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GENERIC FINITENESS OF OUTCOME DISTRIBUTIONS FOR TWO PERSON GAME FORMS WITH THREE OUTCOMES*

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ABSTRACT. A two person game form is given by nonempty finite sets S_1, S_2 of pure strategies, a nonempty set Ω of outcomes, and a function $\theta\colon S_1 \times S_2 \to \Delta(\Omega)$, where $\Delta(\Omega)$ is the set of probability measures on Ω. This note proves that if the set of outcomes contains just three elements, generically, there are finitely many distributions on Ω induced by Nash equilibria.

1. INTRODUCTION

As Debreu (1970) argues, a mathematical model that tries to explain economic equilibrium must have a nonempty set of solutions. In principle, one would want the solution to be unique. However, models with multiplicity of solutions should be allowed for, since they also provide a satisfactory explanation of equilibrium as long as solutions remain locally unique. If the set of solutions is compact, local uniqueness coincides with finiteness of equilibrium points. When there can be constructed models with infinitely many equilibria, one can at best prove that outside a small set of models, the number of equilibria is finite.

Harsanyi (1973) proves that for a generic set of normal form payoffs, the number of Nash equilibria is finite (and odd). We can think of the normal form of a game as being derived from an extensive form game, therefore, the space of extensive form utilities is a linear subspace of the space of normal form utilities, and the generic finiteness in the subspace does not follow from the generic finiteness in the larger space.

Kreps and Wilson (1982) prove that for almost all utilities over the set of final nodes of an extensive game, the set of probability distributions on the set of final nodes (the set of paths) induced by Nash equilibria is finite. However, the previous reasoning can be repeated again. Two different ending nodes in an extensive form game must be considered "equivalent" if they correspond to the same economic outcome.

Consider a set of players $N = \{1, \dots, n\}$, finite sets of strategies $S_1, \dots, S_n$, a nonempty finite set of outcomes Ω, and a function $\theta\colon S \to \Delta(\Omega)$, where $S = S_1 \times \cdots \times S_n$, and $\Delta(\Omega)$ denotes the set of probability distributions on Ω. Utility functions $u_1, \dots, u_n$ over Ω induce payoff functions $v_1, \dots, v_n$ over S that correspond

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to $u_1 \circ \theta, \ldots, u_n \circ \theta$. Therefore, we can talk about the induced normal form game and about the set of equilibria of such an induced game.

The conjecture that for a generic set of utility functions $u_1, \ldots, u_n$ the set of probability distributions on Ω induced by Nash equilibria of the derived game is finite, was refuted by means of a counterexample by Govindan and McLennan (2001). Such a counterexample needs at least three players and six outcomes. In a recent paper, Kukushkin et al. (2007) produce a new counterexample that only needs two players and four outcomes.

Notwithstanding the previous negative results, we have available a number of positive ones. When the number of players is equal to two, Mas-Collel (1998) shows that for generic payoffs, there are finitely many pairs of expected Nash equilibrium payoffs. Govindan and McLennan (1998) prove that in the space of zero sum utilities (i.e., those such that $u_1 = -u_2$) and also in the space of common interest utilities (i.e., those such that $u_1 = u_2$) it is the case that generically there are finitely many distributions on Ω induced by Nash equilibria. For any number of players, Govindan and McLennan (2001) also prove that the conjecture holds when there are just two outcomes. Similar results have been proved that put restrictions in θ.[1]

Most of the recent methodology used to tackle the problem relies on semi-algebraic geometry. Semi-algebraic geometry has been proved a powerful tool to analyze the structure of game theoretic equilibrium correspondences. Using this methodology, Blume and Zame (1994) proved that, given an extensive form, for almost all assignments of payoffs to ending nodes, the sets of sequential and perfect equilibrium strategy profiles coincide.[2] Govindan and Wilson (2001) provide direct proofs of the generic finiteness of Nash equilibrium outcomes for normal and extensive form games. Kohlberg and Mertens (1986) demonstrate that the set of Nash equilibria of any game has only a finite number of connected components. The positive results on the generic determinacy of the Nash equilibrium correspondence contained in Govindan and McLennan (1998, 2001) also make extensive use of semi-algebraic geometry.

The next section proves that if a two person game form has a set of outcomes with just three elements, for almost all assignments of payoffs to outcomes, there is a finite set of probability distributions on outcomes induced by Nash equilibria. The result highlights that the counterexample given by Kukushkin et al. (2007) is the minimal one in the sense that it uses four outcomes.

