9]

Notice that each pure strategy profile leads to a different element in $\mathcal{G}^d$: each player has 2^{N-1} pure strategies, and there are $2^{N(N-1)}$ undirected networks. Therefore, we are in the case of normal form payoffs where the generic finiteness of equilibria is guaranteed.

Suppose now that if a player i wants to receive a link from player j, player j needs to declare that she wants to send a link to player i for it to be created. To accommodate for this case, let the strategy set of player i be $S_i = S_i^r \times S_i^s = \mathcal{P}(N \setminus \{i\}) \times \mathcal{P}(N \setminus \{i\})$. A strategy $s_i \in S_i$ has two components, s_i^r and s_i^s. We interpret s_i^r as the set of players other than i from whom player i wishes to receive a link, and s_i^s as the set of players other than i to whom player i wishes to send a link. Suppose that the pure strategy profile s is played. The link ij is created only if $j \in s_i^r$ and $i \in s_j^s$.

A similar proof to the one used in Section 4 establishes the generic determinacy of the Nash equilibrium concept under this setting. The key step that we must change is the following: Let $T = T_1 \times \cdots \times T_n$ where $T_i \subset S_i$ for all i. Consider the normal form game Γ_T. If there exists a strategy $t_i \in T_i$ such that $j \in t_i^r$ (such that $j \in t_i^s$) and there does not exist a strategy $t_j \in T_j$ such that $i \in t_j^s$ (such that $i \in t_j^r$), replace strategy t_i with $t_i' = (t_i^r \setminus \{j\}, t_i^s)$ (with $t_i' = (t_i^r, t_i^s \setminus \{j\})$). Finally, repeat the same procedure for every $t_i, t_i', \ldots$ and for every i until the hypothesis of Theorem 1 holds.

5.3. **A Extensive Form Game of Network Formation.** We have focused on normal form games of network formation. However, there exists a prominent extensive game of network formation due to Aumann and Myerson (1989). They proposed the first explicit formalization of network formation as a game. It relies

[9]We can assume, for instance, that the arrowhead tells which is the direction of the flow of information.

on an exogenously given order over possible links. Let $(i_1j_1, \ldots, i_mj_m)$ be such a ranking.

The game has m stages. In the first stage players i_1 and j_1 play a simultaneous move game to decide whether or not they form link i_1j_1. Each of them chooses an action from the set $\{yes, not\}$. The link i_1j_1 is established if and only if both players choose *yes*. Once the decision on link i_1j_1 is taken, every player gets informed about it, and the play of the game moves to the decision about link i_2j_2. The game evolves in the same fashion, and finishes with the stage where players i_m and j_m decide upon link i_mj_m.[10] The resulting network is formed by the set links i_kj_k such that both players i_k and j_k chose *yes* at stage k. Although in the argument we work with undirected networks, the game can be applied to the formation of directed networks.

The argument that follows is a modification of the one used by Govindan and McLennan (2001) to prove that, for a given assignment of outcomes to ending nodes in an extensive game of perfect information, and for utilities such that no player is indifferent between two different outcomes, every Nash equilibrium induces a degenerate probability distribution in the set of outcomes. Such an argument is, in turn, a generalization of the one used by Kuhn (1953) to prove his "backwards induction" theorem that characterizes subgame perfect equilibria for games of perfect information.

Consider the generic set of utilities

$$U_G = \left\{ u \in \left(\mathbb{R}^{\mathcal{G}}\right)^N : u_i(g_1) \neq u_i(g_2) \text{ for all } i \in N \text{ and all } g_1, g_2 \in \mathcal{G} \right\}.$$

The claim is that if the utility vector is $u \in U_G$, every Nash equilibrium induces a probability distribution on $\mathcal{G}$ that assigns probability one to some $g \in \mathcal{G}$.

Let S_i denote the set of pure strategies of player i, where now a pure strategy is a function that assigns one element of $\{yes, not\}$ to each information set of player i. As usual, $\Sigma_i = \Delta(S_i)$ and $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$.

Let $\sigma \in \Sigma$ be a Nash equilibrium for $u \in U_G$. The appropriate modification of σ, say $\bar{\sigma}$, is a completely mixed Nash equilibrium of the extensive form game obtained by eliminating all information sets and branches that occur with zero probability in case σ is played. In this reduced game, every information set has a well defined conditional probability over networks and, obviously, $\bar{\sigma}$ induces the same probability distribution on $\mathcal{G}$ as σ.

If there is a stage where a player randomizes between *yes* and *not* and the other player does chooses *yes* with positive probability 1, there must be a last such stage. But at this last stage, say i_hj_h, such an agent, say i_h, cannot be optimizing, since she is not indifferent between $g \setminus \{i_hj_h\}$ and $g \cup \{i_hj_h\}$ for any $g \in \mathcal{G}$.

We can adapt the previous argument to the case where mutual consent is not needed to create a link. Let $(i_1j_1, \ldots, i_mj_m)$ be an oder of links. At stage k, player i_k decides whether or not to create link i_kj_k. Her decision becomes publicly

[10] If players get informed about which has been the terminal position in the simultaneous move game of every stage, the same argument offered below also goes through.

