Equilibrium Equity Premium, and Interest Rate of a Large-Firm Economy in the Presence of Moral Hazard

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Abstract

We present an equilibrium model of a moral-hazard economy with a very large firm and financial markets, where a stock and bonds are traded. We show that optimal contracts necessitate the principal to forbid the agent to trade the stock, and that moral hazard problems can result in low interest rates and sometimes in high equity premium. We also obtain a number of striking results: the second best cost of capital for a risky production asset can be lower than that of the first best; and the market price of risk depends on agent’s productivity as well as the risk aversion of both the principal and agent, and the volatility of the production asset. Moreover, comparative statics for both the first- and second-best cases suggest that the larger the social wealth, the lower the interest rate and the higher the market price of risk; and that low interest rates can also be resulted from low agent’s effort efficiency, low-profit production opportunities, and high production risk.


Keywords: equity premium, interest rate, cost of capital, moral hazard, general equilibrium, optimal contract.

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1 Introduction

We present an equilibrium model of product and capital markets under moral hazard, where the representative principal/investor (she) optimally invests in a production technology and hires an agent (he) to manage it, and then she manages her wealth/portfolio in financial markets where the stock of the production asset and risk-free bonds are traded. In real life, our model can be viewed as a modest benchmark close to economies with very large firms whose corporate decisions nontrivially affect not only the aggregate products but systematic risks of their own economies.¹

In this paper, we examine how moral hazard problems can affects interest rates, equity premium, asset prices, production investment decisions and agents’ compensation contracts, when both product and capital markets are in equilibrium. The importance of these issues should be self-obvious. Fundamental sources of values of asset prices lie in productions whose risks and profitabilities are affected by both the principal’s real investment decisions and agency problems, which are in turn affected by asset prices. Thus, any attempts to understand asset prices ignoring productions decisions and potential agency problems can be at best incomplete.

The economy in this paper is structured with problems of contracting, portfolio management and general equilibrium. In order to obtain specific economics of moral hazard in general equilibrium, we have our economy much more simplified than Prescott-Townsend [1984] and others do. (Shortly, we shall discuss some of the moral-hazard general equilibrium literature in relation to this paper.) In particular, we simplify the economy by allowing only numeraire goods, and by assuming both principal’s and agent’s preferences are represented by utility functions exhibiting constant absolute risk aversion (CARA).

However, unlike most general equilibrium moral-hazard models in the literature, we allow the principal to make an irreversible initial real investment (project selection/capital budgeting) decision, to hire an agent to manage the real asset, and to issue one share on her real asset at securities markets at time zero. Then, the agent continuously produces numeraire consumption goods using the real asset, and all investors including the principal but excluding the agent continuously trade in securities markets. We provide a closed-form solution to our model, and believe that ours is the first closed-form solution integrating optimal contracts into capital/financial markets in general equilibrium.

The assumption of the agent trading restriction is frequently made in the literature for tractability. See, for instance, Albuquerque and Wang [2008], and Gorton and He [2007]. We, however, show that this assumption is without loss of generality, because if the agent were allowed to trade the asset he manages, he could optimally (yet partially in general) undo the contract in securities market and exert suboptimal, although not necessarily zero, effort, which could result in a decrease in the principal’s expected utility. See Section A in the Appendix.

The irreversibility of the principal’s initial real investment can arise, because unlike financial investment decisions, real investment decisions like building factories may not be adjusted continuously. We believe, it is important for a general equilibrium model to have both initial real investment and subsequent production decisions endogenized, particularly when the model is intended to compare asset prices of a moral hazard economy with those of a classical economy.

¹Real life examples may include Chinese Petroleum in China and Samsung Group in Korea. Samsung Group accounts for more than 30 percent of the entire market capitalization of the KOSPI Index (Korea Composite Stock Price Index). Chinese Petroleum explains about 25 percent of Shanghai Composite Index, according to an announcement made by Shanghai Stock Exchange in November 2007.
The reason is that both decisions can affect not only the dollar-productivity but dollar-risk levels of involved real assets on which stock prices critically depend. We shall show that when the decisions are endogenized, a number of results for otherwise similar economies can be easily reversed.

