INTEREST RATE FUTURES: ESTIMATION OF VOLATILITY PARAMETERS IN AN ARBITRAGE-FREE FRAMEWORK

Ramaprasad Bhar and Carl Chiarella
School of Finance & Economics
University of Technology, Sydney
P.O. Box 123, Broadway, NSW 2007
AUSTRALIA
Tel. +61 2 9514 3631 Fax. +61 2 9514 3636

Hedging interest rate exposures using interest rate futures contracts requires some knowledge of the volatility function of the interest rates. Use of historical data as well as interest rate options like caps and swaptions to estimate this volatility function have been proposed in the literature. In this paper the interest rate futures price is modelled within an arbitrage-free framework for a volatility function which includes a stochastic variable, the instantaneous spot interest rate. The resulting system is expressed in a state space form which is solved using extended Kalman filter. The residual diagnostics indicate suitability of the model and the bootstrap resampling technique is used to obtain small sample properties of the parameters of the volatility function.

Keywords: Interest rate futures, Heath-Jarrow-Morton model, Arbitrage-free, Kalman filter, Bootstrap.

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1. Introduction

In most major markets around the world interest rate futures contracts have enjoyed tremendous growth over the last decade or so. The flexibility of these contracts and lower transaction costs have made these attractive to portfolio managers attempting to change risk return characteristics of their holdings. For example, the duration of a fixed income portfolio can be decreased or increased with the help of suitable interest futures contracts.

Many research articles examined the theoretical pricing models and the difference that result from marking-to-market that are special characteristics of the futures markets. For example, Cox-Ingersoll-Ross (CIR) (1981) finds that the size and the difference between the prices of forward and futures contracts is an increasing function of the covariance between the underlying asset and a stochastic interest rate. On the other hand, Flesaker (1993) shows that, for contract maturities up to six months, interest rate futures prices do not differ significantly from the corresponding forward prices. But, for longer maturities the difference may be significant and depends on the maturity as well as the level of the interest rate volatility. This author also points out that the differences in pricing due to continuous marking-to-market and daily marking-to-market is very small. Flesaker (1993) uses the arbitrage-free framework of Heath-Jarrow-Morton (HJM) (1992) under the assumption of constant volatility of the forward rate.

In a recent paper, Jarrow and Turnbull (1994) develop the theoretical framework for delta and gamma hedging with interest rate futures contracts in a multi-factor HJM setting. They also point out the fact that this area is far less developed compared to equity and foreign currency derivatives. To implement the

theory developed by Jarrow and Turnbull (1994), it is necessary to compute the parameters of the volatility function that has to be specified for the HJM model. They specify the following form of the volatility function:

$$\sigma_f = a_0 e^{-\lambda(T - t)} \tag{1}$$

where, σ_f is the volatility of the forward rate f(t,T) at time t for T, a_0 and λ are the parameters of the specification. It is suggested that practitioners normally estimates these parameters by fitting the theoretical models to market prices of such instruments as caps and swaptions. Musiela et al (1992) also suggests an algorithm that uses historical data.

This paper presents an alternative way to infer these parameters using non-linear filtering algorithm. The procedure relies upon a prior specification of the form of the volatility function in the HJM framework, that is more general than in Flesaker (1993) and differs from the equation (1) by a multiplicative factor which is a simple function of the instantaneous spot interest rate, r(t). In this construction, the futures price, with continuous marking-to-market, is represented by a stochastic differential equation system of dimension three. The appropriate discretisation method, the filtering algorithm, and the computational details are also described. The empirical results are presented based on the data on short term interest rate futures contracts from both the Sydney Futures Exchange (SFE) and the Tokyo International Financial Futures Exchange (TIFFE).

2. The Futures Price Model

The starting point in the HJM, (1992) model of the term structure of interest rates is the stochastic integral equation for the forward rate,

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)\, du + \int_0^t \sigma_f(u,T)\, dW(u), \qquad (0 \le u \le T), \tag{2}$$

where f(t, T) is the forward rate at time t applicable to time T (>t). The single source of noise dW(u) is the increment of a standard Wiener process generated by a probability measure Q. The function α(u, T) and σ_f (u, T) are the instantaneous drift and volatility functions at time u for the forward rate f(t, T). According to HJM (1992) avoidance of riskless arbitrage opportunity and application of Girsanov's theorem transforms the equation (2) to,

$$f(t,T) = f(0,T) + \int_0^t \sigma_f(u,T) \int_u^T \sigma_f(u,y)dy\, du + \int_0^t \sigma_f(u,T) d\tilde{W}(u) \tag{3}$$

where $d\tilde{W}(u)$ is the increment of a standard Wiener process generated by an equivalent probability measure $\tilde{Q}$. The Radon-Nikodym theorem relates the two probability measures Q and $\tilde{Q}$, but the details are not necessary for the purposes of this paper. The spot rate process, by definition, is given by r(t) = f(t,t) and thus satisfies the stochastic integral equation,

$$r(t) = f(0,t) + \int_0^t \sigma_f(u,t) \int_u^t \sigma_f(u,y)dy\, du + \int_0^t \sigma_f(u,t)\, d\tilde{W}(u)\ . \tag{4}$$

In the HJM (1992) context, the price, P(t,T), of a pure discount bond at time t, paying $1 at time T is given by,

$$P(t,T) = \exp\left(-\int_t^T f(t,\ s)\, ds\right), \quad 0 \le t \le T, \tag{5}$$

which, on application of Ito's lemma gives the dynamics of the bond price, (Jarrow and Turnbull, (1994))

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left[-\int_t^T ds \int_0^t \sigma_f(u,s) \int_u^s \sigma_f(u,v)\, dv\, du - \int_t^T ds \int_0^t \sigma_f(u,s)\, d\tilde{W}(u)\right]. \quad (6)$$

Let F(t; n) denote the futures price at time t ($\leq$ n) of a futures contract which matures at time n. The futures contract is written on a discount instrument which matures at time T($\geq$ n). When the futures contract matures,

$F(n; n) = P(n, T)$,

and at time t ($\leq$ n)

$F(t; n) = E[\, P(n, T) \mid \vartheta_t \,]$, where ϑ_t denotes the information at time t.

Using equation (6), it follows that,

$$F(t;\ n) = F(0;\ n) \exp\left[-\frac{1}{2}\int_0^t du \left(\int_n^T \sigma_f(u,s)\, ds\right)^2 - \int_0^t d\tilde{W}(u) \int_n^T \sigma_f(u,s)\, ds\right]. \quad (7)$$

It is assumed that σ_f (u, T) has the following form (equation (8)), i.e. the product of the equation (1) and a stochastic term dependent on the instantaneous spot interest rate,

$$\sigma_f(t,T) = a_0\, e^{-\lambda(T-t)}\, r(t)^{\gamma}. \quad (8)$$

It is shown in the following paragraphs that in this situation the futures price process can be modelled as a three dimensional stochastic differential equation system. Equation (8) is motivated by the empirical literature on the short rate process, in particular, Chan et al (1992) and Brenner, Harjes and Kroner (1996). As we see

below the form (8) implies a process for r(t) whose diffusion term is proportional to r^{γ} and the cited empirical literature lends support to such a dependence. Furthermore, Chiarella and El-Hassan (1996a, 1997) show that with the volatility function (8) the HJM model reduces to the Hull and White (1990) extended Vasicek model when $\gamma = 0$ and extended CIR (1985) model when $\gamma = 0.5$.

