A Feedback Model for Trend-Chasing Behaviour

Sian Owen*

School of Banking and Finance, The University of New South Wales, Sydney, NSW 2052, Australia

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Abstract

In this paper a model is developed for share price behaviour in a system where there is just one new item of information released to market participants. This determines the initial price of the share and thereafter, the only information that is used in the model is the new share price and the rate of change with which the price alters. This results in a model for trend chasing behaviour in share prices in which investors are influenced solely by these two pieces of information.

Using a closed feedback system with only the share price and the rate of price change as inputs, a model is derived for the share price. Under these circumstances, it is possible for the price to settle to a stable value but, in many cases, the price behaviour is unstable and either rises or falls excessively.

Keywords: Trend chasing, Positive and Negative Feedback.

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* E-mail: Sian.Owen@unsw.edu.au, Telephone: +61 (0)2 9385 4412, Fax: +61 (0)2 9365 6347

1. *Introduction*

Conventional finance theory holds that all investors are rational individuals who are intent on maximising their utility. With rational expectations, investors would only trade when new information is released that alters their perception of the value of the share under consideration. There are, however, some instances in which share prices alter when there is little or no new information released that could be used to justify the changes. One explanation for these price changes is that they are the product of investment decisions made by irrational or uninformed investors who base their investment decisions on observations about the actions of others and on the behaviour of the share price itself.

One such type of investor is a trend-chaser who trades by following alterations in share price. When prices are rising, the trend-chasers will buy and, when they perceive that prices are falling, they will sell. This behaviour, assuming that the volume of their trades is sufficiently large to alter the price, can lead to excessive volatility in the share price and push prices away from their fundamental values. For uninformed traders, there are particularly strong signals in the price of the share and the rate with which it alters. This is reflected in the findings of papers such as Easley, Kiefer and O'Hara (1997) who suggested that uninformed traders are strongly influenced by the recent trades and are more likely to buy (sell) when previous trades have been buys (sells).

In this paper, a feedback model is developed that uses the share price and the rate of change in the price as inputs and the output makes it possible to estimate how the share will behave if investors are basing their decisions solely on previous movements in the share price and have no new information that can be used to accurately value the stock. This represents an extension on the existing research on positive feedback trading by constructing a formal framework for the model, which does not appear in the existing literature. A further extension is introduced by allowing the two inputs in the model to exhibit either positive or negative feedback independently. Using this more formal framework for modelling feedback trading it is possible to infer the overall impact of the inputs on the share price and to determine whether the behaviour of the prices series will be stable or unstable. This paper is constructed as follows: Section 1 introduces the theory of control systems and feedback loops. Section 2 reviews the relevant literature and Section 3 details the derivation of the model. Finally, there is the conclusion.

2. Control Theory and Feedback Loops.

Control systems first appeared in simple mechanical devices[1] and are now widely used in a variety of electro-mechanical systems from domestic to aerospace applications. A control system is developed from four basic components: a reference, a sensor, a comparator and an actuator. The reference data reflects the desired state of the system whilst the sensor determines the actual state of the system at the current time. Once this is known, the comparator evaluates the current state with respect to the reference data and a decision is made about how the behaviour of the system should be altered. This change is then carried out by the actuator. This is fed into the sensor as a new input which determines the revised state of the system. Under this scenario, a closed loop is formed and a feedback system is produced, as illustrated in Figure 1.

[Insert Figure 1 Here]

In many such systems the objective is to control the behaviour of the system, to get as close as possible to the reference and to minimise the changes made at each stage by the actuator. This scenario usually results in a negative feedback loop and is used when a stable system is required. In other cases, there is no intention to control the behaviour of the system as the primary purpose is just to monitor how it behaves. If no attempt is made to minimise the changes made by the actuator then the system can deviate substantially away from the initial input and the reference serves only to help

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[1] The earliest recognised example of a control system was a water clock devised by Ktesibios of Alexandria around 300 B.C. Ktesibios controlled the flow of water in and out of the clock resulting in the first known instance of accurate time keeping. More recently, control systems were used in the regulation of furnace temperatures for incubating chicken eggs (attributed to Cornelius Drebbel, circa 1620) and the centrifugal flyball governor used to regulate the speed of a steam engine (James Watt, 1788). The use of control systems in electrical engineering was spearheaded with the development of feedback amplifiers in the 1920's.

measure the magnitude of the change. This generates instability the system and is usually the result of a positive feedback loop.

It is important to note that it is possible for feedback models to have multiple inputs within a closed loop, as is the case here. In this situation, each of the inputs may have either a positive or negative influence on the system as a whole and the sign of these inputs is independently determined. The overall impact on the system will be determined by the aggregate effect of the inputs.

3. Existing Research

Control theory and feedback loops have been used in papers by Mar and Bakken (1981), Klein (1989) and Edwards (1992) amongst others, which are concerned with such topics as organisational behaviour, individual’s reactions to workplace stimuli or stressful scenarios such as redundancy. In the majority of papers of this sort, the feedback is negative as the system is ultimately expected to reach a stable condition. For example, in the work by Klein (1989) negative feedback loops are used to analyse the reaction of individual employees to motivation in the workplace. The desired outcome is to have each individual on track to meet their pre-determined goals, which enter the feedback loop as the reference term. The input will denote the current direction and intensity of the individual’s performance and the output will determine how far away they are from their goals. As a result, an individual who is responding well to the stimuli will generate a stable (negative) feedback loop in which the output of the system will be close to the reference, resulting in small changes at the actuator. In papers such as these, the feedback structure is precisely defined as it will be here. This formal structure is necessary for the accurate measurement of the behaviour of the participants.

Positive feedback has been used, in an informal setting, in papers examining the behaviour of trend-chasing investors who are making investment decisions based on the actions of other investors rather than on publicly available information. This literature has concentrated predominantly on aspects such as the potential profitability of such a strategy (Jegadeesh and Titman, 2001), determining whether this activity takes place across different types of investors (Mei and Saunders, 1997; Bange, 2000), different markets (Grinblatt and Keloharju, 2000; Koutmos and Siadi, 2001; Watanabe, 2002) and the impact trading of this sort can have on the volatility of the market. Research in the latter field is typified in the paper by De Long, Shleifer, Summers and Waldmann (1990) which analyses the impact of trend-chasing investors on the stability of share prices. In the article by De Long et al, serial correlation in the price series is used as an indicator of the presence of trend chasing investors, which is the approach most commonly used in the existing papers concerning feedback trading. The feedback occurs because potential investors observe the fact that the share price is rising and decide to purchase in the hope that the price will continue to rise. The authors defined positive feedback in this paper as the situation in which the price is trending upwards and away from the original value of the series. This definition, whilst convenient and simple to utilise has some serious shortcomings. It does not allow for negative feedback systems, which can exist even when investors are trend-chasing, nor is there any facility for the precise evaluation of the impact that feedback has on the investor. The De Long et al paper is typical of the existing research into the behaviour of feedback traders. In the literature to date, there does not seem to be any formal attempt to model the feedback system or to place it within a defined framework. The presence of positive feedback is determined, in most cases, by the presence of serial correlation in the series of share prices and it is not necessary to determine how the series will behave or whether it will be stable or unstable in the long run. Negative feedback is not considered relevant in these papers although is it the dominant feature in the behaviour of many share prices which do not move excessively away from the fundamental value.

4. A Model of Share Price Behaviour

In principle, the decision to invest should be related to the expectations that an individual has for the future performance of a particular stock. If the investors believe that the stock will appreciate in value in the future, then this provides them with an incentive to buy, and vice versa. This simple decision process is illustrated in Figure 2.

[Insert Figure 2 Here]

In this figure, N_0 is the share price when the trading process begins. The investor makes a decision about the value of the stock at the point marked Σ (the comparator) and that decision is acted upon at the point M. The ensuing trade, assuming it is sufficiently large, will result in a new value for the share. This is denoted N_t.

If it is assumed that there is absolutely no feedback then the investor cannot observe the outcome N_t. Under these circumstances, the system will almost certainly be stable. This is due to the fact that, in the absence of any new information, and unable to observe the outcome of the last trade, any rational investor would stop after trading once. Naturally, this situation rarely occurs in reality as the outcome is usually visible. Not only will the first investor be able to see the responses generated by their own trading but other investors will also be able to observe both the new share price and the speed with which it has altered. These investors will use this information to make decisions about the value of the share and, thus, determine their own trading strategy. Assuming that no other information is released about the company, then the new price and the speed at which it has changed are the only inputs to the system. This results in the creation of a closed feedback loop, as illustrated in Figure 3, which echoes the simple loop shown in Figure 1.

[Insert Figure 3 Here]

From this system[2], the share price after trading, N_t, is derived as in equation 1.

[2] This figure is derived from standard control theory principles. The output is influenced solely by the initial step input and the feedback values. An example of a step input is switching on a light bulb. The system goes from one state, off, to another, illuminated, in a single step. An example of feedback is adjusting the volume of water coming from a tap. To reduce the flow, a little is subtracted from the original setting using negative feedback to reach an equilibrium situation. Conversely, the volume of water can be raised by increasing the original setting; an application of positive feedback.

$$N_t = M\left(N_0 + N_t\left[k_1 + k_2\frac{d}{dt}\right]\right) \tag{1}$$

where N_0, Σ, M and N_t are defined as before and time is denoted as t where $t = 0,1,2,...,\infty$.

In this equation k_1 and k_2 are coefficients that represent the impact of the new price and the speed with which it alters, respectively. This information is fed back to the investor at Σ, who evaluates the data at this point and makes a decision about their next trade. As a result the trading process repeats itself starting from this point. In this closed system there is no new information available to the investor after the initial input and, as a result, the only influence on the way that traders behave is the interpretation that they place on the behaviour of the share price.

In this model, the system may be stable or it may exhibit instability. In the derivation of the model, it was decided to opt for a positive feedback arrangement at the comparator and then to allow the signs associated with the coefficients k_1 and k_2 to determine whether the feedbacks were positive or negative. The impact that these feedbacks have in aggregate will determine the overall state of the system.

Rearranging this expression gives equation 2, which represents the new price as determined by the feedback system:

$$N_t = \frac{MN_0}{1 - M\left[k_1 + k_2\frac{d}{dt}\right]} \tag{2}$$

The point at which trading in these shares begins can be considered as a step input to the system and, thus, the solution to this equation can be easily calculated using the appropriate Laplace transform[3]. From this simple system the new share price, N_t, is derived as in equation 3.

$$N_t = N_0 . \left(\frac{M}{1 - Mk_1} \right) . \left[1 - e^{\left(\frac{t}{k_2} \left[\frac{1 - Mk_1}{M} \right] \right)} \right] \tag{3}$$

For simplicity, it will be initially assumed that $M = 1$[4], which is the maximum possible value for this term. This means that the investor's decision is processed immediately and without any friction that may be caused by the presence of transaction costs, delays in processing the order to trade or any other impediments. This gives equation 4.

$$N_t = \frac{N_0}{1 - k_1} \left[1 - e^{\left(\frac{t}{k_2} [1 - k_1] \right)} \right] \tag{4}$$

This solution, however, assumes that the share price begins trading at zero at $t = 0$ which is obviously unrealistic and contrary to the observed behaviour of stock exchanges. Exchanges insist on minimum values for share floatations and will usually suspend any share that falls below a pre-determined value making it highly unlikely that a share will ever have a value of zero. To reflect this, the equation will be adjusted to reset the origin and elevate the initial value, N_0, above zero. The share price, N_t, is now represented by equation 5[5].

$$N_t = \frac{N_0}{1 - k_1} \left[1 - k_1 . e^{\left(\frac{t}{k_2} [1 - k_1] \right)} \right] \tag{5}$$

Depending on the size and sign of the coefficients k_1 and k_2, this series will demonstrate different types of behaviour. These values will determine whether the

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[3] The complete solution to this equation can be found in Appendix 1.
[4] The situation that arises when this assumption does not hold is discussed later in this paper.
[5] The complete derivation of this equation can be found in Appendix 2.

system as a whole demonstrates positive or negative feedback and the prices will behave differently in each case. When positive feedback is the dominant feature in the system then there will be a tendency towards unstable behaviour and the share price will either rise or fall dramatically. Negative feedback is usually characterised by stability and the prices will tend towards the value $N_0(1-k_1)^{-1}$, which can be either higher or lower than the initial value N_0 depending on the influence of k_1 and k_2. The six possible combinations of k_1 and k_2 are illustrated in Figure 4[6].