Also, unlike most exiting general equilibrium models with moral hazard, we obtain closed-form solutions to our first-best and second-best general-equilibrium problems, which enable us to focus on properties of solutions beyond the existence of equilibria. Then, we examine how moral hazard can affect the cost of capital, interest rates and equity premia in equilibria, taking into account the principal’s decisions on real investment, contracting and portfolio management.\(^2\)

Let us first start with one of most important yet misunderstood issues in corporate finance: the cost of capital in the presence of agency problems. It is popularly believed that, because of agency costs, the second-best cost of capital for a risky investment is higher than that of the first best, and thus, the second-best real investment level is lower than the first best.\(^3\) However, we argue for the opposite, namely, the second-best cost capital can be lower than that of the first best.

As is well known, in the presence of financial markets, the optimal real investment decision rule is to equate the marginal (net) product to the marginal cost of capital. This rule remains unchanged even in the second best world. The marginal product is determined in the product market by the agent effort, and the marginal cost of capital is determined in the capital market through competition among investors. In this paper, since investors are represented by the principal, the cost of capital is reflective of the principal’s valuation of risks or the principal’s risk premium.

In order to understand differences in the first and second-best real investment decisions, let us first recall that in the first best, the cost of capital for a risky real (production) asset is the riskfree rate plus the market price of the total risk of the asset. However, in the second best, the cost of capital is the riskfree rate plus the market price of the residual-claim risk. The reason is that the principal in the second best is interested in the residual claim or her share of the total risk, whereas in the first best, she is, in effect, only concerned with the size of the total risk or the 'whole pie size.'\(^4\) Since the residual claim is only a fraction of the total production, if both the riskfree rate and the market price of (unitary) risk were the same across the first best and second best economies, then for a given level of real investment, the second best cost of capital would be lower than that of the first best. In fact, given a real investment level, the low second-best cost of capital still holds in equilibrium, because, as will be discussed shortly, both the second-best interest rate and the market price of risk are lower than those of the first best.

Then, would the low second-best cost of capital imply that the second best level of real investment should be higher than that of the first best? The answer is not necessary affirmative.

\(^2\)See, for example, Cochrane [2001] and references therein for determinants of interest rates and equity premia in classical economies.

\(^3\)Stulz [1995], in a keynote speech, expounds that Japanese investors find their cost of capital lower than Americans do, because Japanese have lower agency costs of managerial discretion due to differences in the organization of firms and/or investment opportunities. Romer [1996] argues that “the agency costs arising from asymmetric information raise the cost of external finance, and therefore discourage investment.”

\(^4\)In the first best, the principal is still interested in the residual claim. However, as shall be seen in the text, the residual claim risk depends only on the total risk, agent’s risk aversion and market price of risk, and is independent of the agent’s effort productivity, because in the first best only risk-sharing matters and is independent of the agent’s effort productivity. Therefore, as far as real asset risk is concerned, the price-taking principal behaves as if she were only interested in the total risk.
The reason is that the cost of capital simply determines the right hand side of the equation. In the left hand side, there is marginal product. Given a real investment level, the second best net production is always lower than the first best because of the agency cost which is the risk premium to be paid to the agent for the incentive compatibility.\(^5\) Thus, in spite of the low second best cost of capital, the second best investment level can be lower than that of the first best, if the marginal agency cost is sufficiently high. Otherwise, an overinvestment problem can arise, i.e., the second best investment can be larger than that of the first best.

As is the case with most of the existing literature on the cost of capital including Stulz [1995] and Romer [1996], the above argument, however, is based on partial equilibrium because capital-market-determined interest rates and market price of risk are held the same across the first and second-best economies. In full equilibrium, the comparison between the first- and second-best real investment decisions becomes even more complicated as the decisions can be drastically affected not only by the presence of the marginal agency cost, but by differences in interest rates and market prices of risk across the first- and second-best economies. In order to obtain a concrete insight into full equilibrium, we consider a special case of our economy to argue that in equilibrium with all of interest rates, the market price of risk and investment levels endogenized, both under- and over-investment problems can occur in the second best economy, even when the marginal agency cost is zero. We discuss each of those endogenized variables, in turn.