In order to transform the equation (7) to the differential form, it is expressed as,

$$F(t;\ n) = F(0;\ n)\exp[-X1 - X2]$$

$$X1 = \frac{1}{2}\int_0^t du \left(\int_n^T \sigma_f(u,s)\ ds\right)^2 \tag{9}$$

$$X2 = \int_0^t d\tilde{W}(u)\int_n^T \sigma_f(u,s)ds$$

which leads to,

$$dX1 = \frac{1}{2}\left(\int_n^T \sigma_f(t,s)ds\right)^2 dt \tag{10}$$

$$dX2 = \left(\int_n^T \sigma_f(t,s)ds\right) d\tilde{W}(t)\ . \tag{11}$$

Now, let Y = -X1 - X2, so that,

$$dY = -\frac{1}{2}\left(\int_n^T \sigma_f(t,s)ds\right)^2 dt - \left(\int_n^T \sigma_f(t,s)ds\right) d\tilde{W}(t) \tag{12}$$

$$\equiv \mu_Y\, dt + \sigma_Y\, d\tilde{W}(t). \tag{13}$$

Rewriting equation (7) as, F(t; n) = F(0; n) exp(Y), then by Ito's lemma,

$$dF(t;n) = \left[F(0;n)\ e^{Y}\mu_Y + F(0;n)\ e^{Y}\frac{1}{2}\sigma_Y^2\right]dt + F(0;\ n)\ e^{Y}\sigma_Y\ d\tilde{W}(t),$$

which simplifies to (by use of the definition of μ_Y and σ_Y in the equation (13))

$$dF(t;\ n) = F(t;\ n)\left(\int_n^T \sigma_f(t,s)ds\right) d\tilde{W}(t). \tag{14}$$

The stochastic differential equation (14), thus, describes the evolution of the futures price in the arbitrage free framework as a function of the volatility of the forward rate. As it has been assumed that the forward rate volatility is given by the equation (8), it implies that the dynamics of the futures price in equation (14) is, in fact, driven by the dynamics of the spot interest rate given by the equation (4). It can be seen from the equation (4) that the spot rate process, in general, is non-Markovian due to the presence of the last integral which represents the accumulated effect of the shocks since t=0. It has been shown in Bhar and Chiarella (1995) that this spot rate process can be converted to a Markovian system with the expansion of states for a general class of volatility function. That result is summarised below for the volatility function assumed in this paper:

$$dr(t) = [f_2(0,t) + \lambda f(0,t) - \lambda r(t) + a_0^2 \phi(t)]\, dt + a_0 r(t)^{\gamma} d\tilde{W}(t), \text{ and} \tag{15}$$

$$d\phi(t) = \left[r(t)^{2\gamma} - 2\lambda\phi(t)\right] dt \tag{16}$$

where $f_2(0,t)$ is the partial derivative of f(0,t) with respect to the second argument and the derivation is summarised in the Appendix (equation (16) uses ϕ instead of $\phi_{0,2}$ as in equation (A7) of the appendix).

The state variable $\phi(t)$, which summarises the characteristics of the path history of the instantaneous spot rate process is not readily observable. It will be further assumed that r(t) itself is not directly observable. Depending on the application, however, some proxy variable e.g. overnight rate or 30-day inter-bank rate may represent r(t). The stochastic differential equations (14) - (16), therefore, jointly determine the futures price in this model. It should, however, be stressed that the

distribution of r(t) in this model is under the equivalent martingale probability measure.

It should be pointed out that similar reductions to Markovian system have been found by Carverhill (1994) and Ritchken and Sankarasubramanian (RS) (1995). It has also been shown in Bhar and Chiarella (1995) that the RS (1995) system can be obtained as a special case of their approach.

3. Non-Linear Filter Estimation Using Milstein Discretisation

With the volatility function given in equation (8), the futures price can be expressed as a Markovian stochastic differential system as:

$$dS(t) = \left[J(t) + H(t)\,S(t)\right] dt + V(t)\, d\tilde{W}(t) \text{ , where} \tag{17}$$

$$S(t) = \left[F(t,\ n),\ r(t),\ \phi(t)\right]' , \tag{18}$$

$$V(t) = \left[F(t;n)\, a_0\, r(t)^{\gamma} \int_n^T e^{-\lambda(s-t)} ds,\ a_0 r(t)^{\gamma},\ 0\right]' , \tag{19}$$

$$J(t) = \left[0,\ f_2(0,t) + \lambda f(0,t),\ 0\right]' , \tag{20}$$

and,

$$H(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\lambda & a_0^2 \\ 0 & r(t)^{2\gamma-1} & -2\lambda \end{bmatrix} . \tag{21}$$

Since the only element of state vector considered observable is F(t; n), the observation vector (a scalar in this case) is introduced as,

$$Y(t) = C\,S(t) + \varepsilon(t), \quad \text{where } C = [1, 0, 0]. \tag{22}$$

The observation error is characterised by $\varepsilon(t) \sim N(0,\sigma_\varepsilon)$, $\sigma_\varepsilon > 0$. It is further assumed that the error sequences in the equation (17) and (22) are independent. The observation error is believed to be introduced by the presence of spread between open-close or high-low quotes. As the observed variable F(t; n) entering the system through the measurement equation (22), the approach to solving this problem is based on non-linear filtering. For ease of exposition the equation (17) is expressed as,

$$dS(t) = F(S(t);\ \theta)\,dt + V(S(t);\ \theta)\,d\tilde{W}(t),\ \theta = [a_0, \lambda, \gamma]. \tag{23}$$

In general F(.) and V(.) will be non-linear in both the state variables as well as the parameters. Estimation of the parameter vector θ will involve some form of discretisation from which the conditional moments over successive time intervals can be calculated. Three different approaches have been described in Bhar and Chiarella (1997). In this paper only the Milstein scheme is adopted. The Milstein scheme which is an order 1.0 strong Taylor scheme[1] (see Kloeden and Platen (1992) for details) discretises the stochastic differential equation (23) within the interval $\delta_k = t_{k+1} - t_k$, as

$$S_{k+1} = \left[S_k + F(S_k;\theta)\delta_k\right] + V(S_k;\theta)\sqrt{\delta_k}\ \tilde{\zeta}$$
$$+ \frac{1}{2}\dot{V}(S_k;\theta)V(S_k;\theta)\delta_k\left[\left(\tilde{\zeta}_k\right)^2 - 1\right],\ \text{where } \tilde{\zeta}_k \sim N(0,1). \tag{24}$$

[1]Strong schemes are also required to generate pathwise approximations for studying estimators of statistical parameters as recommended by Kloeden, Schurz, Platen and Sorensen (1992).