[Insert Figure 4 Here]

As Figure 4 demonstrates, this system exhibits both stable and unstable behaviour and, as both positive and negative feedback can exist in the system, there must be a boundary along which the change from one form of feedback to the other takes place. The boundary can be most easily seen when k_2 is negative, as there is an obvious change in behaviour between Zones 1 and 2. In Zone 1, the system displays positive feedback and, as a result, prices are unstable and continuously rising. Conversely, in Zone 2, the system is stable and prices soon reach an equilibrium value, as the overall feedback here is negative. The boundary between the stable and unstable behaviour clearly occurs along the line where $k_1 = 1$ and an expression for this boundary can be derived from equation 5. By expanding the exponential part of this equation and substituting in the value $k_1 = 1$, the stability boundary can be found. This is given as equation 6[7].

$$N_b = N_0\left[1 - \frac{t}{k_2}\right] \tag{6}$$

where N_b refers to the value of the series N_t at the boundary.

[6] A complete explanation of the behaviour of this series in the six areas detailed in Figure 4 can be found in Appendix 3.

[7] The complete derivation of this equation and equation 7 can be found in Appendix 4

At the boundary, when $k_1 = 1$, it is very difficult to calculate k_2 so the boundary can also be expressed as in equation 7.

$$N_b = N_0 - \frac{t(N_0 - N_1)}{t_1} \tag{7}$$

The stability boundary at $k_1 = 1$ also holds in the regions when k_2 is positive. Here the boundary lies between Zones 4 and 5, although the change in behaviour is less marked in this region. Combining the stability boundary with the six zones illustrated in Figure 4 makes it possible to determine the behaviour of the series, N_t, over time. This is illustrated in Figure 5.

[Insert Figure 5 Here]

The values of k_1 and k_2 will, of course, be unique to each company but it is possible to calculate these coefficients in every case. The calculation relies on the selection of two time points, which will be denoted t_1 and t_2. These points should be selected so that $t_2 > t_1$ and the associated values of the price series, denoted N_1 and N_2, must also be known. The values of k_1 and k_2 can then be calculated using equations 8 and 9[8].

$$k_1 = \frac{N_1 - N_0}{N_1 - N_0 \xi} \tag{8}$$

$$k_2 = \frac{t_1[1 - k_1]}{\ln \xi} \tag{9}$$

where ξ is a positive root of the polynomial given here as equation 10 and $x = e^{\frac{t_1}{k_2}(1-k_1)}$.

[8] The complete derivation for equations 8, 9, 10 and 11 can be found in Appendix 5.

There is a second positive root to this equation which is always equal to one, but the solution of interest here is the root ξ[9].

$$x^{\frac{t_2}{t_1}} - \left[\frac{N_2 - N_0}{N_1 - N_0}\right]x + \left[\frac{N_2 - N_0}{N_1 - N_0}\right] - 1 = 0 \tag{10}$$

An alternative specification for k_2 can also be derived and that is given as equation 11.

$$k_2 = \frac{t_1[1 - k_1]}{\ln\left[\frac{N_0 + N_1(k_1 - 1)}{N_0 k_1}\right]} \tag{11}$$

It is apparent from these equations that k_1 is a dimensionless number whilst k_2 has the same dimensions at those chosen for time and is a function of k_1. The fact that k_2 contains a time element does not make the unit of time chosen important, as k_2 simply adjusts its magnitude accordingly; the value of t/k_2 is unaffected by the units chosen. This does not mean that the frequency of the observations is unimportant, but the term in which time is expressed is irrelevant.

Once the values of k_1 and k_2 are known it is possible to estimate the behaviour of a share price series, once the initial input has been made to the system. The values of the coefficients k_1 and k_2 will determine which of the six possible zones the series will occupy and whether the behaviour of the series will be stable or unstable.

In the work presented here thus far, it has been assumed that $M = 1$ which simplifies the derivation of the model and the analysis of its associated features. If this assumption is relaxed, there is very little change. The series of share prices, N_t can

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[9] An explanation of the proposed method of estimation for these coefficients and a proof of the fact that there can only be two positive roots to the polynomial can be found in Appendix 5.

be expressed as in equation 12[10], where κ_1 is the coefficient on the new share price and κ_2 is the coefficient associated with the rate of change of the share price[11].

$$N_t = \frac{N_0 M}{(1 - M\kappa_1)}\left[1 - \left(1 + \kappa_1 - \frac{1}{M}\right).e^{\frac{t}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right)}\right] \tag{12}$$

As before, the values of the coefficients can be calculated and the equations for these terms are given as equations 13 and 14.

$$\kappa_1 = \frac{N_1 - N_0[M(1-\psi)+\psi]}{M(N_1 - N_0\psi)} \tag{13}$$

$$\kappa_2 = \frac{t_1}{\ln\psi}\left(\frac{1 - M\kappa_1}{M}\right) \tag{14}$$

As before, it is necessary to find the positive root, ψ, of the polynomial below to determine the values of these equations.

$$x^{\frac{t_2}{t_1}} - \left[\frac{N_2 - N_0}{N_1 - N_0}\right]x + \left[\frac{N_2 - N_0}{N_1 - N_0}\right] - 1 = 0 \tag{15}$$

where $x = e^{\frac{t_1}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right)}$

As in the previous case, it is possible to find a second expression for the coefficient associated with the rate of change in the share price and here this is given as equation 16.

[10] The complete derivation for equations 12, 13, 15, 16 and 17 can be found in Appendix 6.

[11] κ_1 and κ_2 are equivalent to the coefficients k_1 and k_2 in the earlier work. The change in notation is used only to differentiate between the two cases

$$\kappa_2 = \frac{t_1}{\ln\left[\frac{N_1(1-\kappa_1 M)-N_0 M}{N_0(1-M-\kappa_1 M)}\right]}.\left(\frac{1-M\kappa_1}{M}\right) \tag{16}$$

The case in which $M \neq 1$ does not introduce many differences. The behaviour of the price series can still be determined in terms of the same six zones as before and the stability boundary is unchanged by this alteration.

5. *Conclusion*

The model derived here uses feedback principles to determine a model for trend-chasing behaviour. Allowing two distinct elements of feedback within the model and allowing these feedbacks to take different signs when appropriate makes it possible to conduct a much more detailed analysis of the various different ways that the series can behave. This model represents a marked extension of previous research in which the possibility of a stable price series was ignored, and the inputs were not clearly defined.

The model is established using three points in the time series, the initial value N_0 at $t = 0$ and then N_1 and N_2 at times t_1 and t_2. This immediately allows the characteristic polynomial equation to be derived and the coefficients, k_1 and k_2, to be calculated. This, then, completes the descriptive equation for the price series, which is

$$N_t = \frac{N_0}{1-k_1}\left[1-k_1.e^{\left(\frac{t}{k_2}[1-k_1]\right)}\right]$$

and this allows the series to be fully and accurately represented.

The applications of this model extend not just to trend-chasing investors but also to any situation in which the market participants are investing without reference to the available information about the firm. For example, many company share prices jump sharply on the first day of trading after an IPO and this model could be used to analyse the behaviour of the price series in this situation, either over very short time scales (hours or days) or over longer periods such as weeks or months. The inclusion of a stability boundary in the equation also makes it possible to assess the risk being taken by investors and developing this feature is another potential area for future research which will be examined in subsequent papers.

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Appendix 1

The Laplace transform is a convenient method for solving a differential equation. The process involves transforming the equation from the time domain, denoted with the subscript t, into the Laplace domain, denoted s. Once in the Laplace domain the equation can be manipulated and easily solved. Finally, the resulting output is returned to the time domain to solve the equation.

This process involves multiplying each term in the equation by e^{-st} and then integrating each term with respect to time from zero to infinity. The result of this integration is the Laplace transform.

Thus, the Laplace transform of some term f_t, which is a function of time, can be expressed in equation A1 and is denoted F_s where

$$F_s = \int_0^\infty f_t . e^{-st} dt \qquad \text{(A1.1)}$$

There are many possible Laplace transforms but only four will be discussed here, as they are the only ones used in this paper.

1. The Step Input u_t

Here the step function is equivalent to setting $f_t = 1$ for $t > 0$. The corresponding Laplace for $t > 0$ is:

$$F_s = \int_0^\infty 1.e^{-st} dt = -\frac{1}{s}\left[e^{-st}\right]_0^\infty = \frac{1}{s} \qquad \text{(A1.2)}$$

2. An exponential function of the form $f_t = e^{-at}$

The Laplace transform in this case is:

$$F_s = \int_0^\infty e^{-at}.e^{-st}dt = \int_0^\infty e^{-(a+s)t}dt = -\frac{1}{s+a}\left[e^{-(s+a)t}\right]_0^\infty = -\frac{1}{s+a}\left[0-1\right] = \frac{1}{s+a} \qquad \text{(A1.3)}$$

3. A function of the form $f_t = 1 - e^{-at}$

The Laplace transform in this case is

$$F_s = \int_0^\infty \left(1-e^{-at}\right)e^{-st}dt = \int_0^\infty \left(e^{-st} - e^{-st}e^{-at}\right)dt = -\frac{1}{s}\left[e^{-st}\right]_0^\infty + \frac{1}{s+a}\left[e^{-(a+s)t}\right]_0^\infty$$

$$F_s = \frac{1}{s} - \frac{1}{(s+a)} = \frac{a}{s(s+a)} \qquad \text{(A1.4)}$$

Before solving equation 2, some simple notation needs to be introduced. Here the relationship $\frac{d}{dt} f_t$ is equivalent to $sF_s - f_0$, where f_0 is the value of the function at $t = 0$ which, in this case, is zero. Thus $\frac{d}{dt} f_t \rightarrow sF_s$

It is now possible to find the solution to equation 2. Recalling equation 2, denoted A1.5, here, as derived from the feedback system,

$$N_t = \frac{MN_0}{1 - M\left[k_1 + k_2 \frac{dN}{dt}\right]} \qquad \text{(A1.5)}$$

This is transformed into the Laplace domain as in equation A1.6. The step input of N_0 is represented in the Laplace domain by the term $\frac{N_0}{s}$ and the differential term $\frac{dN}{dt}$ is replaced with the operator s.

$$N_S = \frac{N_0}{s} \cdot \frac{M}{1 - M[k_1 + k_2 s]} \tag{A1.6}$$

Rearranging this equation into the standard forms given in equations A1.2 and A1.4, requires a few lines of algebra:

$$N_S = \frac{N_0}{s} \cdot \frac{M}{1 - M[k_1 + k_2 s]} = N_0 M \cdot \frac{1}{s(1 - Mk_1 - Mk_2 s)} = N_0 M \cdot \frac{1}{s([1 - Mk_1] - Mk_2 s)}$$

$$N_S = N_0 M \cdot \frac{-1}{s(Mk_2 s - [1 - Mk_1])} = \frac{N_0 M}{Mk_2}(-1)\frac{1}{s\left(s - \left[\frac{1 - Mk_1}{Mk_2}\right]\right)}$$

$$N_S = \frac{N_0}{k_2}(-1)\frac{1}{s\left(s + \left[\frac{Mk_1 - 1}{Mk_2}\right]\right)} = \frac{N_0}{k_2}(-1)\frac{Mk_2}{(Mk_1 - 1)}\frac{\left[\frac{Mk_1 - 1}{Mk_2}\right]}{s\left(s + \left[\frac{Mk_1 - 1}{Mk_2}\right]\right)}$$

This is now in the form $c.\frac{a}{s(s+a)}$, in which c is a constant and $\frac{a}{s(s+a)}$ is the standard Laplace transformation for the time function of the constant, c, multiplied by $1 - e^{-at}$. This is given as equation A1.7

$$N_S = N_0 \frac{M}{(1 - Mk_1)} \cdot \frac{\left[\frac{Mk_1 - 1}{Mk_2}\right]}{s\left(s + \left[\frac{Mk_1 - 1}{Mk_2}\right]\right)} \tag{A1.7}$$

Completing the Laplace transform and returning to the time domain gives the solution as equation A1.8, which is also equation 3 in the main text.

$$N_t = N_0\left(\frac{M}{1-Mk_1}\right)\left[1-e^{\left(\frac{t}{k_2}\left[\frac{1-Mk_1}{M}\right]\right)}\right] \qquad \text{(A1.8)}$$

Appendix 2

Equation 4, reprised here as equation A2.1, represents the solution to the first order differential equation derived in the main body of the paper. This is an exponential growth curve.

$$N_t = \frac{N_0}{1-k_1}\left[1-e^{\left(\frac{t}{k_2}[1-k_1]\right)}\right] \tag{A2.1}$$

This solution can be represented as a curve of the form given in equation A2.2

$$y = A_m\left(1-e^{-pt}\right) \tag{A2.2}$$

where A_m is the maximum value towards which the curve tends asymptotically. This curve takes the standard shape of an exponential growth curve, as given in figure A2.1.