We argue that the second-best equilibrium interest rate is lower than the first best. This result provides an additional explanation of the well-known “riskfree rate puzzle,” presented by Weil [1989] who argues that empirically observed (riskfree) interest rates are too low to justify investors’ high risk aversion. Weil conjectures that high equity premia and low interest rates may be caused by high idiosyncratic risks of consumption risks of individual investors. We show that moral hazard problems can also be a cause of low interest rates.

To see this intuitively, note that, given (monetary) endowment, equilibrium interest rates depend on demand for capital which hinges on the marginal cost of capital and the marginal product.\(^6\) For simplicity, let us assume the marginal agency cost is zero. Then, the demand for capital can be expressed in terms of a growth-adjusted cost of capital which is a market-determined interest rate plus the growth-adjusted risk premium that is a market-determined risk premium minus the growth rate of future cash flows from the asset. However, it can be shown that the capital market (investors) demands higher growth-adjusted risk premia on risky real assets in the second best economy than it does in the first best, mainly because the second best cash-flow growth rates are lower than the first best.\(^7\)

Thus, if the interest rate were held constant, the demand for capital in the second best would be lower than that of the first best. This means, given a fixed amount of endowment, higher current consumption in the second best than it does in the first best, and as a result, the principal expects low future incomes from her real investment. Motivated to smooth out

\(^5\)Note that the agency cost is incurred in the product market, reducing expected future cashflows from the asset. Since the agency cost is not incurred in the capital market, it is not correct to say that the cost of capital increases because of the agency cost.

\(^6\)We ignore other motivations for money demand such as precautionary balances, as our economy has complete and frictionless financial markets.

\(^7\)A caution: the risk premium on a real asset (the cost of capital minus the interest rate) can be different from the equity premium on a stock (the cost of equity capital), because the stock is only a partial claim to the total cash flow from the real asset as the total cash flow is shared between the principal and agent. When the principal makes her project selection decision, she is concerned with the adjusted cost of total-asset capital, not with the cost of equity or the equity premium.
her intertemporal consumption plan, the principal is willing to give up some of the current consumption for future consumption by supplying extra capital to the market. With additional capital supplied, the second-best interest rate becomes lower than that of the first best.

However, unlike the above interest-rate comparison, we find that the second-best equity premium can be either higher or lower than the first best. This result contrasts with numerical examples provided by both Kahn [1990] and Kocherlakota [1998]. Kahn argues that moral hazard can help explain the high equity premium puzzle first raised by Mehra and Prescott [1985], whereas Kocherlakota claims that moral hazard can deepen the puzzle with too high riskfree rate and too low equity premium. However, economic interpretations/intuitions about their numerical results were not provided. We show that agency problems can result in low riskfree interest, and does not necessarily deepen the equity premium puzzle.

In order to understand our result intuitively, note that the equity premium on the real asset (or the market portfolio) can be expressed as a discounted dollar-risk premium on the risk of the principal’s future residual claim divided by the current stock price. Thus, the equity premium is positively related to the dollar-risk premium, and inversely related to both the interest (discount) rate and the stock price. However, all these three factors interact with each other in a highly complicated manner.

Note that (see Section 6.3.3) holding the real-investment level constant, the second-best dollar-risk premium is lower than the first best, because the principal’s residual claim in the second best is a smaller fraction of the real asset than it is in the first best. Thus, the dollar risk-premium factor can contribute to a decrease in the second best equity premium. Second, the low second-best interest rate contributes to an increase in the second-best equity premium. Therefore, holding the real investment level constant, if the second-best interest rate is sufficiently lower than those of the first best, then the second-best equity premium can be higher than that of the first best. Otherwise, the opposite can result.

Together with the riskfree rate, the other fundamental capital market variable is the market price of risk. In the first-best (classical) world, the market price of risk is proportional to the two factors: the risk aversion of the principal and agent, and the total production risk. The second-best market price of risk depends not only on the same two factors, but on the agent’s effort efficiency because of the agent’s incentive compatibility condition. The reason is that in the second best, the principal (investors) is concerned with the residual claim risk which is affected by the sensitivity of the compensation contract which in turn depends on the agent’s effort efficiency.