The derivative $\dot{V}$ in the third term of the equation (24) represents the matrix of derivatives of the elements of the vector V with respect to the elements of the state vector, S. Using V^i to denote the ith element of the vector V, this derivative is evaluated as,

$$\begin{bmatrix} \frac{\partial V^1}{\partial F} & \frac{\partial V^1}{\partial r} & \frac{\partial V^1}{\partial \phi} \\ \frac{\partial V^2}{\partial F} & \frac{\partial V^2}{\partial r} & \frac{\partial V^2}{\partial \phi} \\ \frac{\partial V^3}{\partial F} & \frac{\partial V^3}{\partial r} & \frac{\partial V^3}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \frac{a_0 r(t)^{\gamma}\eta}{\lambda} & \frac{F(t;n)a_0\gamma\, r(t)^{\gamma-1}\eta}{\lambda} & 0 \\ 0 & a_0\gamma\, r(t)^{\gamma-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{25}$$

where, $\eta = \left[e^{-\lambda(n-t)} - e^{-\lambda(T-t)}\right]$.

Examining the equation (24) the first two conditional moments can be written as,

$$E\left(S_{k+1}\middle|S_k\right) = S_k + F(S_k;\theta)\delta_k \tag{26}$$

$$\begin{aligned} Cov\left(S_{k+1}\middle|S_k\right) = {} & V(S_k;\theta)V'(S_k;\theta)\delta_k \\ & + \frac{1}{2}(\dot{V}VV' + VV'\dot{V}')\delta_k^{3/2}E(\tilde{\zeta}^3) \\ & + \frac{1}{4}\dot{V}VV'\dot{V}\delta_k^2(E(\tilde{\zeta}^4) - 1) \end{aligned} \tag{27}$$

From equation (26) the best forecast of S at t_{k+1} made at t_k (knowing Y_k at t_k) is,

$$\hat{S}_{k+1|k} = S_k + F(S_k;\theta)\delta_k \tag{28}$$

and the best forecast of variance of S_{k+1} is,

$$P_{k+1\,|\,k} = P_{k\,|\,k} + Q_{k+1} \tag{29}$$

where, Q_{k+1} is given by (27). The estimation error is, therefore, given by

$$Y_{k+1} - C\hat{S}_{k+1\,|\,k}, \tag{30}$$

and the variance of estimation error is,

$$\nu_{k+1} = C P_{k+1\,|\,k} C' + \sigma_{\varepsilon} \ . \tag{31}$$

It is also assumed that the prior value of S i.e. $S_0 \sim N(\hat{S}_0, P_0)$. The updating equation for the state vector is given by,

$$\hat{S}_{k+1\,|\,k+1} = \hat{S}_{k+1\,|\,k} + K_{k+1}[Y_{k+1} - C\hat{S}_{k+1\,|\,k}], \tag{32}$$

where the Kalman gain matrix,

$$K_{k+1} = P_{k+1\,|\,k}\ C'\ \nu_{k+1}^{-1}. \tag{33}$$

(Note that ν_{k+1} in this case is a scalar).

The recursion for the error covariance completes the specifications,

$$P_{k+1\,|\,k+1} = [I - K_{k+1}C]\,P_{k+1\,|\,k}[I - K_{k+1}C]' + K_{k+1}RK'_{k+1}. \tag{34}$$

Under the assumption of normal distribution as incorporated in the equations (26) and (27), the transition probability density function for the state vector S_k to S_{k+1} can be written for a given set of observations (T) with the help of the updating equations (32) - (34). Following the argument in Harvey (1989) and Tanizaki (1993) the prediction error decomposition form of the likelihood function is given by,

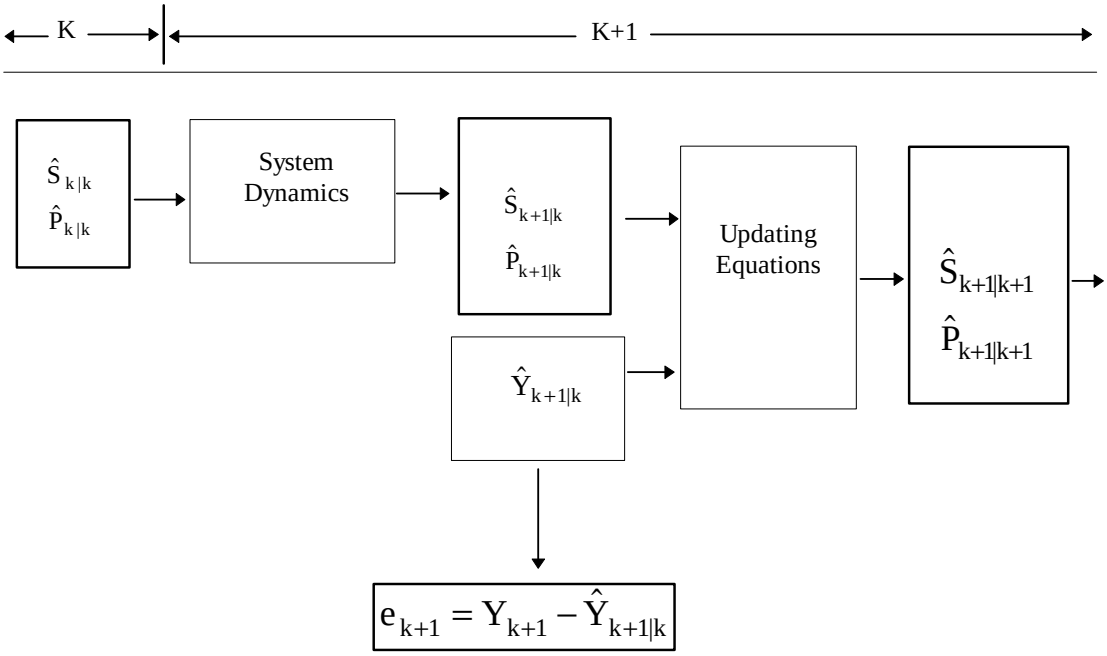
$$\log L = -\frac{T}{2}\log 2\pi - \frac{1}{2}\sum_{k=1}^{T}\log|\nu_k| - \frac{1}{2}\sum_{k=1}^{T} e_k^2 \nu_k^{-1}, \tag{35}$$

where, $e_k = Y_k - \hat{Y}_{k\,|\,k-1}$. 1

To estimate the parameter vector θ of the volatility function of the forward rates, the likelihood function in (35) can be maximised using a suitable numerical optimisation procedure. Maximising L with respect to θ yields consistent and asymptotically efficient estimators $\hat{\theta}$ (see Lo (1988)). That is, $\operatorname*{p\,lim}_{T\to\infty}\hat{\theta} = \theta$ and $\sqrt{T}\left(\hat{\theta}-\theta\right) \sim N\left(0, I^{-1}(\theta)\right)$, where the asymptotic covariance matrix I(θ) is given by,

$$I(\theta) = \lim_{T\to\infty}\frac{1}{T}\sum_{R=1}^{\infty} E\left[\frac{\partial^2 \log L}{\partial\theta\partial\theta'}\right].$$