[Insert Figure A2.1 Here]

It is easy to alter the point at which this curve passes the vertical axis without altering the constants or changing the curve in any way. This is illustrated in Figure A2.2 in which the origin is moved to the right to lie under the point F on the curve. The amplitude of the curve is reduced from A_m to $A_m - F$ whilst the constant value F is added to maintain the original values. Thus the equation of the curve takes the form of equation A2.3.

[Insert Figure A2.2 Here]

$$y = F + \left(A_m - F\right).\left(1-e^{-pt}\right) \tag{A2.3}$$

With some simple re-arrangement this gives equation A2.4.

$$y = A_m\left(1 - e^{-pt}\right) + F.e^{-pt} \tag{A2.4}$$

Expressing equation 4 in this manner gives equation A2.5

$$N_t = \frac{N_0}{1-k_1}.\left[1 - e^{\left(\frac{t}{k_2}[1-k_1]\right)}\right] + N_0.e^{\left(\frac{t}{k_2}[1-k_1]\right)} \tag{A2.5}$$

This equation is then simplified as follows:

$$N_t = \frac{N_0}{1-k_1}\left[1 - e^{\left(\frac{t}{k_2}[1-k_1]\right)}\right] + N_0.e^{\left(\frac{t}{k_2}[1-k_1]\right)} = \frac{N_0\left[1 - e^{\left(\frac{t}{k_2}[1-k_1]\right)}\right] + N_0.e^{\left(\frac{t}{k_2}[1-k_1]\right)}.(1-k_1)}{1-k_1}$$

$$N_t = \frac{N_0 - N_0.e^{\left(\frac{t}{k_2}[1-k_1]\right)} + N_0.e^{\left(\frac{t}{k_2}[1-k_1]\right)} - N_0.e^{\left(\frac{t}{k_2}[1-k_1]\right)}k_1}{1-k_1}$$

$$N_t = \frac{N_0}{1-k_1}\left[1 - k_1.e^{\left(\frac{t}{k_2}[1-k_1]\right)}\right] \tag{A2.6}$$

This gives equation A2.6, which is used in the main text as equation 5. This represents the equation for N_t allowing the shares to start trading at the initial price, N_0, instead of zero.

Appendix 3

The series, N_t, is represented in the main text by equation 5, repeated here as equation A3.1.

$$N_t = \frac{N_0}{1-k_1}\left[1-k_1.e^{\left(\frac{t}{k_2}[1-k_1]\right)}\right] \tag{A3.1}$$

The behaviour of N_t in each of the six zones can be explained by considering the way that equation A3.1 behaves. If that expression is differentiated with respect to time, *t*, the result is equation A3.2.

$$\frac{dN_t}{dt} = N_0.\left(\frac{-k_1}{k_2}\right).e^{\left(\frac{t}{k_2}[1-k_1]\right)} \tag{A3.2}$$

It is now possible to examine the behaviour of the series, N_t in each of the six zones marked in Figure 4.

<u>Zone 1:</u> $k_1 > 1$ and $k_2 < 0$

In this region, the differential of the curve with respect to t, $\frac{dN_t}{dt}$, is positive and tends towards $+\infty$ as t grows. The price series, N_t, remains positive throughout, rising above the initial value, N_0, and the behaviour of this series in this zone can be typified as in Figure A3.1.

[Insert Figure A3.1 Here]

Zone 2: $0 < k_1 < 1$ and $k_2 < 0$.

In this region, the differential of the curve, $\frac{dN_t}{dt}$, is positive and tends towards $+0$ as t grows. The values of the series, N_t, remain positive but the series settles to a stable value which is larger than N_0. The behaviour of the series is typified in Figure A.3.2.

[Insert Figure A3.2 Here]

Zone 3: $k_1 < 0$ and $k_2 < 0$

In this part of the system, negative feedback is clearly prevalent and the series, obviously, will be stable. The differential with respect to time, $\frac{dN_t}{dt}$, is negative and tends towards -0 as t increases. The values of N_t remain positive and the series settles at a stable value given by $\frac{N_0}{1-k_1}$ which is below the initial price N_0. The behaviour of the series in this zone is illustrated in Figure A3.3.

[Insert Figure A3.3 Here]

Zone 4: $k_1 > 1$ and $k_2 > 0$

In this region, $\frac{dN_t}{dt}$ is negative and tends towards -0 as t grows. The series will eventually settle at a value equal to $\frac{N_0}{1-k_1}$, which will be negative in this zone. For small values of t, the series will be positive, and, using equation A3.1, it is possible to determine the range of values for which this is true.

From equation A3.1, it is clear that the series N_t can only be positive when the inequality denoted A3.3 holds.

$$k_1.e^{\left(\frac{t}{k_2}[1-k_1]\right)} > 1 \tag{A3.3}$$

Using this expression it is possible to find the range of values of t for which the series will be positive. Rearranging A3.3 gives:

$$k_1.e^{\left(\frac{t}{k_2}[1-k_1]\right)} > 1 \quad \Rightarrow \quad e^{\left(\frac{t}{k_2}[1-k_1]\right)} > \frac{1}{k_1} \quad \Rightarrow \quad \frac{t}{k_2}[1-k_1] > \ln\left(\frac{1}{k_1}\right)$$

In this zone, however, $(1-k_1)$ is negative and dividing the expression throughout by this term therefore reverses the inequality which becomes:

$$\frac{t}{k_2} < \frac{1}{(1-k_1)}.\ln\left(\frac{1}{k_1}\right) \quad \Rightarrow \quad t < \frac{k_2}{(1-k_1)}.\ln\left(\frac{1}{k_1}\right)$$

Thus, for N_t to be positive, t must lie in the range given by equation A3.4.

$$t < \frac{k_2}{(1-k_1)}.\ln\left(\frac{1}{k_1}\right) \tag{A3.4}$$

The series changes from positive to negative when $N_t = 0$ which occurs when t takes the value given by equation A3.5

$$t = \frac{k_2}{(1-k_1)}.\ln\left(\frac{1}{k_1}\right) \tag{A3.5}$$

The behaviour of the series, N_t, in this zone is typified in Figure A3.4.

[Insert Figure A3.4 Here]

In reality, of course, it is not possible to have a negative share price but the series will tend towards zero until it is suspended by the Stock Exchange. However, the description of the behaviour of the share price as given here will be accurate until the point at which suspension occurs.

<u>Zone 5:</u> $0 < k_1 < 1$ and $k_2 > 0$

In this zone, the differential of N_t with respect to t is negative and will tend towards $-\infty$ as t grows. As in Zone 4, the series is positive for small values of t and these values can be calculated.

Again, equation A3.1 is the starting point and this time N_t can only be positive when the inequality denoted A3.6 holds.

$$k_1.e^{\left(\frac{t}{k_2}[1-k_1]\right)} < 1 \tag{A3.6}$$

Using the permissible range of values for k_1 in this zone, it is possible to determine the range of values of t for which this inequality holds.

$$k_1.e^{\left(\frac{t}{k_2}[1-k_1]\right)} < 1 \quad \Rightarrow \quad e^{\left(\frac{t}{k_2}[1-k_1]\right)} < \frac{1}{k_1} \quad \Rightarrow \quad \frac{t}{k_2}[1-k_1] < \ln\left(\frac{1}{k_1}\right)$$

$$\frac{t}{k_2} < \frac{1}{1-k_1}.\ln\left(\frac{1}{k_1}\right) \quad \Rightarrow \quad t < \frac{k_2}{1-k_1}.\ln\left(\frac{1}{k_1}\right)$$

Thus, the series will be positive in this zone for the range of values given by the inequality, denoted equation A3.7

$$t < \frac{k_2}{1-k_1}.\ln\left(\frac{1}{k_1}\right) \tag{A3.7}$$

The behaviour of the series in this zone is illustrated in Figure A3.5 and the same caveat concerning negative share prices applies here as it does in Zone 4.

[Insert Figure A3.5 Here]

Zone 6: $k_1 < 0$ and $k_2 > 0$

In this zone, the differential with the respect to time, $\frac{dN_t}{dt}$, is positive and will tend towards $+\infty$ as t increases. Throughout this zone, the series, N_t, remains positive. The gradient here is much steeper than in Zone 1 as can be demonstrated by calculating $|1-k_1|$ in both zones. In Zone 6, where $k_1 < 0$, the value of the modulus will be greater than in Zone 1, where $k_1 > 1$, provided that the same value of k_1 is used in both cases.

The behaviour of the series N_t in this area of Figure 4 is illustrated in Figure A3.6

[Insert Figure A3.6 Here]

Appendix 4

The behaviour of the series, N_t, is represented by equation 5, which is represented here as equation A4.1.

$$N_t = \frac{N_0}{1-k_1}\left[1-k_1.e^{\left(\frac{t}{k_2}[1-k_1]\right)}\right] \qquad \text{(A4.1)}$$

The value of this equation when $k_1 = 1$ is difficult to calculate with the equation in this form, so the exponential part of the equation can be expanded using the standard series for e^x given in equation A4.2.

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \qquad \text{(A4.2)}$$

Using this expansion, the exponential part of equation A4.1 takes the form of equation A4.3.

$$e^{\left(\frac{t}{k_2}[1-k_1]\right)} \approx 1 + t\left(\frac{1-k_1}{k_2}\right) + \frac{t^2}{2!}\left(\frac{1-k_1}{k_2}\right)^2 + \frac{t^3}{3!}\left(\frac{1-k_1}{k_2}\right)^3 + \frac{t^4}{4!}\left(\frac{1-k_1}{k_2}\right)^4 + \ldots \qquad \text{(A4.3)}$$

Further expanding equation A4.1 using the exponential expansion and some simple algebra gives an expression for the entire equation.

$$k_1.e^{\left(\frac{t}{k_2}[1-k_1]\right)} \approx k_1 + k_1 t\left(\frac{1-k_1}{k_2}\right) + \frac{k_1 t^2}{2!}\left(\frac{1-k_1}{k_2}\right)^2 + \frac{k_1 t^3}{3!}\left(\frac{1-k_1}{k_2}\right)^3 + \frac{k_1 t^4}{4!}\left(\frac{1-k_1}{k_2}\right)^4 + \ldots$$

$$1-k_1.e^{\left(\frac{t}{k_2}[1-k_1]\right)} \approx 1 - k_1 - k_1 t\left(\frac{1-k_1}{k_2}\right) - \frac{k_1 t^2}{2!}\left(\frac{1-k_1}{k_2}\right)^2 - \frac{k_1 t^3}{3!}\left(\frac{1-k_1}{k_2}\right)^3 - \frac{k_1 t^4}{4!}\left(\frac{1-k_1}{k_2}\right)^4 - \ldots$$

$$\frac{N_0}{1-k_1}\left(1-k_1.e^{\left(\frac{t}{k_2}[1-k_1]\right)}\right)\approx\frac{N_0}{1-k_1}\left(1-k_1-k_1t\left(\frac{1-k_1}{k_2}\right)-\frac{k_1t^2}{2!}\left(\frac{1-k_1}{k_2}\right)^2-\frac{k_1t^3}{3!}\left(\frac{1-k_1}{k_2}\right)^3-\ldots\right)$$

$$\frac{N_0}{1-k_1}\left(1-k_1.e^{\left(\frac{t}{k_2}[1-k_1]\right)}\right)\approx N_0\left(1-\frac{k_1t}{k_2}-\frac{k_1t^2}{2!}\left(\frac{1-k_1}{k_2^2}\right)-\frac{k_1t^3}{3!k_2^3}(1-k_1)^2-\ldots\right)$$

At the boundary, $k_1=1$, this expression becomes equation A4.4

$$N_b=N_0\left(1-\frac{t}{k_2}\right) \tag{A4.4}$$

This boundary is a straight line and equation A4.4 conforms to the standard form, $y=mx+c$ where the intercept is N_0 and the gradient is $\frac{-N_0}{k_2}$.

In addition to the relationship represented in equation A4.4, the boundary must pass through N_0 when $t=0$ and through N_1 when $t=t_1$. This means that the gradient must also be equal to $\frac{N_1-N_0}{t_1}$

Equating the two expressions for the gradient makes it possible to derive an equation for k_2, denoted equation A4.5

$$\frac{-N_0}{k_2}=\frac{N_1-N_0}{t_1}$$

$$-N_0t_1=k_2(N_1-N_0)$$

$$N_0t_1=k_2(N_0-N_1)$$

$$k_2 = \frac{N_0 t_1}{(N_0 - N_1)} \text{ when } k_1 = 1 \tag{A4.5}$$

Substituting this term into equation A4.4 gives the second equation for the stability boundary when $k_1 = 1$, equation A4.6.

$$N_b = N_0 \cdot \left(1 - \frac{t}{\left[\frac{N_0 t_1}{(N_0 - N_1)} \right]} \right) = N_0 \cdot \left(1 - \frac{t(N_0 - N_1)}{N_0 t_1} \right)$$

$$N_b = N_0 - N_0 \frac{t(N_0 - N_1)}{N_0 t_1} = N_0 - \frac{(N_0 - N_1)t}{t_1} \tag{A4.6}$$

Appendix 5

The calculation of the values of k_1 and k_2 requires two values of t and the equivalent values from the price series, N_t. The relationship between the times and the prices are illustrated in Figure A5.1.