Above discussions are based on our closed-form solutions to equilibrium interest rates, equity premia, real investment levels, and market prices of risk. The closed-form solutions also enable us to obtain a number of comparative statics with respect to various model parameters. For example, we show that in both the first- and second-best economies, the larger the current social wealth (such as GDP (gross domestic product) or GNI (gross national income)), the lower the interest rate and the higher the market price of risk. Our comparative statics further confirm somewhat intuitive results: high interest rates can be resulted from high managerial effort efficiency, or from either profitable or safe production opportunities. These results are intuitively clear since the demand for capital increases with the managerial effort efficiency, the expected profitability of production opportunities, and decreases in the riskiness of the asset.

This paper is related to the classical theory of investment by Fisher (1930) and Hirshleifer

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8Recently, Gorton and He [2006] also show that the market price of risk can depend on the risk aversion of agents and the ownership structure, which is consistent with one of our results.
(1958, 1970), better known as the Fisher separation theorem of consumption and investment decisions. Extending the Fisherian world of consumption and investment decisions by incorporating moral hazard and portfolio management problems into the principal’s production and consumption decisions, our model enables us to investigate effects of moral hazard on production decisions, interest rates, and equity premium in general equilibrium.

This paper is also related to the literature on general equilibrium with moral hazard. Prescott and Townsend [1984] present a general equilibrium model of moral hazard where a central planner designs contracts for all agents, and argue that a constrained competitive equilibrium can exist and implement a (constrained) Pareto-efficient allocation. There are extensions of Prescott-Townsend’s seminal work. Bisin and Gottardi [1999] introduces financial markets to a Prescott-Townsend economy. Zame [2007] generalizes Prescott-Townsend, allowing each agent to work for many firms. However, Prescott-Townsend economies do not model conflicts of interest among members of the economies that may arise in designing contracts. Citanna and Villanacci [2002] incorporate contracting problems into a general equilibrium framework for an economy with commodity markets but without financial markets, and show the existence of equilibria.9

In spite of their important contributions to general equilibrium theories with moral hazard, the above studies do not provide clear guidance about properties beyond the existence of their equilibria, nor do they explain how moral hazard can affect production decisions and publicly traded asset prices in financial markets. Unlike those studies, we mainly focus on implications of product and capital equilibria on interest rates, equity premia/asset prices, production investment decisions and optimal contracts.

This paper is organized as follows: Section 2 describes the economy to be analyzed. In Section 3, we state the first- and second-best problems and provide a BSDE (Backward Stochastic Differential Equation) representation of expected exponential utility. Section 4 and 5 offer general characterization of equilibrium interest rate, market price of risk and stock price for the first- and second-best worlds. In Section 6, we consider a spacial case for which we solve both the first- and second-best problems in closed forms. Most of our economic intuitions are built on this section. We summarize main results of the paper in Section 7. Finally, in the Appendix, we discuss implications of restricting the agent from stock trading. The Appendix also contains most proofs of the results in the paper.

2 The Economy

Consider an economy with one representative investor (the principal) and one agent consuming numeraire goods. One may visualize an economy consisting of one large firm and many small investors who are represented by the representative investor, where the large firm is managed by the agent. In real life, the economy may be associated with conglomerate-driven economies like China and Korea.

The investor is endowed with initial wealth $M_P$ and a production opportunity. The agent is endowed with initial wealth $M_A$ and some human capital. We assume the production op-

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9Albuquerque and Wang [2008] present a general equilibrium model for an economy comprised of controlling and noncontrolling shareholders and argue that agency problems can lead to overinvestment problem larger risk premia and higher interest rates. However, they do not consider contracting problems. In contrast, our equilibrium agency problems result in lower (as opposed to ‘higher’) interest rates and can produce both over- and underinvestment problems.
portunity is so proprietary that the ownership cannot be transferred to the agent at or before time 0, although it can be traded after time 0. The economy is uncertain with the source of uncertainty given by \( B^0_t \), a standard Brownian motion on probability space \((\Omega, P^0)\). Let \( F_t \) be an augmented sigma algebra generated by \( \{B^0_s : 0 \leq s \leq t\} \).