The updating equations (32) - (34) are at the heart of the filtering algorithm and formal derivations of these can be found in Jazwinski (1970, chapter 7). However, we present here an intuitive understanding of the approach, with the help of the accompanying diagram, to gain an insight into the algorithm. With reference to the time slot between k and k+1, we enter the (k+1)th period with a prior belief about the state vector $\hat{S}_{k|k}$ and the covariance matrix $\hat{P}_{k|k}$ and then the system dynamics produce $\hat{S}_{k+1|k}$ and $\hat{P}_{k+1|k}$ respectively following equations (28) and (29). At this point equation (30) helps us to generate the prediction error e_{k+1} which is used to construct the log likelihood function. Finally, before leaving the (k+1)th period the updating equations are applied to compute $\hat{S}_{k+1|k+1}, \hat{P}_{k+1|k+1}$ to be ready for the iteration for the next period. Issues relating to specification of the prior corresponding to k = 1 e.g. $P_{0|0}$ is discussed in Section 5.



4. Data Used In This Study

The model developed in the previous section is applied to short-term interest rate futures contracts traded on the Sydney Futures Exchange (SFE) and on the Tokyo International Financial Futures Exchange (TIFFE). The SFE contracts are on 90-day bank bills for Australian dollar denominated borrowing/lending and the TIFFE contracts are on 3-month Euroyen deposits. Both contracts are deliverable in March, June, September and December. The 90-day bank bill futures contracts mature on second Fridays of the delivery months and the 3-month Euroyen contracts mature on third Wednesdays of the delivery months. Although trading of these contracts five or six quarters ahead are quite common, only the near quarter month contracts are included in the analysis in this paper. Also while nearly a year before the delivery dates these contracts are listed in both these two exchanges, the estimation has been carried out using only the last three months of the trading data. This represents the most active part of trading during the lives of these contracts. Therefore, the average

number of observations per contract is 62. The period covered by the data starts from September 1989 contracts and ends with the March 1994 contracts.

The 90-day bank bill contracts trading on the SFE are quoted in terms of an index number obtained from (100 - yield to maturity). Therefore, the futures price per dollar of face value is given by [1/(1 + yield*90/365)], where yield is expressed as a fraction. In case of the 3-month Euroyen contracts trading on the TIFFE, the quoted price is an index given by (100 - discount). Thus, the futures price per 100 units of face value is given by [100 - (100 - discount)*90/360].

5. Empirical Results

The functional form of the volatility of the forward rate given by the equation (8) includes a variable γ. The empirical results presented here has been obtained by setting $\gamma = 0.5$. The objective of the estimation procedure is therefore to obtain the parameter vector, $\theta = \{a_0, \lambda\}$. Also, σ_ε is given a value of 0.00001 based upon an analysis of the spread between the open-close prices of the futures contracts during the time period examined.

The estimation process also requires specification of the initial forward rate function f(0,T). This is done by using a polynomial fitting and is described in detail in Bhar and Hunt (1993). The initial value of the spot rate is obtained from f(0,T) by setting T=0.

Table 1 reports the estimated coefficients together with the values of the log likelihood function for the 19 90-day bank bill futures contracts examined from the SFE. Table 2 similarly reports the results for the 19 3-month euroyen futures contracts from the TIFFE. The standard errors reported in these Tables should be

treated with caution due to small sample size. The bootstrap resampling approach to non-parametric confidence interval estimation is discussed below. These Tables also include the sample statistics of the implied instantaneous spot interest rates. These statistics are simply the mean and standard deviation of the time path for r(t) generated by the state equations with the estimated parameter vector θ. The implied mean rates correspond well with the short-term rates in existence around the period in question. This is indicative of reasonable fit of the model with the estimated parameters with γ set to 0.5. However, the question of the value of γ that fits the data best has not been pursued in this study.

Table 3 gives the residual diagnostics for both sets of data. It is clear that for most of the contracts the hypothesis of serial correlations can be rejected at usual level of significance. This is further proof of statistical significance of the estimated parameters. Regarding normality in the residual, the TIFFE data perform better than the SFE data. This is evident from the Jarque-Bera statistics in Table 3. This probably suggests that the value of $\gamma = 0.5$ is not appropriate for the SFE data set.

In order to be able to comment about the statistical confidence interval of the estimated parameters, bootstrap distribution of these parameters are generated using the March 1994 contracts for both the SFE and the TIFFE data. The steps in the bootstrap procedure are as follows:

i. Using the original data set, the parameters are estimated as explained in the paper by maximising the prediction error decomposition form of the likelihood function.

ii. The standardised prediction error vector ($e_k^s(\theta) = e_k(\theta)\,\nu_k^{-0.5}$) is obtained. This in turn generates the innovations of the state transition equation ($\tilde{\zeta}_k(\theta) = K_k e_k^s(\theta)$). More information regarding this can be found in Anderson and Moore (1979, page 231).

iii. By sampling $\tilde{\zeta}_k(\theta)$, and $e_k^s(\theta)$ using uniformly distributed random numbers one instance of bootstrap innovations for the state transition equation and the measurement equation is obtained.

iv. The innovations from the step (iii) are used for a complete recursion through the equations (22) and (24) in place of the original error sequences. This generates one instance of bootstrap observations $Y^B(k)$. The initial conditions are kept same as in the original maximum likelihood estimation in step (i).

v. The steps (iii) and (iv) are repeated to obtain 1000 bootstrap replications. For each such replication the maximum likelihood estimation is carried out and a histogram of the distribution of the estimated parameters and sample statistics are obtained.

In the Figures 2 and 3 the distribution of these parameters for the SFE data are shown The quantiles of the distributions are also given as a measure for the confidence interval. Similarly, the Figures 6 and 7 relate to the TIFFE data. In case of SFE data the estimated a_0 lies within 1.42 standard deviation of the mean of the bootstrap distribution, where as λ lies within 0.41 standard deviation of the bootstrap distribution. Similarly, in case of TIFFE data the estimated a_0 lies within 0.96 standard deviation of the mean of the bootstrap distribution where as λ lies within 0.48 standard deviation of the bootstrap distribution. On this basis the statistical

significance of the estimates can be claimed. However, these bootstrap distributions are subject to median bias as explained in Efron (1987). This bias problem can be overcome by considering, amongst a number of other methods, a distribution of the pivotal quantity e.g. bootstrap-t. These quantities are described in the Figures 4-5 and in the Figures 8-9.