The three outcomes case is of some relevance since, typically, real life two person games have this many outcomes: either one of them wins, or they tie.

[1]As a matter of fact, Govindan and McLennan (2001) show that the conjecture is true for almost all θ's. Also that if at all mixed strategies, for each player i the set of distributions on Ω that agent i can induce by varying her strategy is $(|S_i| - 1)$-dimensional, then for generic utilities there are finitely many equilibria. There also exist positive results for specific classes of games. See for instance, De Sinopoli (2001) and De Sinopoli and Iannantuoni (2005) for voting games, and Park (1997) for sender-receiver cheap-talk games.

[2]This result is stronger than the one established by Kreps and Wilson (1982). Kreps and Wilson prove that for almost all assignments of payoffs to ending nodes, every perfect equilibrium is sequential and almost all sequential equilibrium strategy profiles are perfect equilibrium profiles.

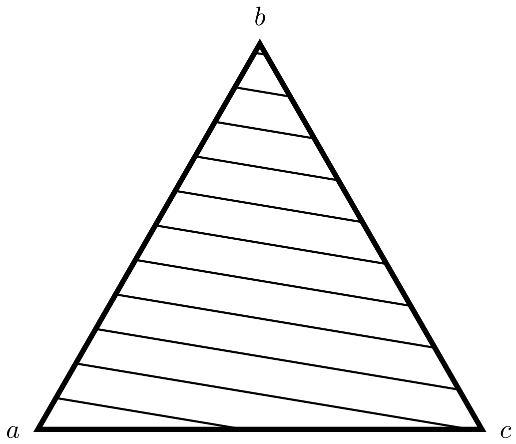

FIGURE 1. Indifference curves of a utility function over lotteries representable by the expected utility form.

2. The Result

Consider two players indexed by $i = 1, 2$, finite set of strategies S_1, S_2, set of outcomes $\Omega = \{a, b, c\}$, and a function $\theta\colon S \to \Delta(\Omega)$. An assignment of payoffs to outcomes is a point in $\left(\mathbb{R}^{\Omega}\right)^N \cong \mathbb{R}^6$.

Proposition. *If the set of outcomes has just three elements, generically, the set of probability distributions on outcomes induced by Nash equilibria is finite.*

Proof. Recall that the set of lotteries over three outcomes coincides with the 2-dimensional simplex. It is represented by means of an equilateral triangle each of whose vertexes represents the degenerate lottery where one outcome is assigned probability one. Preferences over lotteries over three outcomes have straight and parallel indifference curves that lie on this triangle.[3] See Figure 1 for an example.

Consider the following result:

Theorem (Mas-Collel, 1998)**.** *In the class of two person game forms, for generic payoffs, there are finitely many pairs of expected payoffs resulting from Nash equilibria.*

Since the generic finiteness of equilibrium outcome distributions holds for common interest and zero sum games (Govindan and McLennan, 1998) we can restrict attention to the case where indifference curves of players 1 and 2 have different slopes. Given an equilibrium probability distribution on outcomes, the indifference curves of both players that represent the expected utility coming from such a distribution cross at just this one point. Therefore, generic finiteness of pairs of equilibrium payoffs implies generic finiteness of equilibrium distributions. This is illustrated in Figure 2, where, for a given two person game form with three outcomes, only indifference curves over outcomes of both players that represent equilibrium expected utilities are drawn. □

[3]As long as there is not complete indifference among the three outcomes.

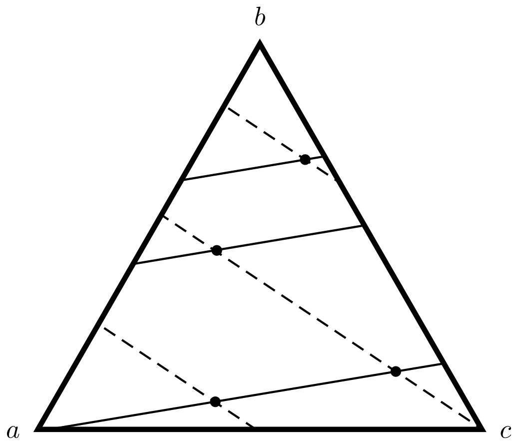

FIGURE 2. Generic finiteness of probability distributions induced by Nash equilibria for two person game forms with three outcomes

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