At time zero, the principal establishes a firm based on her production opportunity to produce numeraire goods with her initial investment of \( I_0 \), and she hires the agent to manage the firm by signing a contract, or a compensation scheme \( C_T \). Initially, the principal assumes the full ownership of the firm by holding one share of the firm.

After time zero, the accounting value (cumulative production) process of the firm over time before compensation to the agent, denoted by \( \{D_t\} \), evolves according to the following dynamics:

\[
dD_t = g(I)dB^0_t,
\]

with \( D_0(I) \) being the initial value. We assume that the process \( \{D_t\} \) is public information, and that both \( g \) and \( D_0 \) are increasing and concave in \( I \).

The agent exerts effort \( \{\mu_t\} \) during the contract period \([0, T]\) to change the probability measure of \( \{D_t\} \) from \( P^0 \) to \( P^\mu \) such that

\[
dP^\mu \over dP^0 = M^\mu_T = \exp \left\{ -\frac{1}{2} \int_0^T \left( \frac{f(\mu_s, I)}{g(I)} \right)^2 ds + \int_0^T \frac{f(\mu_s, I)}{g(I)} dB^0_s \right\},
\]

where \( f(\mu_s, I) \) is concave in \((\mu_s, I)\). Then, by the Girsanov theorem,

\[
B^\mu_t = B^0_t - \int_0^t \frac{f(\mu_s, I)}{g(I)} ds
\]

is a standard Brownian motion under \( P^\mu \), and

\[
dD_t = f(\mu_t, I)dt + g(I)dB^\mu_t.
\]

However, \( P^\mu \) is neither observable nor verifiable.\(^{10}\) These dynamics of the cumulative cash flow process can be interpreted as an outcome process that can be affected by both the agent’s effort \( \mu_t \) and the principal’s real investment/project selection decision, \( I.\)\(^{11}\) For the instantaneous effort \( \mu dt, \) the agent incurs personal instantaneous monetary cost of \( h(\mu_t)dt, \) where \( h \) is increasing and convex. His personal value of the total cumulative cost of effort during the contract period is \( \int_0^T h(\mu_t)dt. \)

There are capital markets where one stock and bonds are traded. The stock represents claims to \( D_T - C_T \) with a total of one share outstanding. Thus the market value of the stock depends on investors’ belief on agents’ optimal effort levels. Let their belief be \( \{\mu^*_t\}. \) We shall examine general equilibrium where this investors’ belief is fulfilled. Given the belief, the capital market is complete with the risk-neutral measure \( Q \) such that\(^{12}\)

\[
\frac{dQ}{dP^\mu} = Z^\mu_T = \exp \left\{ -\frac{1}{2} \int_0^T \theta^2_s ds - \int_0^T \theta_s dB^\mu_s \right\},
\]

\(^{10}\)In the literature, this formulation is called the weak formulation. See Sch"attler and Sung [1993].

\(^{11}\)See Sung [1995] for the project-selection interpretation. Of course if \( I \) is fixed, then the dynamics of the outcome becomes similar to that of Holmstrom and Milgrom [1987].

\(^{12}\)It is well-known that when markets are complete, moral hazard problems cannot arise under strong formulations of agency problems where the agent directly chooses the drift and/or volatility of the outcome. The reason is that since markets are complete, each sample path of the Brownian motion, i.e. \( \omega \in \Omega, \) can be objectively verified, and so can the agent’s controls of drifts and/or volatilities. Hence agency problems become trivial. However, this trivialization can be avoided under our weak formulation where the agent chooses a probability measure, i.e. he only indirectly chooses the drift. Thus given a realized sample path, there is no way for the principal to objectively verify the drift. Therefore, nontrivial agency problems still exist under our weak formulation.
Then, $\theta$ is called the market price of risk. Under $Q$, again by the Girsanov theorem,

$$B^\theta_t = B^{\theta^*}_t + \int_0^t \theta_s ds$$

is a standard Brownian motion. Note that the above risk-neutral system is also equivalent to saying that the capital market is complete under the original observable probability space $(\Omega, P^0, \mathcal{F}_t)$ with the following risk-neutral measure $Q$ such that

$$\frac{dQ}{dP^0} = \exp\left\{ -\frac{1}{2} \int_0^T \phi_s^2 ds - \int_0^T \phi_s dB^0_s \right\},$$

where $\phi_s = \theta_s - \frac{f(\mu^*_s, I)}{g(I)}$ and

$$B^\theta_t = B^0_t + \int_0^t \left( \theta_s - \frac{f(\mu^*_s, I)}{g(I)} \right) ds.$$