This pivotal quantity for the 90-day bank bill futures data for a_0, λ are -1.42 and -0.41 respectively. Referring to the Figures 4 and 5 it is clear that both these values are within the 95% confidence intervals. Similarly, these pivotal quantities for a_0, λ for the 3-month euroyen futures data are -0.96 and -0.48 respectively. Both these values also lie within the 95% confidence interval as can be seen from the Figures 8 and 9. The bootstrap distributions also clearly indicate that only a certain combinations of the parameters a_0 and λ have high probability of occurring. This provides further support to the appropriateness of the estimation method developed here.

Implementation of a recursive algorithm like Kalman filter requires careful specification of the prior values, particularly, the prior covariance matrix of the state vector. The error in the estimate of the state vector is not only dependent on the data but also on P_0 since $P_{k+1|k}$ is partly determined by P_0 . It is therefore interesting to determine the effect of the change in P_0 on subsequent $P_{k+1|k}$. This is achieved by defining the quantity sensitivity in the Figure 1. It can be seen from this figure that the effect of a change in the prior covariance specification settles down within one iteration. This indicates that the system is not overly sensitive to the prior covariance specification. However, in the absence of sufficient knowledge, it has been suggested (see Harvey (1989), page 121) that $P_0 = wI$ be specified, where I is the identity matrix

and w is large scalar, e.g. 10000. In this way the filter is initialised with diffuse prior information.

Before concluding this section a few comments regarding the numerical procedure adopted in this paper are in order. Maximisation of the log likelihood function is carried out by the OPTMUM routine of the Gauss™ package. For the type of data sets used in this study convergence time is only a few seconds per futures contract. However, like any non-linear optimisation the convergence time is sensitive to the appropriate initial value specification. Results presented here are robust to different initial value specifications.

6. Conclusions

This paper presents a technique to estimate the parameters of the volatility function of the interest rate process determining the prices of interest rate futures in an arbitrage-free framework. The volatility function includes a stochastic variable represented by the instantaneous spot interest rate as well as a deterministic function of time. It is shown that the resulting system of stochastic differential equations can be represented in the state space form which can be estimated using non-linear filter.

The technique has been applied to short-term interest rate futures contracts from both the Sydney Future Exchange and the Tokyo International Financial Futures Exchange. The effectiveness of the estimation method has been analysed using the residual series and are found to have the desirable properties. The significance of the estimated parameters have also been established using non-parametric bootstrap simulations.

The question of whether the estimated volatility function consistently price other derivative securities has not been addressed in this paper. However, some of these issues following from the estimation method presented here have been addressed in Chiarella and El-Hassan (1996b, 1997).

An extension of this methodology developed here is to apply it to interest rate futures contracts of more than one delivery dates. For example, the parameters of the volatility function may be estimated using bank bill futures contracts deliverable over two consecutive quarters. The same concept is applicable to the euroyen futures contracts as well.

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Appendix:

Using the definition forward rate volatility in equation (8), the stochastic integral equation (4) for r(t) may be expressed as,

$$\begin{aligned} r(t) &= f(0,t) + \int_0^t a_0^2\, r(u)^{2\gamma} \int_u^t e^{-\lambda[(t-u)+(y-u)]} dy\, du + \int_0^t a_0 r(u)^{\gamma} e^{-\lambda(t-u)} d\tilde{W}(u). \\ &= f(0,t) + \frac{a_0^2}{\lambda}\int_0^t r(u)^{2\gamma} e^{-\lambda(t-u)} du - \frac{a_0^2}{\lambda}\int_0^t r(u)^{2\gamma} e^{-2\lambda(t-u)} du \\ &\quad + a_0 \int_0^t r(u)^{\gamma} e^{-\lambda(t-u)} d\tilde{W}(u) \end{aligned} \quad . \tag{A1}$$

Defining the subsidiary variables,

$$\phi_{0,1}(t) = \int_0^t r(u)^{2\gamma} e^{-\lambda(t-u)} du, \tag{A2}$$

$$\phi_{0,2}(t) = \int_0^t r(u)^{2\gamma} e^{-2\lambda(t-u)} du, \tag{A3}$$

$$z_0(t) = \int_0^t r(u)^{\gamma} e^{-\lambda(t-u)} d\tilde{W}(u). \tag{A4}$$

With the help of (A2) - (A4), (A1) can be written as,

$$r(t) = f(0,t) + \frac{a_0^2}{\lambda}\phi_{0,1}(t) - \frac{a_0^2}{\lambda}\phi_{0,2}(t) + a_0 z_0(t). \tag{A5}$$

Taking differentials of (A) - (A4),

$$d\phi_{0,1}(t) = [r(t)^{2\gamma} - \lambda\phi_{0,1}(t)]dt, \tag{A6}$$

$$d\phi_{0,2}(t) = [r(t)^{2\gamma} - 2\lambda\phi_{0,2}(t)]dt, \tag{A7}$$

$$dz_0(t) = -\lambda z_0(t) + r(t)^{\gamma} d\tilde{W}(t). \tag{A8}$$

Hence, equation (A5) can now be written in differential form using (A6) - (A8),

$$dr(t) = [f_2(0,t) - a_0^2\phi_{0,1}(t) + 2a_0^2\phi_{0,2}(t) - \lambda a_0 z_0(t)]dt + a_0 r(t)^{\gamma} d\tilde{W}(t). \tag{A9}$$

However from (A5),

$$\lambda r(t) - \lambda f(0,t) + a_0^2 \phi_{0,2}(t) = a_0^2 \phi_{0,1}(t) + \lambda a_0 z_0(t) \tag{A10}$$

which when substituted in (A9) gives,

$$dr(t) = [f_2(0,t) + \lambda f(0,t) + a_0^2 \phi_{0,2}(t) - \lambda r(t)]dt + a_0 r(t)^{\gamma} d\tilde{W}(t). \tag{A11}$$