[Insert Figure A5.1 Here]

Using these terms and the initial value of the price series, N_0 it is possible to derive expressions for the coefficients. The values of k_1 and k_2 can be calculated using the equation for N_t, which is given as equation 5 in the main text. Using this equation and the values in Figure A5.1 it is possible to create expressions for N_t when $t = t_1$ and $t = t_2$, with the condition that $t_2 > t_1$. These expressions are denoted equations A5.1 and A5.2 respectively.

When $t = t_1$, $N_t = N_1$

$$N_1 = \frac{N_0}{1-k_1}\left[1-k_1.e^{\left(\frac{t_1}{k_2}[1-k_1]\right)}\right] \quad \text{(A5.1)}$$

When $t = t_2$, $N_t = N_2$

$$N_2 = \frac{N_0}{1-k_1}\left[1-k_1.e^{\left(\frac{t_2}{k_2}[1-k_1]\right)}\right] \quad \text{(A5.2)}$$

Using equation A5.1, an expression can be found for the difference between N_1 and N_0. This is as follows:

$$N_1 - N_0 = \frac{N_0}{1-k_1}\left[1 - k_1.e^{\left(\frac{t_1}{k_2}[1-k_1]\right)}\right] - N_0 = \frac{N_0 - N_0 k_1.e^{\left(\frac{t_1}{k_2}[1-k_1]\right)} - N_0(1-k_1)}{1-k_1}$$

$$N_1 - N_0 = \frac{N_0 k_1\left(1 - e^{\left(\frac{t_1}{k_2}[1-k_1]\right)}\right)}{1-k_1}$$

This expression can be simplified by substituting in the term $x = e^{\frac{t_1}{k_2}[1-k_1]}$. This gives equation A5.3. This substitution means that the value of the term x, which will be evaluated later, must always be a positive number.

$$N_1 - N_0 = \frac{N_0 k_1 (1-x)}{1-k_1} \tag{A5.3}$$

Repeating this process for equation A5.2 gives a similar expression, which will be denoted A5.4.

$$N_2 - N_0 = \frac{N_0}{1-k_1}\left[1 - k_1.e^{\left(\frac{t_2}{k_2}[1-k_1]\right)}\right] - N_0 = \frac{N_0 - N_0 k_1.e^{\left(\frac{t_2}{k_2}[1-k_1]\right)} - N_0(1-k_1)}{1-k_1}$$

$$N_2 - N_0 = \frac{N_0 k_1\left(1 - e^{\left(\frac{t_2}{k_2}[1-k_1]\right)}\right)}{1-k_1} \tag{A5.4}$$

The exponential expression in this equation can be represented as $e^{\left(\frac{t_1}{k_2}[1-k_1]\frac{t_2}{t_1}\right)} = x^{\frac{t_2}{t_1}}$ allowing the equation A5.4 to be re-expressed as equation A5.5

$$N_2 - N_0 = \frac{N_0 k_1 \left(1 - x^{\frac{t_2}{t_1}}\right)}{1 - k_1} \qquad \text{(A5.5)}$$

The ratio of equations A5.3 and A5.5 can then be found to be equation A5.6

$$\frac{N_2 - N_0}{N_1 - N_0} = \frac{\left[\dfrac{N_0 k_1 \left(1 - x^{\frac{t_2}{t_1}}\right)}{1 - k_1}\right]}{\left[\dfrac{N_0 k_1 (1 - x)}{1 - k_1}\right]} = \frac{1 - x^{\frac{t_2}{t_1}}}{1 - x} \qquad \text{(A5.6)}$$

Rearranging this expression gives equation A5.7 as below

$$N_2 - N_0 = \left[\frac{1 - x^{\frac{t_2}{t_1}}}{1 - x}\right] . (N_1 - N_0) \qquad \text{(A5.7)}$$

Equating equations A5.5 and A5.7 gives the following expression (A5.8):

$$N_2 - N_0 = \frac{N_0 k_1 \left(1 - x^{\frac{t_2}{t_1}}\right)}{1 - k_1} = \left[\frac{1 - x^{\frac{t_2}{t_1}}}{1 - x}\right] . (N_1 - N_0) \qquad \text{(A5.8)}$$

which can be re-arranged as follows to give an equation for the coefficient k_1, denoted equation A5.9.

$$\frac{N_0 k_1 \left(1 - x^{\frac{t_2}{t_1}}\right)}{1 - k_1} = \frac{[N_1 - N_0] \left(1 - x^{\frac{t_2}{t_1}}\right)}{1 - x}$$

$$\frac{N_0 k_1}{1-k_1} = \left[\frac{N_1 - N_0}{1-x}\right]$$

$$N_0 k_1 (1-x) = [N_1 - N_0](1-k_1)$$

$$N_0 k_1 - N_0 k_1 x = N_1 - k_1 N_1 - N_0 + k_1 N_0$$

$$k_1 (N_0 - N_0 x + N_1 - N_0) = N_1 - N_0$$

$$k_1 = \frac{N_1 - N_0}{N_1 - N_0 x} \qquad \text{(A5.9)}$$

Recalling that $x = e^{\frac{t_1}{k_2}[1-k_1]}$, an expression for k_2 can be found, denoted here as equation A5.11.

$$x = e^{\frac{t_1}{k_2}[1-k_1]}$$

$$\ln x = \frac{t_1}{k_2}[1-k_1]$$

$$k_2 = \frac{t_1[1-k_1]}{\ln x} \qquad \text{(A5.10)}$$

It is also possible to find a second expression for k_2, in which this coefficient is expressed entirely in terms of the price series, time and the coefficient k_1. Deriving this equation first requires a rearrangement of equation A5.9 to derive an expression for x, equation A5.10, as follows:

$$k_1 (N_1 - N_0 x) = N_1 - N_0$$

$$N_1k_1 - N_0k_1x = N_1 - N_0$$

$$x = \frac{N_0 - N_1 + N_1k_1}{N_0k_1} = \frac{N_0 + N_1(k_1 - 1)}{N_0k_1} \qquad \text{(A5.11)}$$

Substituting equation A5.11 into equation A5.10 gives a second expression for k_2, given here as equation A5.12.

$$k_2 = \frac{t_1[1 - k_1]}{\ln\left[\frac{N_0 + N_1(k_1 - 1)}{N_0k_1}\right]} \qquad \text{(A5.12)}$$

As equation A5.12 demonstrates, it is possible to construct an expression for k_2 that does not involve the term x. It is not possible, however to evaluate k_1 without evaluating x. This necessitates an alternative approach to equation A5.6, which is rearranged as follows, resulting in equation A5.13.

$$\frac{N_2 - N_0}{N_1 - N_0} = \frac{1 - x^{\frac{t_2}{t_1}}}{1 - x}$$

$$\left[\frac{N_2 - N_0}{N_1 - N_0}\right].(1 - x) = 1 - x^{\frac{t_2}{t_1}}$$

$$x^{\frac{t_2}{t_1}} - x\left[\frac{N_2 - N_0}{N_1 - N_0}\right] + \left[\frac{N_2 - N_0}{N_1 - N_0}\right] - 1 = 0 \qquad \text{(A5.13)}$$

This polynomial needs to be solved to find the roots of the equation. It is clear that one root will always be $x = 1$ but there will be other roots as well. The relationship between the terms t_1 and t_2 will determine the number of roots that equation A5.13 can have. For example, if the time points are chosen such that $t_2 = 2t_1$ then equation

A5.13 is a quadratic equation[12] and will have just two roots. For higher order expressions, however, there could be many roots to this equation but the only solutions that are of interest here are those where x is positive, as a negative solution would violate both the initial definition of this term and equation A5.10.[13]

It can be demonstrated that there are only ever two positive roots to this equation by examining the polynomial itself. As Geary, Lowry and Hayden (1955) noted, "if an equation $f(x)=0$ can be written in the form $f_1(x)=f_2(x)$ its real roots are the abscissae of the points of intersection of the graphs of $f_1(x)$) and $f_2(x)$, because the graphs have the same ordinate at a point of intersection." (Geary, Lowry and Hayden, 1955, page 303) Recalling equation A5.13 it is possible to rearrange that equation as:

$$x^{\frac{t_2}{t_1}} = x\left[\frac{N_2 - N_0}{N_1 - N_0}\right] - \left[\frac{N_2 - N_0}{N_1 - N_0}\right] + 1$$

which gives two functions, $y = f_1(x) = x^{\frac{t_2}{t_1}}$

and $y = f_2(x) = x\left[\frac{N_2 - N_0}{N_1 - N_0}\right] - \left[\frac{N_2 - N_0}{N_1 - N_0}\right] + 1$

The graphs of these two functions can be seen in Figures A5.2 and A5.3.

[Insert Figure A5.2 Here]

[Insert Figure A5.3 Here]

12 $x^2 - x\left[\frac{N_2 - N_0}{N_1 - N_0}\right] + \left[\frac{N_2 - N_0}{N_1 - N_0}\right] - 1 = 0$ from which the root is easily extracted as $\left[\frac{N_2 - N_0}{N_1 - N_0}\right]$

13 It is, of course, impossible to have a negative value for an exponential or to take the logarithm of a negative number.

The two functions both pass through the point $(1,1)$ so if Figure A5.3 is overlaid on Figure A5.2 as in Figure A5.4 it is clear that there are only two possible points of intersection between the two functions $f_1(x)$ and $f_2(x)$, limiting the number of positive roots to two. As has already been noted, it is clear that one solution will always be $x = 1$ but there will be another solution which will be denoted ξ.

[Insert Figure A5.4 Here]

There are three cases to examine here. The differentials with respect to x of the two functions are used in each of the following cases, and are given as equations A5.14 and A5.15, below.

$$\frac{df_1(x)}{dx} = \frac{t_2}{t_1} x^{\frac{t_2}{t_1}-1} \tag{A5.14}$$

$$\frac{df_2(x)}{dx} = \left[\frac{N_2 - N_0}{N_1 - N_0}\right] \tag{A5.15}$$

<u>Case 1.</u> The gradient of $f_1(x) = x^{\frac{t_2}{t_1}}$ is greater than the gradient of $f_2(x) = x\left[\frac{N_2 - N_0}{N_1 - N_0}\right] - \left[\frac{N_2 - N_0}{N_1 - N_0}\right] + 1$ at the point $x = 1$.

The minimum possible value of the root ξ occurs at the point where the function $f_2(x) = x\left[\frac{N_2 - N_0}{N_1 - N_0}\right] - \left[\frac{N_2 - N_0}{N_1 - N_0}\right] + 1$ crosses the horizontal axis; when $y = 0$. This value of the minimum possible root is, therefore, $\frac{N_2 - N_1}{N_2 - N_0}$.

If the gradient of $f_1(x)$ is greater than the gradient of $f_2(x)$ at the point $x = 1$, then the relationships between the differentials of these two functions can be expressed as:

$$\frac{t_2}{t_1} x^{\frac{t_2}{t_1}-1} > \left[\frac{N_2 - N_0}{N_1 - N_0}\right]$$

and, since $x = 1$, this expression is equivalent to $\frac{t_2}{t_1} > \left[\frac{N_2 - N_0}{N_1 - N_0}\right]$

From this, it is possible to infer that the second intersection between the two functions must lie somewhere between the point $x = 1$ and the minimum possible value of the root. Thus the root ξ must lie in the range defined as:

$$\left[\frac{N_2 - N_1}{N_2 - N_0}\right] < \xi < 1$$

This case is illustrated in Figure A5.5

[Insert Figure A5.5 Here]

For some simple polynomials, specifically the quadratic and the cubic, it may be possible to find ξ manually but, for the majority of cases, it will be necessary to use some form of iterative process to determine the value of the root. Any simple iterative process would be sufficient to find these values, provided a suitable starting point is used. It has already been noted that the minimum possible value for ξ is $\frac{N_2 - N_1}{N_2 - N_0}$ and this could be a suitable starting point for the iterative process in this case.

<u>Case 2.</u> The gradient of $f_1(x) = x^{\frac{t_2}{t_1}}$ is less than the gradient of $f_2(x) = x\left[\frac{N_2 - N_0}{N_1 - N_0}\right] - \left[\frac{N_2 - N_0}{N_1 - N_0}\right] + 1$ at the point $x = 1$.