Let $E$, $E^\mu$ and $E^Q$ be expectation operators under probability measures $P, P^\mu$ and $Q$, respectively. That is, for a random variable $\xi$,

$$E^Q[\xi] = E^\mu[Z_T \xi] = E[M_T Z_T \xi].$$

Throughout the paper, we assume that risk-free interest (short) rate $r_t$ at time $t \in [0, T]$ is deterministic function of time $t$. Let

$$R_t := \exp\left( \int_0^t r_s ds \right) > 1.$$ 

Then, no arbitrage implies that the stock price can be computed as follows.

$$R_t^{-1} S_t = E^Q [R_T^{-1} (D_T - C_T) | \mathcal{F}_t].$$

That is, $R_t^{-1} S_t$ is a $Q$-martingale. Thus, by the martingale representation theorem, there exists a unique square integrable process $\tilde{\sigma}^S$ such that

$$dR_t^{-1} S_t = \tilde{\sigma}^S_t dB^\theta_t.$$ 

Note that $\tilde{\sigma}^S$ can be affected by the contract $C_T$. Let

$$\sigma^S_t := R_t \tilde{\sigma}^S_t.$$ 

Then, the the above stock price dynamics can also be written as follows.

$$dS_t = S_t r_t dt + \sigma^S_t dB^\theta_t.$$ 

Since we cannot guarantee $D_T > C_T$ in our model, in this paper, the stock price is not necessarily positive, and there is no exponential representation of the stock price.

### 2.1 Budget Constraints

Both the principal and agent consume at discrete dates 0 and $T$. Let $c^i_0$ and $c^i_T$ denote, respectively, the initial and terminal consumptions of an individual $i = P, A$. Right after the capital investment and their initial consumptions, (fractional) shares of the asset (the firm stock) as well as risk-free bonds are traded in the capital market. We assume that the principal can
freely trade in both stock and bond markets, and that the agent can freely trade in the bond market, i.e., to borrow and lend at (riskfree) interest rates. However, the agent is prohibited from trading the stock including all derivative markets related to the stock. The reason for this prohibition is to prevent the agent from undoing his contract in capital markets. See Proposition A.12 where it is be shown that it is optimal for the principal not to allow the agent to trade the stock. Although it is one of our main results, the proposition is provided in the Appendix for ease of exposition.

Let \( W_i^t, i = P, A \) be wealth levels at time \( t \) of individual \( i \) from capital market transactions after time 0. Then, the initial wealth levels for capital market transactions after the capital investment and their initial consumptions are as follows.

\[
W_P^0 = M_P - I - c_P^0 \\
W_A^0 = M_A - c_A^0
\]

Since he is not allowed to trade in the stock market, dynamics of the agent’s self-financed wealth process are simply given by

\[
dW_A^t = r_tD_W^t, \quad \text{or} \quad d(R_t^{-1}W_A^t) = 0.
\]

On the other hand, recall that the principal starts with one share of the firm as she issues one share to herself by investing \( I \) in the firm. Let \( \bar{\pi}_t \) be the additional number of shares held at time \( t \) by the principal. The principal chooses \( \bar{\pi}_t \) after the managerial contract is signed. (One may alternatively view the principal as a group of identical investors who are endowed with an aggregate endowment of one share and trade among themselves after the contract with the agent is signed.) Then, dynamics of the principal’s self-financed wealth processes are as follows:

\[
dW_P^t = (W_P^t - \bar{\pi}_tS_t)r_tdt + \bar{\pi}_tS_tdB_t
\]

which implies

\[
dR_t^{-1}W_P^t = \pi_t^P dB_t^P
\]

where

\[
\pi_t^P := R_t^{-1}\bar{\pi}_tS_t.
\]