Table 1
Results of Estimation Using 90-Day Bank Bill Futures Data

Contract	a_0	λ	Log Likelihood	Implied Spot Rate Mean	Implied Spot Rate Std. Dev.
Sep. 1989	0.00187	-11.57442	-431.69	0.1269	0.0633
	(0.00001)	(0.26943)			
Dec. 1989	0.00889	-6.29805	-448.75	0.0760	0.0158
	(0.00001)	(0.28785)			
Mar. 1990	0.01573	-5.85906	-392.61	0.0773	0.0159
	(0.00001)	(0.30718)			
Jun. 1990	0.00056	-13.84868	-446.92	0.0836	0.0352
	(0.00040)	(1.91649)			
Sep. 1990	0.13102	4.33495	-473.78	0.0680	0.0091
	(0.05313)	(1.89958)			
Dec. 1990	0.08770	1.66162	-429.31	0.0851	0.0113
	(0.01101)	(0.53510)			
Mar. 1991	0.06770	2.14568	-385.95	0.0791	0.0118
	(0.15908)	(9.50070)			
Jun. 1991	0.07987	3.42275	-470.58	0.0811	0.0117
	(0.01842)	(0.86161)			
Sep. 1991	0.25339	6.94233	-441.16	0.0757	0.0143
	(0.01487)	(0.40827)			
Dec. 1991	0.14790	5.15174	-428.05	0.0979	0.0144
	(0.01723)	(0.59136)			
Mar. 1992	0.00252	-9.88192	-423.41	0.0643	0.0143
	(0.00206)	(2.58737)			
Jun. 1992	0.01996	-2.20689	-451.78	0.0529	0.0073
	(0.01733)	(3.29487)			
Sep. 1992	0.14419	4.30487	-451.23	0.0520	0.0097
	(0.02909)	(1.12774)			
Dec. 1992	0.00355	-8.87842	-464.97	0.0606	0.0117
	(0.00166)	(1.48274)			
Mar. 1993	0.04240	0.88222	-429.60	0.0491	0.0064
	(0.00862)	(0.78970)			
Jun. 1993	0.01935	-2.38675	-460.29	0.0503	0.0065
	(0.00527)	(0.97900)			
Sep. 1993	0.03015	0.07056	-478.99	0.0504	0.0065
	(0.00786)	(1.00500)			
Dec. 1993	0.01298	-3.11939	-488.15	0.0509	0.0065
	(0.00516)	(1.42447)			
Mar. 1994	0.01398	0.33134	-496.79	0.0577	0.0076
	(0.00038)	(0.39318)			

Data represent 90-day bank bill futures contracts traded on the Sydney Futures Exchange deliverable in the near expiry month i.e. March, June, September or December. The numbers in parentheses below the coefficients are standard errors obtained from the covariance matrix generated by the inverse of the last Hessian computed during the optimisation process.

Table 2
Results of Estimation Using 3-Month Euroyen Futures Data

Contract	a_0	λ	Log Likelihood	Implied Spot Rate Mean	Implied Spot Rate Std. Dev.
Sep. 1989	0.02554	0.15839	-426.99	0.0486	0.0068
	(0.00009)	(0.40451)			
Dec. 1989	0.00791	-7.56936	-415.26	0.0556	0.0086
	(0.00001)	(0.29568)			
Mar. 1990	0.01604	-3.39943	-433.12	0.0475	0.0064
	(0.00003)	(0.33377)			
Jun. 1990	0.00334	-7.62007	-457.84	0.0577	0.0098
	(0.00005)	(0.30590)			
Sep. 1990	0.06639	2.23176	-479.35	0.0390	0.0075
	(0.00005)	(0.33775)			
Dec. 1990	0.00689	-6.05398	-442.80	0.0548	0.0080
	(0.00003)	(0.31957)			
Mar. 1991	0.02231	-3.21829	-408.67	0.0486	0.0065
	(0.00006)	(0.34614)			
Jun. 1991	0.02215	-1.46132	-451.60	0.0469	0.0063
	(0.00006)	(0.34614)			
Sep. 1991	0.03269	-0.54846	-465.08	0.0500	0.0064
	(0.00005)	(0.34715)			
Dec. 1991	0.00397	-6.94571	-467.26	0.0558	0.0085
	(0.00005)	(0.30688)			
Mar. 1992	0.03274	-0.99355	-417.11	0.0520	0.0070
	(0.00004)	(0.37844)			
Jun. 1992	0.01012	-2.06550	-485.44	0.0485	0.0064
	(0.00013)	(0.35077)			
Sep. 1992	0.06612	2.11638	-460.90	0.0527	0.0070
	(0.00004)	(0.38723)			
Dec. 1992	0.01176	-1.95498	-486.95	0.0481	0.0063
	(0.00013)	(0.34432)			
Mar. 1993	0.00433	-4.98060	-485.76	0.0530	0.0073
	(0.00045)	(0.04786)			
Jun. 1993	0.00267	-6.33375	-480.14	0.0540	0.0079
	(0.00028)	(0.04564)			
Sep. 1993	0.03741	2.16001	-505.41	0.0495	0.0064
	(0.00011)	(0.39089)			
Dec. 1993	0.01307	-2.55519	-453.77	0.0516	0.0069
	(0.00009)	(0.34764)			
Mar. 1994	0.04941	0.61786	-436.10	0.0449	0.0007
	(0.00005)	(0.36037)			

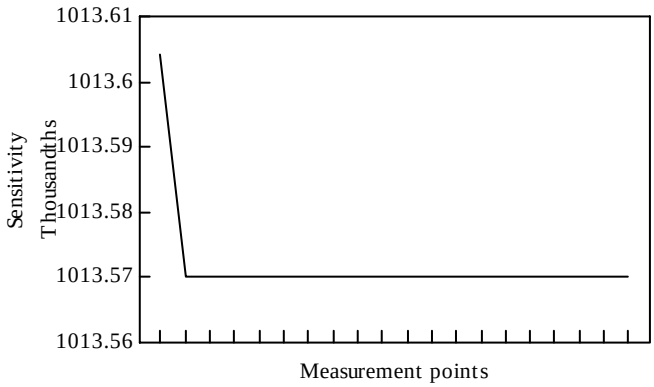
Data represent 3-month euroyen futures contracts traded on the Tokyo International Financial Futures Exchange deliverable in the near expiry month i.e. March, June, September or December. The numbers in parentheses below the coefficients are standard errors obtained from the covariance matrix generated by the inverse of the last Hessian computed during the optimisation process.

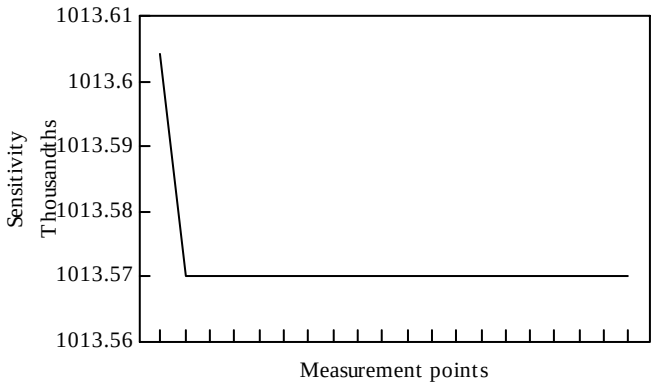
Table 3
Residual Diagnostics for Both Datasets