If the gradient of $f_1(x)$ is less than the gradient of $f_2(x)$ at the point $x=1$, then the relationships between the differentials of these two functions will can be expressed as:

$$\frac{t_2}{t_1} x^{\frac{t_2}{t_1}-1} < \left[\frac{N_2 - N_0}{N_1 - N_0}\right]$$

and, as before, this expression is equivalent to $\frac{t_2}{t_1} < \left[\frac{N_2 - N_0}{N_1 - N_0}\right]$

Thus the root ξ must lie in the range $\xi > 1$

This case can be illustrated as Figure A5.6

[Insert Figure A5.6 Here]

Case 3. The gradient of $f_1(x) = x^{\frac{t_2}{t_1}}$ is equal to the gradient of $f_2(x) = x\left[\frac{N_2 - N_0}{N_1 - N_0}\right] - \left[\frac{N_2 - N_0}{N_1 - N_0}\right] + 1$ at the point $x=1$.

If the gradients of the two functions are equal at the point $x=1$, then this means that $\frac{t_2}{t_1} = \left[\frac{N_2 - N_0}{N_1 - N_0}\right]$ and there are two positive roots, but they both lie at the point $x=1$.

Recalling that $x = e^{\frac{t_1}{k_2}[1-k_1]}$, this implies that $\frac{t_1}{k_2}[1-k_1] = 0$ [14].

There are two possible conditions under which this can be true:

1. $k_1 = 1$. If this is the case, then the solution to the polynomial equation lies on the stability boundary between Zones 1 and 2 and Zones 4 and 5 as pictured in Figure 4.

2. $t_1 = 0$. In this case, the first time point is the start conditions when the system first begins to operate. This corresponds to the point N_0 in the price series.

As has been demonstrated, in all three cases there are just two positive roots, $x = 1$ and $x = \xi$, where ξ can take any value including 1. This second root is the one that is required to find the values of the coefficients k_1 and k_2. To find the value of ξ will necessitate solving the polynomial equation given previously as equation A5.13, reprised below.

$$x^{\frac{t_2}{t_1}} - x\left[\frac{N_2 - N_0}{N_1 - N_0}\right] + \left[\frac{N_2 - N_0}{N_1 - N_0}\right] - 1 = 0 \tag{A5.13}$$

To find the roots of this polynomial equation some simple iteration is required.

Once ξ is known the coefficients can be calculated using this value, and equations A5.9 and A5.11 can now be written as A5.16 and A5.17, below:

$$k_1 = \frac{N_1 - N_0}{N_1 - N_0\xi} \tag{A5.16}$$

$$k_2 = \frac{t_1[1 - k_1]}{\ln \xi} \tag{A5.17}$$

---

[14] $e^0 = 1$

Appendix 6

Assuming that $M \neq 1$, the equation for the price series is given in the form of equation A6.1, which appears as equation 3 in the main text. For ease of comparison and to avoid confusion, the coefficients here are denoted κ_1 and κ_2 but their definitions are the same as before and interpretation of these values remains unchanged.

$$N_t = N_0 . \left(\frac{M}{1 - M\kappa_1} \right) . \left[1 - e^{\frac{t}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right)} \right] \tag{A6.1}$$

As previously, the first step is to modify this equation so that the initial value is greater than zero. Following the steps laid out in Appendix 2, this gives a second equation for N_t which takes the form of equation A6.2, below

$$N_t = N_0 \left(\frac{M}{1 - M\kappa_1} \right) \left[1 - e^{\frac{t}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right)} \right] + N_0 . e^{\frac{t}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right)} \tag{A6.2}$$

This equation can be rearranged as follows to give equation A6.3 which is a more concise expression for the price series.

$$N_t = \frac{N_0 M \left[1 - e^{\frac{t}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right)} \right] + N_0 (1 - M\kappa_1) e^{\frac{t}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right)}}{1 - M\kappa_1}$$

$$N_t = \frac{N_0 M - N_0 M . e^{\frac{t}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right)} + N_0 (1 - M\kappa_1) e^{\frac{t}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right)}}{1 - M\kappa_1}$$

$$N_t = \frac{N_0 M}{(1 - M\kappa_1)} \left[1 - e^{\frac{t}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right)} + \frac{1}{M} e^{\frac{t}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right)} - \kappa_1 e^{\frac{t}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right)} \right]$$

$$N_t = \frac{N_0 M}{(1 - M\kappa_1)}\left[1 - \left(1 + \kappa_1 - \frac{1}{M}\right).e^{\frac{t}{\kappa_2}\left(\frac{1 - M\kappa_1}{M}\right)}\right] \tag{A6.3}$$

The equivalent expression, when $M = 1$, was given as equation 5 in the main text and is reprised here as equation A6.4.

$$N_t = \frac{N_0}{(1 - k_1)}\left[1 - k_1.e^{\frac{t}{k_2}(1 - k_1)}\right] \tag{A6.4}$$

These two equations represent the same curve and so it is valid to equate all parts of these equations as follows:

$$k_2 = \kappa_2$$

$$\frac{N_0}{(1 - k_1)} = \frac{N_0 M}{(1 - M\kappa_1)}$$

$$k_1 = 1 + \kappa_1 - \frac{1}{M}$$

and

$$1 - k_1 = \frac{1 - M\kappa_1}{M}$$

These are identical statements and either can be rearranged to give the following equation, A6.5.

$$\kappa_1 - k_1 = \frac{1}{M} - 1 \tag{A6.5}$$

Recalling that M represents the processing of the investors decision and has a maximum possible value of 1, the right hand side of equation A6.5 can be expressed as the equality denoted A6.6, below.

$$\frac{1}{M} - 1 \geq 0 \tag{A6.6}$$

which implies that κ_1 will be bigger than k_1 but the difference is easily calculable. For example, if $M = 0.95$ then the corresponding difference between the coefficients is 0.053.

Maintaining the assumption that $M \neq 1$ will require some slight modification of the equations used to calculate the coefficients. Recalling the methodology detailed in Appendix 5, the equations for the coefficients κ_1 and κ_2 are derived in the following manner.

Again, two values of t are required along with the equivalent values from the price series, N_t. The relationship between the times and the prices are the same as previously illustrated.

Deriving expressions for the price series at times t_1 and t_2 using equation A6.3 now gives equations A6.7 and A6.8 as below.

When $t = t_1$, $N_t = N_1$

$$N_1 = \frac{N_0 M}{\left(1 - M\kappa_1\right)} \left[1 - \left(1 + \kappa_1 - \frac{1}{M} \right) . e^{\frac{t_1}{\kappa_2}\left(\frac{1 - M\kappa_1}{M}\right)} \right] \tag{A6.7}$$

When $t = t_2$, $N_t = N_2$

$$N_2 = \frac{N_0 M}{(1 - M\kappa_1)}\left[1 - \left(1 + \kappa_1 - \frac{1}{M}\right).e^{\frac{t_2}{\kappa_2}\left(\frac{1 - M\kappa_1}{M}\right)}\right] \qquad \text{(A6.8)}$$

Using equation A6.7, an expression can be found for the difference between N_1 and N_0. This is denoted equation A6.9 and is derived as follows:

$$N_1 - N_0 = \frac{N_0 M}{(1 - M\kappa_1)}\left[1 - \left(1 + \kappa_1 - \frac{1}{M}\right).e^{\frac{t_1}{\kappa_2}\left(\frac{1 - M\kappa_1}{M}\right)}\right] - N_0$$

$$N_1 - N_0 = \frac{N_0 M\left[1 - \left(1 + \kappa_1 - \frac{1}{M}\right).e^{\frac{t_1}{\kappa_2}\left(\frac{1 - M\kappa_1}{M}\right)}\right] - N_0(1 - M\kappa_1)}{(1 - M\kappa_1)}$$

$$N_1 - N_0 = \frac{N_0 M\left[1 - \left(1 + \kappa_1 - \frac{1}{M}\right).e^{\frac{t_1}{\kappa_2}\left(\frac{1 - M\kappa_1}{M}\right)}\right] - N_0 + N_0 M\kappa_1}{(1 - M\kappa_1)}$$

$$N_1 - N_0 = \frac{N_0 M\left[1 - \left(1 + \kappa_1 - \frac{1}{M}\right).e^{\frac{t_1}{\kappa_2}\left(\frac{1 - M\kappa_1}{M}\right)} - \frac{1}{M} + \kappa_1\right]}{(1 - M\kappa_1)}$$

$$N_1 - N_0 = \frac{N_0 M\left[1 + \kappa_1 - \frac{1}{M} - \left(1 + \kappa_1 - \frac{1}{M}\right).e^{\frac{t_1}{\kappa_2}\left(\frac{1 - M\kappa_1}{M}\right)}\right]}{(1 - M\kappa_1)}$$

$$N_1 - N_0 = \frac{N_0 M}{(1 - M\kappa_1)}\left(1 + \kappa_1 - \frac{1}{M}\right)\left(1 - e^{\frac{t_1}{\kappa_2}\left(\frac{1 - M\kappa_1}{M}\right)}\right) \tag{A6.9}$$

This expression can be simplified by substituting in the term $\psi = e^{\frac{t_1}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right)}$. This gives equation A6.10. As before, this substitution means that the value of the term ψ, must always be positive.

$$N_1 - N_0 = \frac{N_0 M}{(1 - M\kappa_1)}\left(1 + \kappa_1 - \frac{1}{M}\right)(1 - \psi) \tag{A6.10}$$

Repeating this process for the difference between N_2 and N_0 gives a similar equation which will be denoted A6.11, below

$$N_2 - N_0 = \frac{N_0 M}{(1 - M\kappa_1)}\left[1 - \left(1 + \kappa_1 - \frac{1}{M}\right).e^{\frac{t_2}{\kappa_2}\left(\frac{1 - M\kappa_1}{M}\right)}\right] - N_0$$

$$N_2 - N_0 = \frac{N_0 M}{(1 - M\kappa_1)}\left(1 + \kappa_1 - \frac{1}{M}\right)\left[1 - e^{\frac{t_2}{\kappa_2}\left(\frac{1 - M\kappa_1}{M}\right)}\right] \tag{A6.11}$$

The exponential expression in this equation can be represented as $\psi^{\frac{t_2}{t_1}}$ giving equation A6.12

$$N_2 - N_0 = \frac{N_0 M}{(1 - M\kappa_1)}\left(1 + \kappa_1 - \frac{1}{M}\right)\left[1 - \psi^{\frac{t_2}{t_1}}\right] \tag{A6.12}$$

The ratio of equations A6.12 and A6.10 is given as equation A6.13

$$\frac{N_2 - N_0}{N_1 - N_0} = \frac{\dfrac{N_0 M}{(1 - M\kappa_1)}\left(1 + \kappa_1 - \dfrac{1}{M}\right)\left[1 - \psi^{\frac{t_2}{t_1}}\right]}{\dfrac{N_0 M}{(1 - M\kappa_1)}\left(1 + \kappa_1 - \dfrac{1}{M}\right)(1 - \psi)}$$

$$\frac{N_2 - N_0}{N_1 - N_0} = \frac{\left[1 - \psi^{\frac{t_2}{t_1}}\right]}{(1 - \psi)} \qquad \text{(A6.13)}$$

Rearranging this equation to give an expression for $N_2 - N_0$ and equating it to equation A6.12 makes it possible to derive an equation for κ_1, denoted equation A6.14.

$$\frac{N_0 M}{(1 - M\kappa_1)}\left(1 + \kappa_1 - \frac{1}{M}\right)\left[1 - \psi^{\frac{t_2}{t_1}}\right] = \left[\frac{1 - \psi^{\frac{t_2}{t_1}}}{1 - \psi}\right].(N_1 - N_0)$$

$$\frac{N_0 M}{(1 - M\kappa_1)}\left(1 + \kappa_1 - \frac{1}{M}\right) = \frac{(N_1 - N_0)}{1 - \psi}$$

$$\frac{N_0 M + N_0 M\kappa_1 - N_0}{(1 - M\kappa_1)} = \frac{(N_1 - N_0)}{1 - \psi}$$

$$N_0 M + N_0 M\kappa_1 - N_0 = \frac{N_1 - N_1 M\kappa_1 - N_0 + N_0 M\kappa_1}{1 - \psi}$$

$$N_0 M + N_0 M\kappa_1 - N_0 - N_0 M\psi - N_0 M\kappa_1\psi + N_0\psi = N_1 - N_0 + M\kappa_1(N_0 - N_1)$$

$$\kappa_1 M(N_1 - N_0\psi) = N_1 - N_0[M(1 - \psi) + \psi]$$

$$\kappa_1 = \frac{N_1 - N_0[M(1 - \psi) + \psi]}{M(N_1 - N_0\psi)} \qquad \text{(A6.14)}$$

Recalling the substitution used previously, it is possible to derive an equation for κ_2 denoted here as A6.15.

$$\psi = e^{\frac{t_1}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right)}$$

$$\ln\psi = \frac{t_1}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right)$$

$$\kappa_2 = \frac{t_1}{\ln\psi}\left(\frac{1-M\kappa_1}{M}\right) \qquad \text{(A6.15)}$$

Using the expression developed previously $1-k_1 = \frac{1-M\kappa_1}{M}$, this can be re-expressed as:

$$\kappa_2 = \frac{t_1}{\ln\psi}(1-k_1)$$

which is the same equation as was developed in Appendix 5 (Equation A5.10) to represent k_2 which reaffirms that $k_2 = \kappa_2$.