**Definition 2.1** We say that both the product and capital markets are in equilibrium if and only if the optimal expected utilities of the principal and agent exist for all \( t \in [0, T] \), and both product and financial markets clear as follows.

\[
M_P = I + c_P^0 + W_P^0 \quad (2.2) \\
M_A = c_A^0 + W_A^0 \quad (2.3) \\
D_T = c_T^P + c_T^A \quad (2.4) \\
c_T^P = W_T^P + D_T - C_T, \quad (2.5) \\
c_T^A = W_T^A + C_T, \quad \text{and} \quad (2.6) \\
\bar{\pi}_S^P \equiv 0. \quad (2.7)
\]
Eq.’s (2.2) to (2.6) are for market clearing at time zero and $T$, and Eq. (2.7) is to ensure the equity market clear for all $t \in [0, T]$. Note that Eq.'s (2.4) to (2.6) imply

$$0 = W^P_T + W^A_T,$$

where

$$W^P_T = R_T W^P_0 + R_T \int_0^T \pi_s^P dB_s^\theta$$

$$= R_T W^P_0 + R_T \int_0^T \pi_s^P \left( \theta - \frac{f(\mu_t, I)}{g(I)} \right) dt + R_T \int_0^T \pi_s^P dB_s^\theta$$

$$W^A_T = R_T W^A_0.$$ 

Moreover, Eq. (2.7) implies

$$\pi_s^P = 0. \quad (2.8)$$

3 Problem Statements

The principal is endowed with an asset and hires an agent to manage the asset. The principal’s first-best problem is stated as follows.

**Problem 1** *(The First Best.)* Choose $C_T$ and $(c^P_0, c^A_0, \mu_t, \pi_t)$ to

$$\max E^\mu \left[ - \exp \left\{ -\gamma_P c^P_0 \right\} - \exp \left\{ -\gamma_P (W^P_T + D_T - C_T) \right\} \right]$$

s.t. $W^P_T = M^P - c^P_0 - I + \int_0^T r_t W^P_t dt + \int_0^T \bar{\pi}_t \sigma_t dB_t^\theta$

$$E^\mu \left[ - \exp \left\{ -\gamma_A c^A_0 \right\} - \exp \left\{ -\gamma_A \left( R_T (M^A - c^A_0) + C_T - \int_0^T h(\mu_t) dt \right) \right\} \right] \geq L.$$

The first constraint is the self-financing condition arising from the presence of capital markets, and the second is the agent’s participation constraint. In the first best world, both the principal and agent observe the agent’s initial consumption and effort levels as well as $\{D_t\}$. Thus, the principal can dictate the agent’s initial consumption level by writing a contract $C_T$ contingent on $c^P_0$.

The principal’s second best problem is as follows.

**Problem 2** *(The Second Best.)* Choose $C_T$ and $(c^P_0, c^A_0, \mu_t, \pi_t)$ to

$$\max E^\mu \left[ - \exp \left\{ -\gamma_P c^P_0 \right\} - \exp \left\{ -\gamma_P (W^P_T + D_T - C_T) \right\} \right]$$

s.t. $W^P_T = M^P - c^P_0 - I + \int_0^T r_t W^P_t dt + \int_0^T \bar{\pi}_t \sigma_t dB_t^\theta$

$$\mu \in \arg \max \limits_\mu E^\mu \left[ - \exp \left\{ -\gamma_A \left( R_T (M^A - c^A_0) + C_T - \int_0^T h(\mu_t) dt \right) \right\} \right]$$

$$E^\mu \left[ - \exp \left\{ -\gamma_A c^A_0 \right\} - \exp \left\{ -\gamma_A \left( R_T (M^A - c^A_0) + C_T - \int_0^T h(\mu_t) dt \right) \right\} \right] \geq L.$$
that the agent’s effort levels be chosen by the agent himself alone. If there are more than one optimal effort levels, then it is assumed that the agent chooses one that can improve the principal’s expected utility. Both the principal and agent are allowed to observe \(c_0^A\), and \(\{D_t\}\). However, the principal cannot observe and verify \(\{\mu_t\}\).