Contract	90-Day bank bill futures		3-Month euroyen futures	
	Q(5)	JB	Q(5)	JB
Sep. 1989	5.03 (0.412)	3.55 (0.169)	13.49 (0.019)	10.02 (0.007)
Dec. 1989	4.29 (0.509)	1.68 (0.434)	5.09 (0.404)	20.98 (0.000)
Mar. 1990	3.75 (0.586)	28.94 (0.000)	7.15 (0.209)	2.72 (0.256)
Jun. 1990	3.55 (0.615)	6.35 (0.049)	1.45 (0.917)	1.10 (0.575)
Sep. 1990	2.65 (0.753)	51.37 (0.000)	5.33 (0.377)	4.55 (0.102)
Dec. 1990	1.98 (0.851)	24.60 (0.000)	6.84 (0.232)	22.65 (0.000)
Mar. 1991	2.88 (0.718)	10.27 (0.006)	4.46 (0.484)	56.96 (0.000)
Jun. 1991	5.24 (0.387)	11.67 (0.001)	2.43 (0.786)	11.83 (0.003)
Sep. 1991	6.35 (0.273)	59.37 (0.000)	2.48 (0.778)	11.23 (0.004)
Dec. 1991	7.15 (0.209)	17.26 (0.000)	4.43 (0.489)	4.50 (0.105)
Mar. 1992	7.34 (0.196)	3.42 (0.181)	8.88 (0.114)	0.28 (0.868)
Jun. 1992	22.28 (0.000)	31.45 (0.000)	1.13 (0.951)	0.13 (0.935)
Sep. 1992	6.43 (0.266)	22.10 (0.000)	2.61 (0.759)	3.67 (0.159)
Dec. 1992	3.61 (0.607)	19.86 (0.000)	2.56 (0.767)	1.24 (0.537)
Mar. 1993	2.31 (0.805)	16.46 (0.000)	3.39 (0.639)	3.20 (0.201)
Jun. 1993	2.39 (0.793)	58.30 (0.000)	5.46 (0.363)	10.91 (0.004)
Sep. 1993	1.41 (0.923)	2.09 (0.350)	8.33 (0.139)	0.55 (0.761)
Dec. 1993	2.11 (0.833)	19.05 (0.000)	10.91 (0.053)	65.05 (0.000)
Mar. 1994	1.42 (0.922)	1.75 (0.417)	2.17 (0.825)	2.25 (0.324)

Numbers in parentheses are p-values. Q(5) represents the Ljung-Box Q-statistic of lag 5 for residual correlations. Under the null hypothesis of no serial correlations this statistic has a Chi-square distribution with 5 degrees of freedom. JB represents Jarque-Bera statistic for test of normality and it has a Chi-square distribution with two degrees of freedom. For p-values less than 0.010, the corresponding null hypothesis can be rejected at 0.010 level of significance.

Figure 1
Extended Kalman Filter: Error Sensitivity



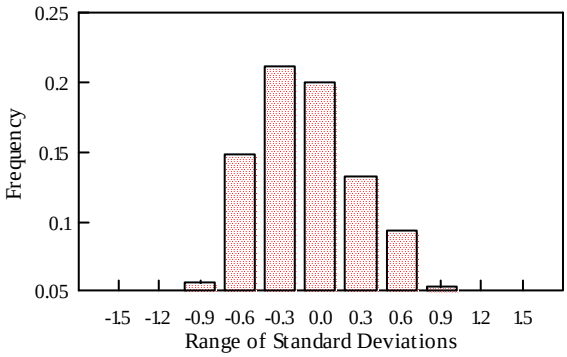


Filter performance analysed using the March 1994 90-day bank bill futures contracts (upper panel) and 3-month euroyen futures contracts (lower panel). In both cases $\gamma = 0.5$. Filter sensitivity is almost identical with both data sets.

Sensitivity is measured as the ratio $\dfrac{\left\|\delta P_{k+1|k}\right\|}{\left\|\delta P_0\right\|}$, where $\|A\| \triangleq \Sigma a_{ij}^2$

P_0 , $P_{k+1|k}$ are prior covariance and forecast covariance of the state vector at k+1 based on the information at k, respectively. δP_0 refers to the two cases where P_0 is specified as I*100 and I*10000, where I is the identity matrix.

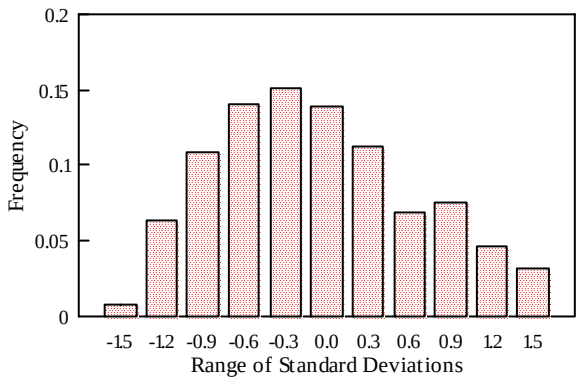
Figure 2
Bootstrap Sampling Distribution (a_0)



Percentile	
0.025	0.02240
0.050	0.02600
0.100	0.02948
0.500	0.04508
0.900	0.07697
0.950	0.09678
0.975	0.11015

Frequency distribution of the parameter from 1000 bootstrap resampling using March 1994 90-day bank bill futures data. Sample mean and standard deviation are 0.04983, 0.02530 respectively.

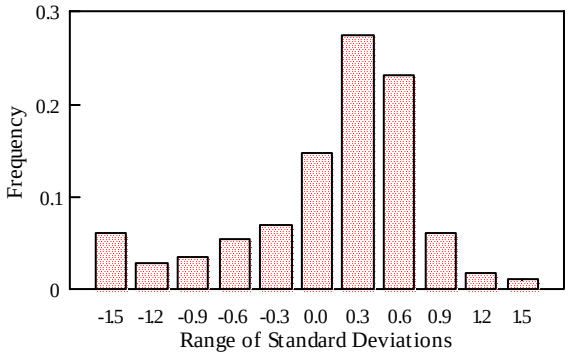
Figure 3
Bootstrap Sampling Distribution (λ)



Percentile	
0.025	-1.70916
0.050	-1.44495
0.100	-1.10233
0.500	0.78716
0.900	4.07776
0.950	5.18079
0.975	6.63965

Frequency distribution of the parameter from 1000 bootstrap resampling using March 1994 90-day bank bill futures data. Sample mean and standard deviation are 1.18781, 2.09292 respectively.