As before, it is again possible to find a second expression for κ_2 by rearranging equation A6.14 and substituting that term into equation A6.15. This derivation is included here, as follows, for completeness.

$$\kappa_1.M(N_1 - N_0\psi) = N_1 - N_0[M - M\psi + \psi]$$

$$N_0\psi - N_0M\psi - \kappa_1MN_0\psi = N_1 - N_0M - \kappa_1MN_1$$

$$\psi(N_0 - N_0M - \kappa_1MN_0) = N_1(1-\kappa_1M) - N_0M$$

$$\psi = \frac{N_1(1-\kappa_1 M) - N_0 M}{N_0(1 - M - \kappa_1 M)} \tag{A6.16}$$

Thus equation A6.15 can be re-written as:

$$\kappa_2 = \frac{t_1}{\ln\left[\frac{N_1(1-\kappa_1 M) - N_0 M}{N_0(1 - M - \kappa_1 M)}\right]}\left(\frac{1 - M\kappa_1}{M}\right) \tag{A6.17}$$

As was discussed in Appendix 5, it is not possible to evaluate these equations for the coefficient values without finding the positive roots of the associated polynomial equation. The method of solution and the locations of the roots are the same as in the case where $M = 1$ and a detailed discussion of the process here can be found in Appendix 5.

The only other substantial impact on the system that results from the situation when $M \neq 1$ is that this influences the equation representing the boundaries between the zones. As was discussed in Appendix 4, the boundary between Zones 1 and 2 and between Zones 4 and 5 occurs when $k_1 = 1$. In the case where $M \neq 1$, this boundary is now positioned such that $M\kappa_1 = 1$. Using the same procedure as was outlined in Appendix 4, an equation can be derived for the stability boundary under these circumstances.

This procedure starts with an expansion of the exponential part of equation A6.3 using the standard series for e^x. Using this expansion, the exponential part of equation A6.3 takes the form of equation A6.18

$$e^{\frac{t}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right)} \approx 1 + \frac{t}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right) + \frac{t^2}{\kappa_2^2.2!}\left(\frac{1-M\kappa_1}{M}\right)^2 + \frac{t^3}{\kappa_2^3.3!}\left(\frac{1-M\kappa_1}{M}\right)^3 + \frac{t^4}{\kappa_2^4.4!}\left(\frac{1-M\kappa_1}{M}\right)^4 + \ldots$$

(A6.18)

Further expanding equation A6.3 using the exponential expansion and some simple algebra gives an expression for the entire equation.

$$\frac{N_0 M}{(1-M\kappa_1)}\left[1-\left(1+\kappa_1-\frac{1}{M}\right).e^{\frac{t}{\kappa_2}\left(\frac{1-M\kappa_1}{M}\right)}\right]$$

$$\approx N_0\left\{1-\left(1+\kappa_1-\frac{1}{M}\right).\left[\frac{t}{\kappa_2}\right]-\left(1+\kappa_1-\frac{1}{M}\right).\left[\frac{t^2}{\kappa_2^2}\right]\left(\frac{1-M\kappa_1}{M}\right)-\ldots\right\}$$

At the boundary, denoted N_β in this case, $M\kappa_1 = 1$ and $\kappa_1 = \frac{1}{M}$, so this expression becomes equation A6.19.

$$N_\beta = N_0\left\{1-\left(1+\kappa_1-\frac{1}{M}\right).\left[\frac{t}{\kappa_2}\right]-0-0\ldots\right\}$$

$$N_\beta = N_0\left\{1-\frac{t}{\kappa_2}\right\} \qquad \text{(A6.19)}$$

It has already been demonstrated that $k_2 = \kappa_2$ so equation A6.19 represents the same stability boundary as was derived in Appendix 4.