Several features can be added to this basic model. For instance, two players can be called to reconsider their decision in case some set of links is formed, or two player may not be allowed to decide upon the link connecting them. At this respect, if players are forming an undirected network, m can be different from $2^{\frac{N(N-1)}{2}}$.

known. It is, consequently, a game of perfect information and the argument given by Govindan and McLennan (2001) covers this case.

Appendix A. Proof of Theorem 1

The current proof is based on the one offered by Govindan and McLennan (2001). It uses some concepts and results of semi-algebraic theory that we will now revise. Expositions of semi-algebraic geometry in the economic literature occur in Blume and Zame (1994), Schanuel et al. (1991) and Govindan and McLennan (2001). Proofs of major results are omitted.

Definition 3. A set A is semi-algebraic if it is the finite union of sets of the form

$$\Big\{x \in \mathbb{R}^m : P(x) = 0 \text{ and } Q_1(x) > 0 \text{ and} \ldots \text{and } Q_k(x) > 0\Big\}$$

where P and $Q_1, \ldots, Q_k$ are polynomials in $x_1, \ldots, x_m$ with real coefficients. A function (or correspondence) $g : A \to B$ with semi-algebraic domain $A \subset \mathbb{R}^n$ and range $B \subset \mathbb{R}^m$ is semi-algebraic if its graph is a semi-algebraic subset of $\mathbb{R}^{n+m}$.

Each semi-algebraic set is the finite union of connected components. Each component is a *semi-algebraic manifold* of a given dimension. A *d-dimensional semi-algebraic manifold* in $\mathbb{R}^m$ is a semi-algebraic set $M \subset \mathbb{R}^m$ such that for each $p \in M$ there exist polynomials $P_1, \ldots, P_{m-d}$ and U, a neighborhood of p, such that $DP_1(p), \ldots, DP_{m-d}(p)$ are linearly independent and

$$M \cap U = \Big\{q \in U : P_1(q) = \ldots = P_{m-d}(q) = 0\Big\}.$$

Theorem 2 (Stratification, Whitney (1957))**.** *If A is a semi-algebraic set, then A is the union of a finite number of disjoint, connected semi-algebraic manifolds A^j with $A^j \subset \text{cl}(A^k)$ whenever $A^j \cap \text{cl}(A^k) \neq \emptyset$.*

Henceforth, the superscript of a set indexes components of a decomposition as per Theorem 2, while a subscript keeps indexing strategy sets by players. Theorem 2 has important consequences. Among those, we will use the following intuitive ones: Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be semi-algebraic sets, then

- the dimension of A, $\dim A$, is equal to the largest dimension of any element of any stratification,
- if A is 0-dimensional then A is finite,
- A is generic if and only if $\dim(\mathbb{R}^m \setminus A) < m$,
- $\dim(A \times B) = \dim A + \dim B$.

We need one additional result. While Theorem 2 decomposes semi-algebraic sets, the following one decomposes semi-algebraic functions.

Theorem 3 (Generic Local Triviality, Hardt (1980))**.** *Let A and B be semi algebraic sets, and let $g \colon A \to B$ be a continuous semi-algebraic function. Then there is a relatively closed semi-algebraic set $B' \subset B$ with $\dim B' < \dim B$ such that each component B^j of $B \setminus B'$ has the following property: there is a semi algebraic set F^j and a semi-algebraic homeomorphism $h \colon B^j \times F^j \to A^j$, where $A^j = g^{-1}(B^j)$, with $g(h(b, f)) = b$ for all $(b, f) \in B^j \times F^j$.*

We can now proceed to prove Theorem 1. Recall that at every completely mixed strategy $\sigma \in \Sigma$, the set of probability distributions on outcomes that player i can induce by varying her strategy is $(|S_i| - 1)$-dimensional.

Proof of Theorem 1. Let $A = \{(\sigma, u) : \sigma \text{ is a completely mixed equilibrium for } u\}$. Let π_Σ be the projection of A onto Σ. Apply Theorem 3 to π_Σ and choose Σ^j such that $\dim A^j = \dim A$.[11] We have that $\dim A = \dim \Sigma^j + \dim F^j \leq \dim \Sigma + \dim F^j$. Let σ belong to Σ^j, then $\dim \pi_\Sigma^{-1}(\sigma) = \dim\{\sigma\} + \dim F^j = \dim F^j$. Now consider a given u, the set $\{\tilde{u}_i \in U_i : \sigma \text{ is a completely mixed equilibrium for } (\tilde{u}_i, u_{-i})\}$ is $(\dim U_i - (|S_i| - 1))$-dimensional. Consequently, the dimension of $\pi_\Sigma^{-1}(\sigma)$ and F^j is equal to $\dim U - \dim \Sigma$, which implies that $\dim A \leq \dim U$.

Now apply Theorem 3 to π_U, the projection of A onto U. Choose U^j to be of the same dimension as U. Therefore, $\dim A^j = \dim U + \dim \pi_U^{-1}(u)$. This implies that $\dim \pi_U^{-1}(u) \leq \dim A - \dim U \leq 0$, i.e. there is a finite set of completely mixed equilibria whenever u belongs to a full dimensional U^j. This concludes the proof since lower dimensional U^j's are nongeneric. □

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[11] Such a Σ^j can be found because we can keep applying Theorem 3 to $\pi_\Sigma : \pi_\Sigma^{-1}(\Sigma') \to \Sigma'$, where Σ' plays the role of B'.

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