Figure 4
Bootstrap Sampling Distribution (Pivotal Quantity)



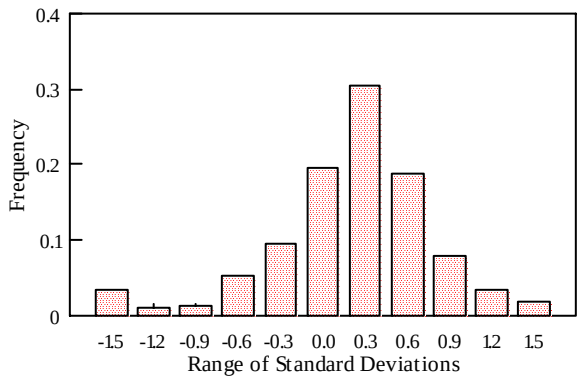
Percentile	
0.025	-10.92
0.050	-8.90
0.100	-6.22
0.500	-0.54
0.900	1.85
0.950	4.08
0.975	7.58

Frequency distribution of the pivotal quantity defined below from 1000 bootstrap resampling using March 1994 90-day bank bill futures data. This bootstrap-t distribution provides improved estimates of confidence interval compared to ordinary bootstrap distribution where median bias may be present. (See Efron (1987)):

$$\text{Pivotal quantity} = \text{Bootstrap-t} = \left(a_o^B - a_0\right) / SE(a_o^B)$$

The superscript B denotes bootstrap estimate and SE stands for standard error.

Figure 5
Bootstrap Sampling Distribution (Pivotal Quantity)



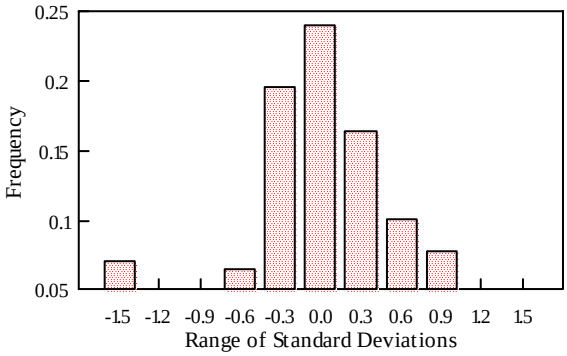
Percentile	
0.025	-9.47
0.050	-5.82
0.100	-4.08
0.500	-0.35
0.900	2.50
0.950	4.01
0.975	5.59

Frequency distribution of the pivotal quantity defined below from 1000 bootstrap resampling using March 1994 90-day bank bill futures data. This bootstrap-t distribution provides improved estimates of confidence interval compared to ordinary bootstrap distribution where median bias may be present. (See Efron (1987)) :

$$\text{Pivotal quantity} = \text{Bootstrap-t} = \left(\lambda^B - \lambda\right) / SE(\lambda^B)$$

The superscript B denotes bootstrap estimate and SE stands for standard error.

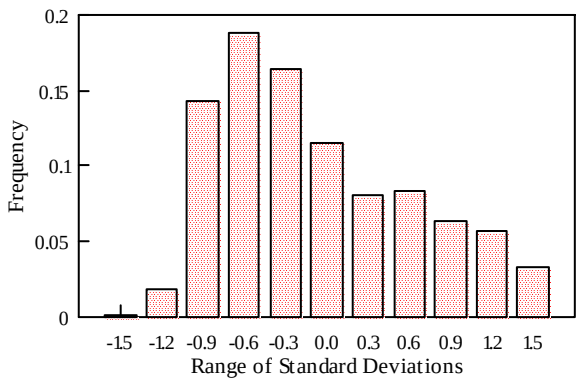
Figure 6
Bootstrap Sampling Distribution (a_0)



Percentile	
0.025	-0.09099
0.050	-0.06727
0.100	0.08015
0.500	0.14820
0.900	0.26931
0.950	0.33377
0.975	0.40010

Frequency distribution of the parameter from 1000 bootstrap resampling using March 1994 3-month euroyen futures data. Sample mean and standard deviation are 0.15667, 0.11135 respectively.

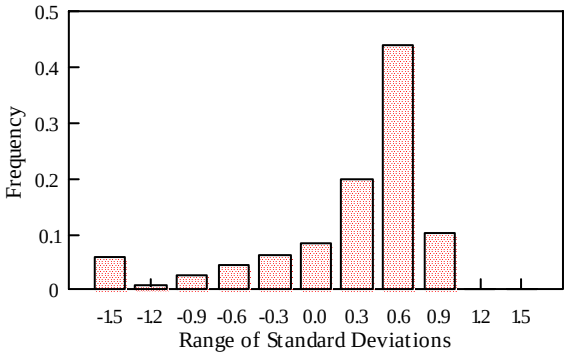
Figure 7
Bootstrap Sampling Distribution (λ)



Percentile	
0.025	-0.96854
0.050	-0.82212
0.100	-0.56603
0.500	1.05958
0.900	4.88134
0.950	6.17853
0.975	7.48682

Frequency distribution of the parameter from 1000 bootstrap resampling using March 1994 3-month euroyen futures data. Sample mean and standard deviation are 1.71662 , 2.26695 respectively.

Figure 8
Bootstrap Sampling Distribution (Pivotal Quantity)



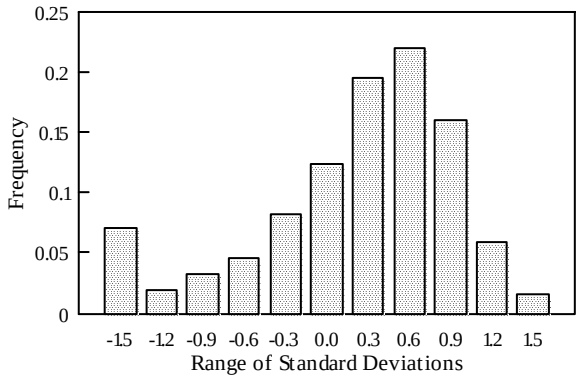
Percentile	
0.025	-25.29
0.050	-14.14
0.100	-8.60
0.500	-0.37
0.900	1.54
0.950	1.82
0.975	1.98

Frequency distribution of the pivotal quantity defined below from 1000 bootstrap resampling using March 1994 3-month euroyen futures data. This bootstrap-t distribution provides improved estimates of confidence interval compared to ordinary bootstrap distribution where median bias may be present. (See Efron (1987)) :

$$\text{Pivotal quantity} = \text{Bootstrap-t} = \left(a_o^B - a_0\right) / SE(a_o^B)$$

The superscript B denotes bootstrap estimate and SE stands for standard error.

Figure 9
Bootstrap Sampling Distribution (Pivotal Quantity)



Percentile	
0.025	-14.56
0.050	-10.77
0.100	-7.44
0.500	-0.76
0.900	2.27
0.950	3.22
0.975	4.02

Frequency distribution of the pivotal quantity defined below from 1000 bootstrap resampling using March 1994 3-month euroyen futures data. This bootstrap-t distribution provides improved estimates of confidence interval compared to ordinary bootstrap distribution where median bias may be present. (See Efron (1987)) :

$$\text{Pivotal quantity} = \text{Bootstrap-t} = \left(\lambda^B - \lambda\right) / SE(\lambda^B)$$

The superscript B denotes bootstrap estimate and SE stands for standard error.