Figure 1. A Simple Feedback Loop

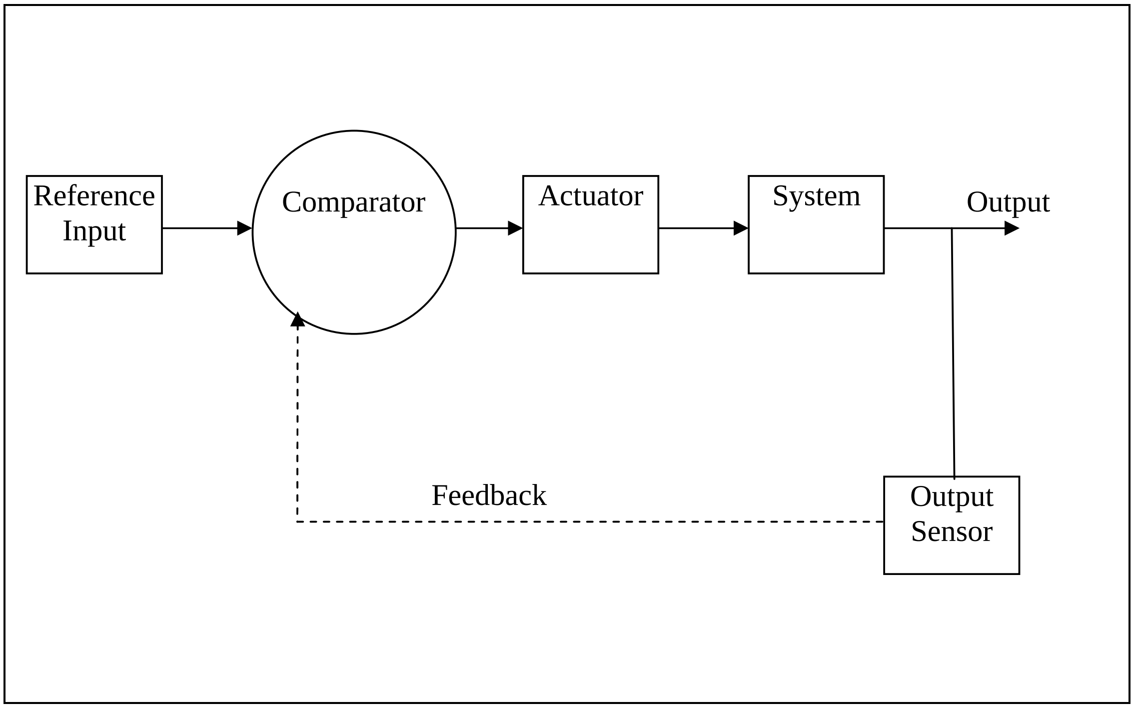

Figure 2. A Simple Decision Process for Share Buying

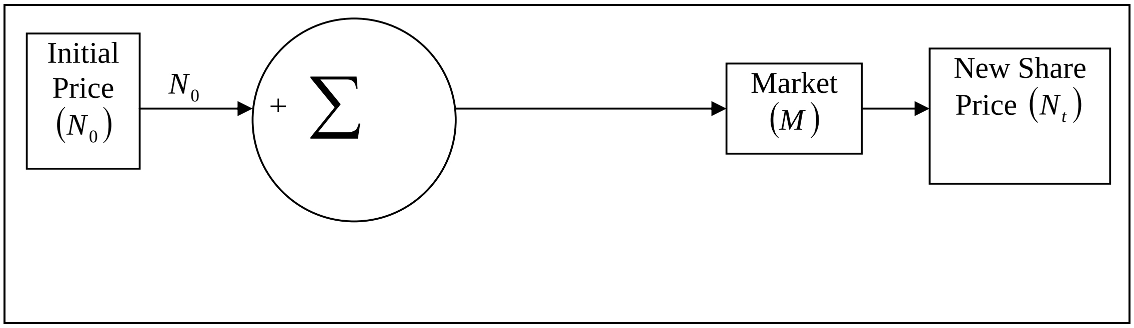

Figure 3. A Closed Feedback Loop for Share Buying

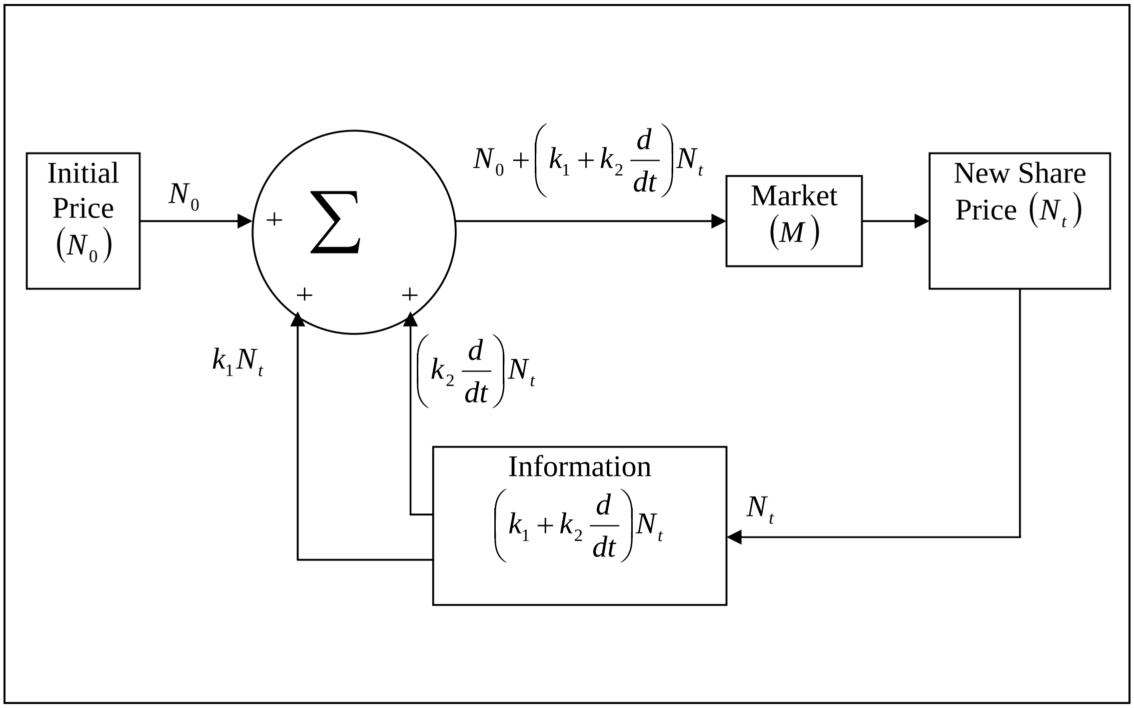

Figure 4. Regions of Stable and Unstable Behaviour

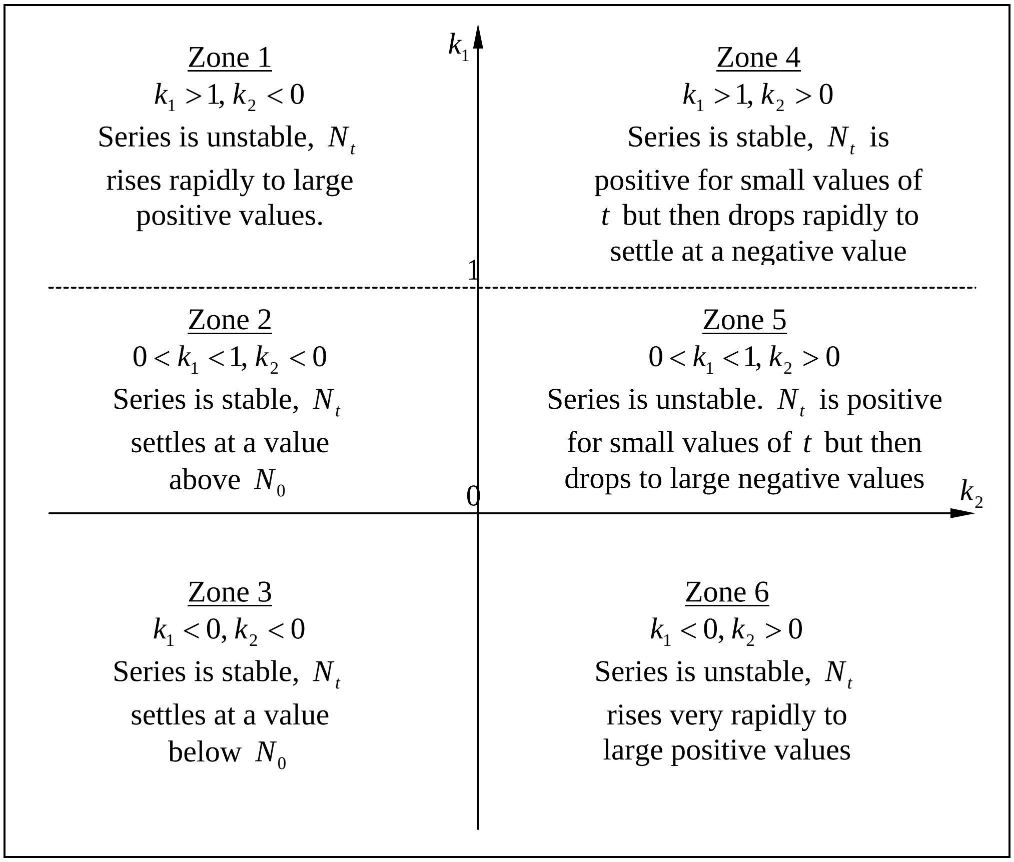

Figure 5. Behaviour of the series N_t over time

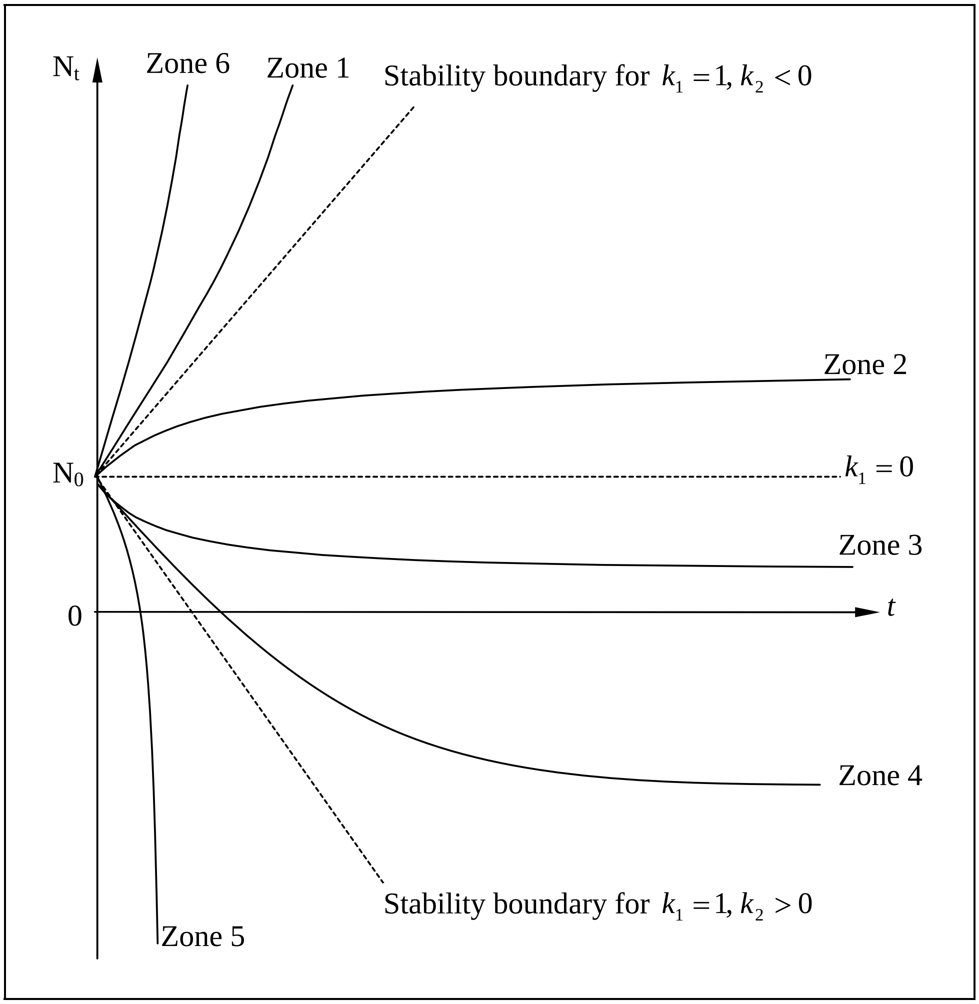

Figure A2.1. An exponential growth curve representing the solution to a first order differential equation

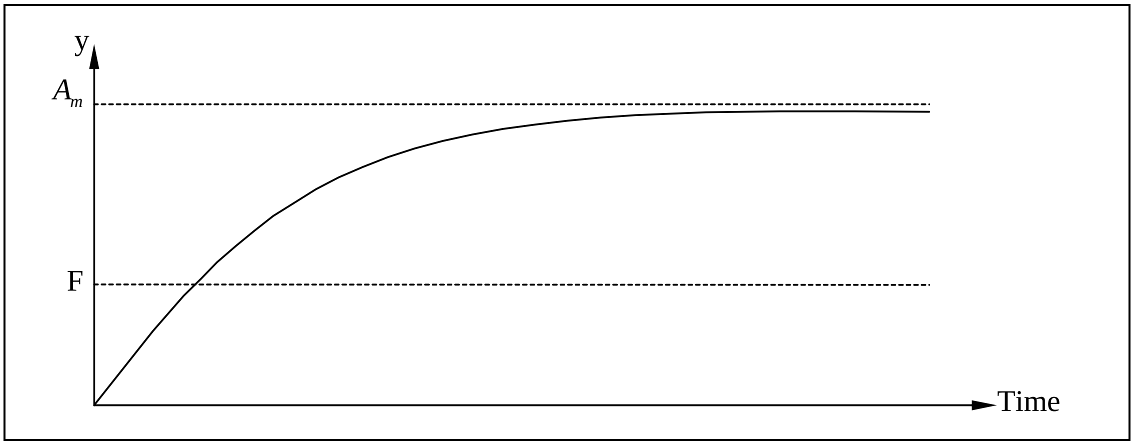

Figure A2.2. Changing the origin of an exponential growth curve

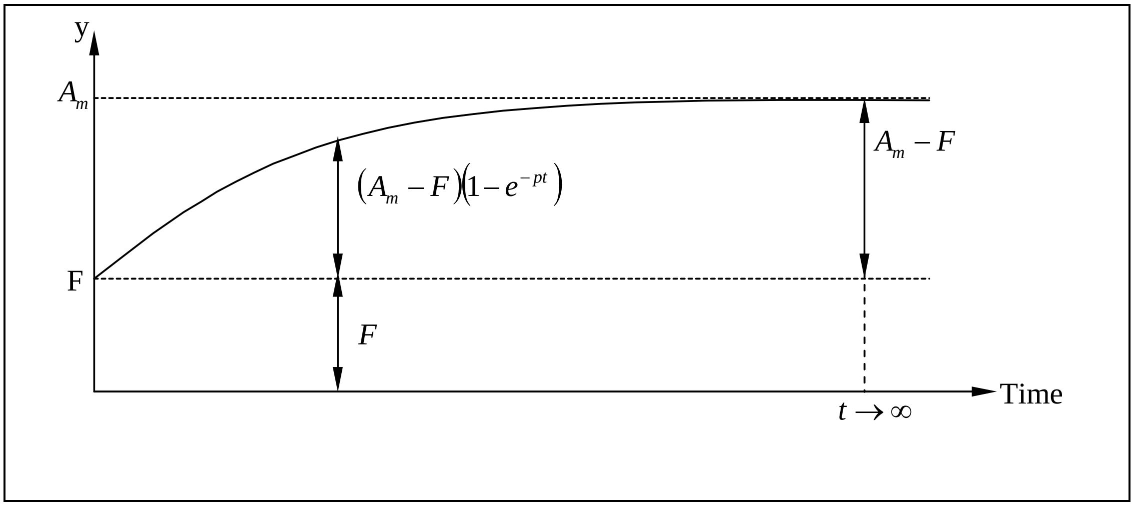

Figure A3.1. Behaviour of N_t in Zone 1

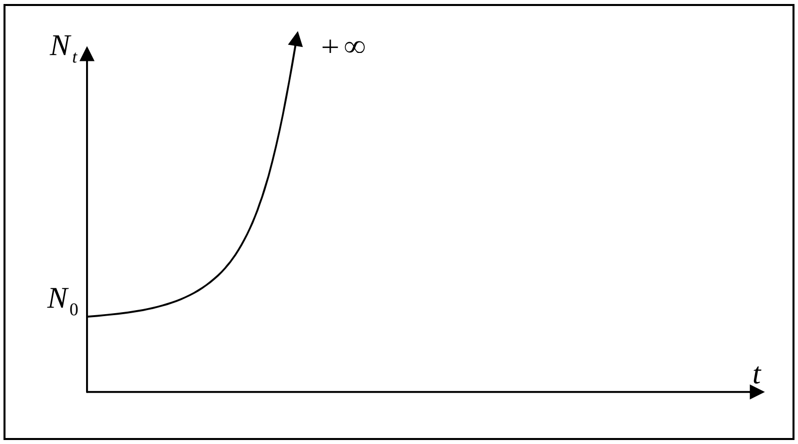

Figure A3.2. Behaviour of N_t in Zone 2

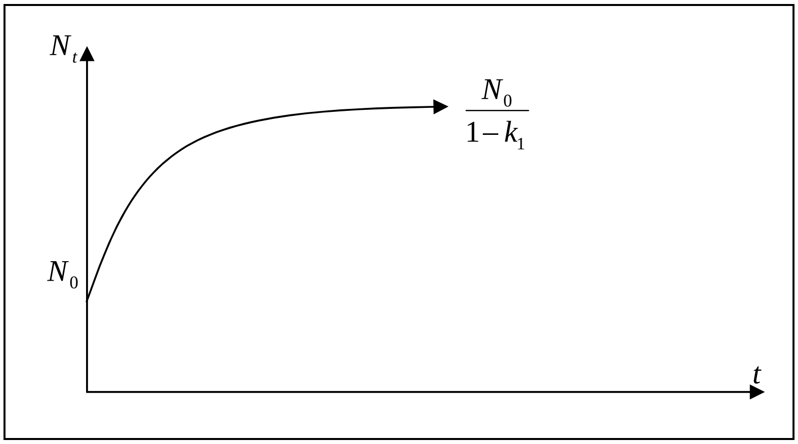

Figure A3.3. Behaviour of N_t in Zone 3

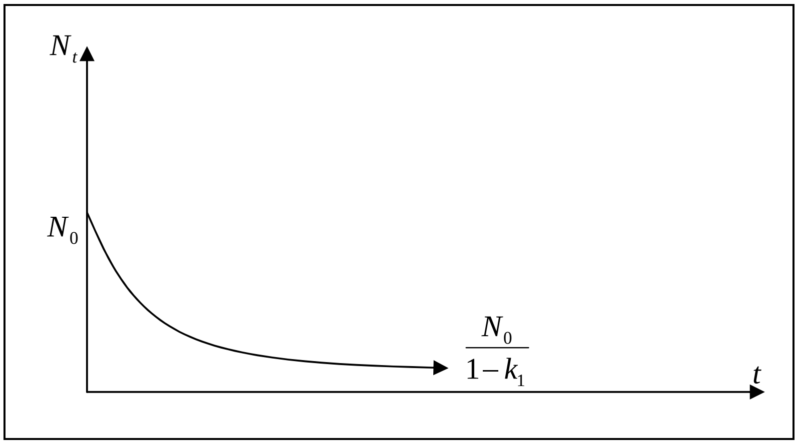

Figure A3.4. Behaviour of N_t in Zone 4

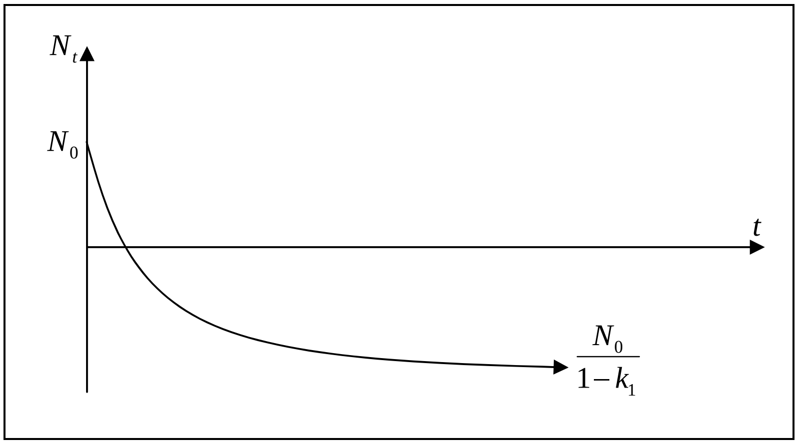

Figure A3.5. Behaviour of N_t in Zone 5

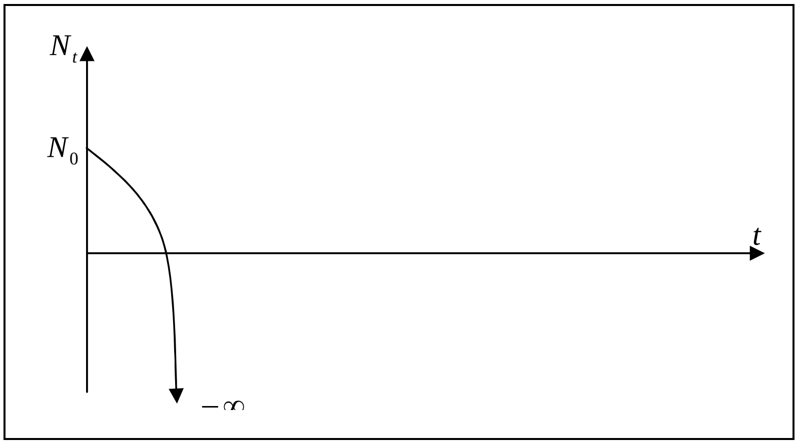

Figure A3.6. Behaviour of N_t in Zone 6

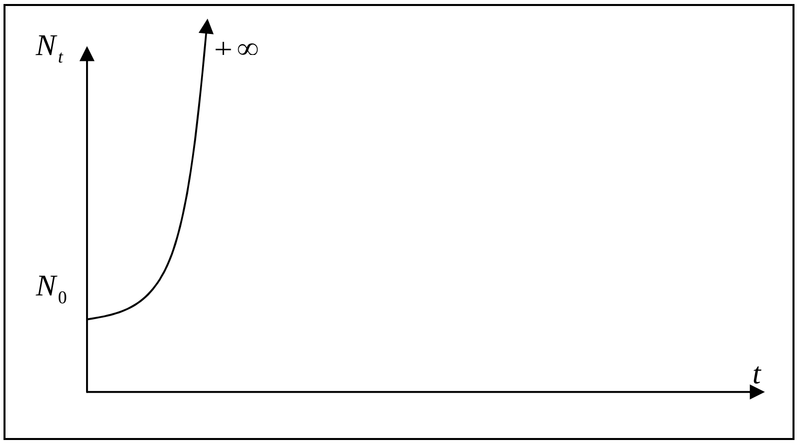

Figure A5.1. Points in the Series N_t used in the Calculation of k_1 and k_2.

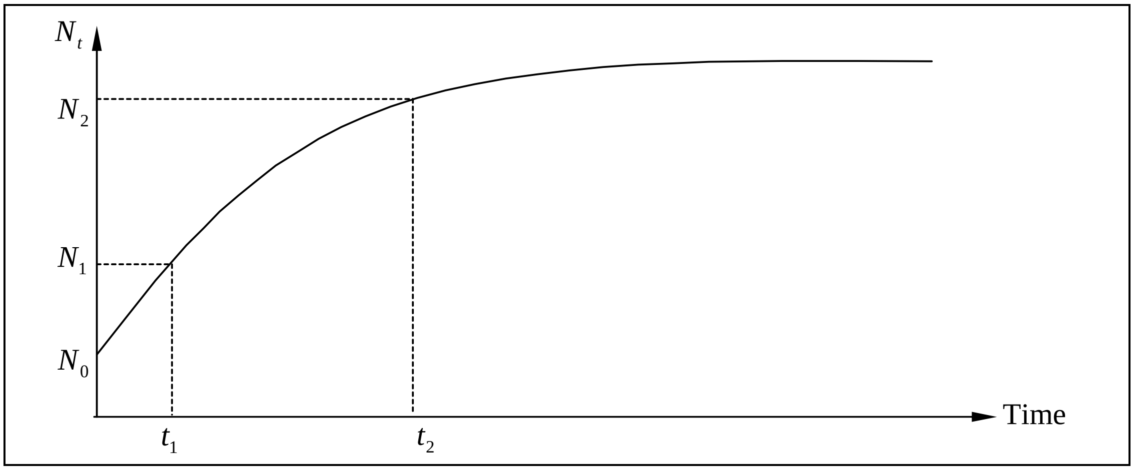

Figure A5.2. Function $y = f_1(x)$

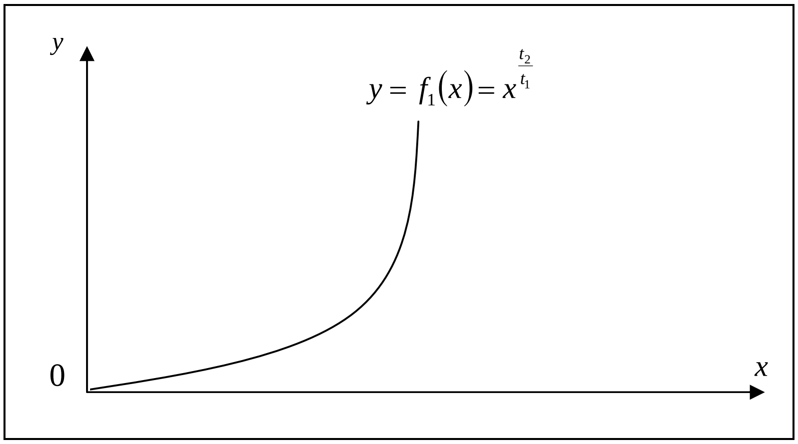

Figure A5.3. Function $y = f_2(x)$

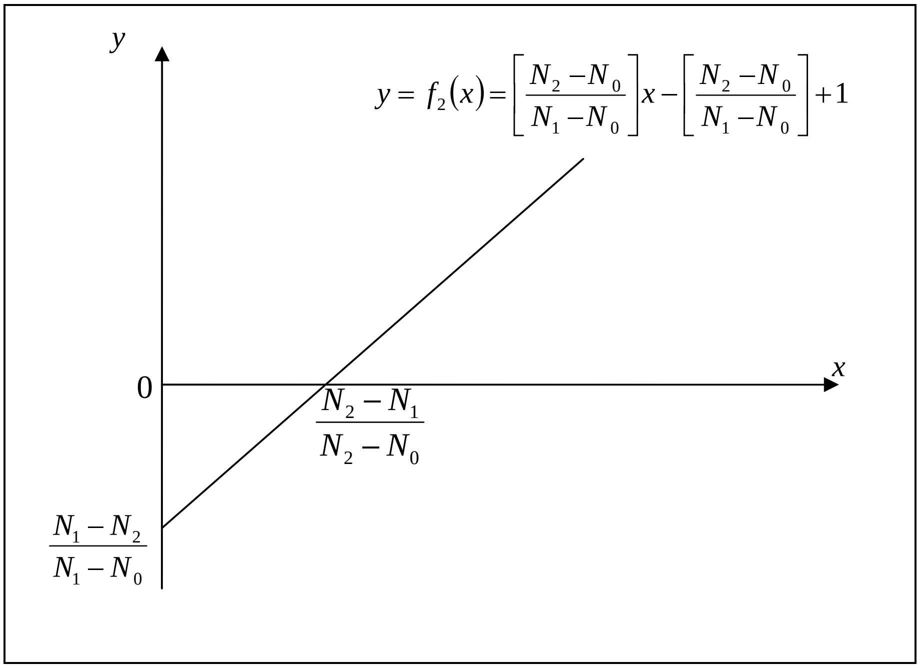

Figure A5.4. Functions $y = f_1(x)$ and $y = f_2(x)$ presented together

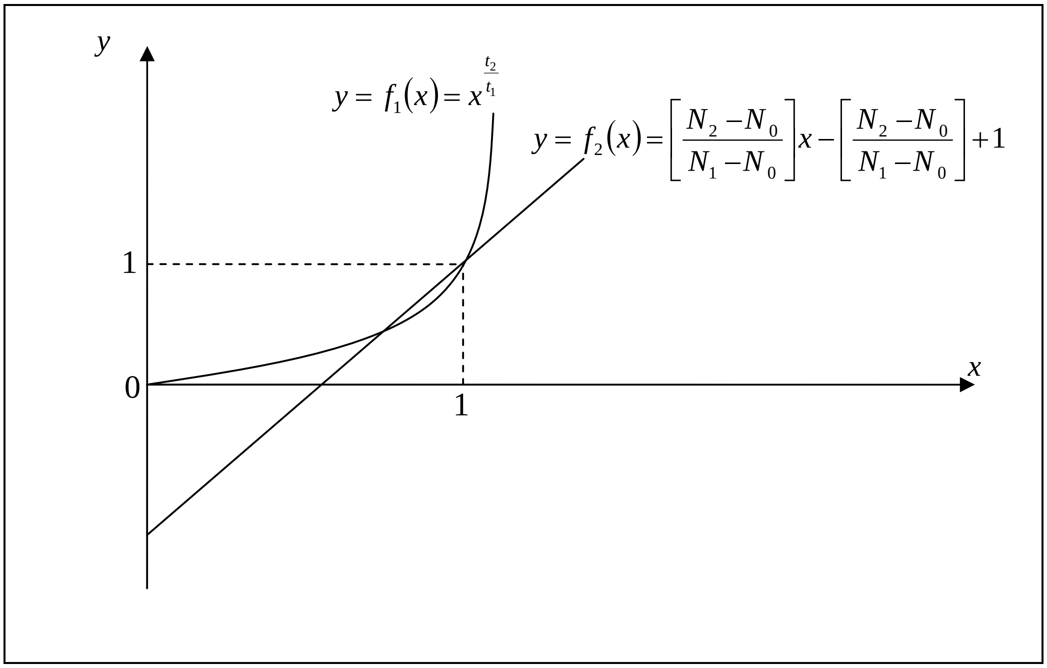

Figure A5.5. Case 1. Gradient of $y = f_1(x)$ is greater than the gradient of $y = f_2(x)$

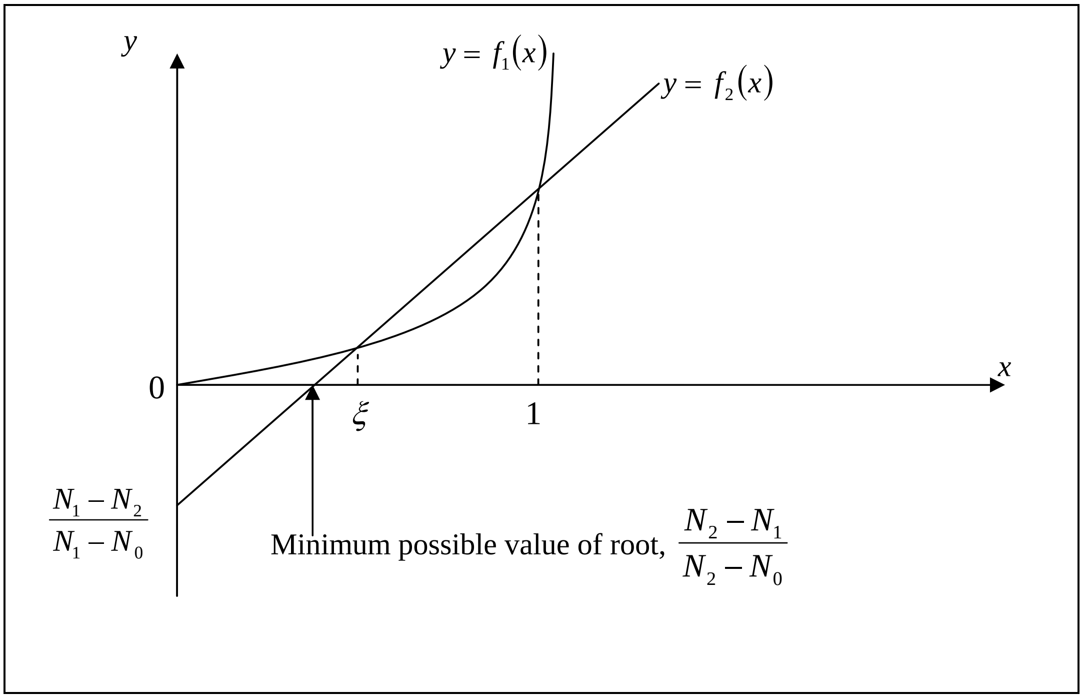

Figure A5.6. Case 2. Gradient of $y = f_1(x)$ is less than the gradient of $y = f_2(x)$