It is convenient to divide the time line into two stages. The first stage occurs at time 0, when the principal and agent consume \((c^P_0, c^A_0)\), and the principal decides on capital investment \(I\) and compensation scheme \(C_T\). The second stage is concerned with period \((0, T]\), during which the agent exerts continuous effort, and at time \(T\), both the principal and agent make another round of consumptions \((c^P_T, c^A_T)\).

For the principal’s problems in two stages, we also divide the agent’s participation constraint into two stages as follows.

\[
E^\mu \left[ -\exp \left\{ -\gamma_A c^A_T \right\} \right] \geq -\exp \left\{ -\gamma_A R \right\},
\]

\[
-\exp \left\{ -\gamma_A c^A_0 \right\} - \exp \left\{ -\gamma_A R \right\} \geq L,
\]

where \(R\) is an arbitrary number for the principal to decide. Given any \(R\), the principal designs \(C_T\) such that \(E^\mu \left[ -\exp \left\{ -\gamma_A c^A_T \right\} \right] \geq -\exp \left\{ -\gamma_A R \right\}\), and then she optimally chooses \(R\) subject to the agent’s participation constraint. Consequently, both the principal and agent’s problem can be solved in two stages backwardly: In the second stage, given the first-stage consumption and production decisions \((c^P_0, c^A_0, I)\) and the agent’s second-stage certainty equivalent wealth \(R\), the principal designs a compensation contract \(C_T(R, .)\) and then trade shares of the stock and bonds, whereas the agent, given \(c^A_0\) and \(C_T\), chooses effort levels as instructed by the principal, and trades bonds. In the first stage, the agent decides on his initial consumption \(c^A_0\), and the principal decides on \((c^P_0, I, R)\).

### 3.1 Backward Representation of Expected Utility

Before proceeding the analysis of problems stated in the last section, we find it convenient to represent both the principal’s and agent’s second-stage utility levels in forms of backward stochastic differential equations (BSDEs). (See Williams [2006] and Civitanić, Wan and Zhang [2007] for BSDE methods for principal-agent problems.) In this paper, the BSDE method turns out to be particularly helpful in characterizing unique equilibrium market-price-of-risk \(\{\theta_t\}\) for both the first- and second-best economies, as will be seen in Propositions 4.1 and 5.3.

We utilize the following lemma for the BSDE representations.

**Lemma 3.1 (Backward representation of expected utility.)** Consider a conditional expected utility function of the following form with its certainty equivalent wealth given by \(V_t\).

\[
-\exp\{-\gamma V_t\} := E^\mu \left[ -\exp \left\{ -\gamma (F(B^T_t) + \int_t^T H(\mu_s, \pi_s)ds + \int_t^T v(\mu_s, \pi_s)dB^0_s) \right\} \right| \mathcal{F}_t,
\]

Then, there exists a unique \(\mathcal{F}_t\)-predictable and square integrable processes \(\{Z_t\}\) such that \(V_t\) satisfies the following BSDE.

\[
V_t = F(B^T_T) + \int_t^T \left( \frac{Z_s}{\gamma} + v(\mu_s, \pi_s) \right) ds - \int_t^T \frac{Z_s}{\gamma} dB^0_s.
\]
4 The First Best

We start with the second-stage problems. We use Lemma 3.1 to transform the principal’s second-stage problem in the form of a BSDE.

**Corollary 4.1** There exists a unique \( \mathcal{F}_t \)-predictable and square integrable processes \( \{Z_t^A\} \) such that the first best salary function \( C_T \) can be represented as follows:

\[
C_T = R - R_T W_0^A - \int_0^T \frac{Z_s^A f}{\gamma A} g - h(\mu_s) - \frac{1}{2\gamma A}(Z_s^A)^2 \, ds + \int_0^T \frac{Z_s^A}{\gamma A} dB_s^0. \tag{4.9}
\]

Moreover, there exists a unique \( \mathcal{F}_t \)-predictable and square integrable processes \( \{Z_t^P\} \) such that the principal’s first-best second-stage problem is equivalent to choosing \( \{\pi_t^P, \mu_t, Z_t^A\} \) to maximize, for all \( t \in [0, T] \),

\[
V_t^P = R_T (W_t^P + W_t^A) + D_0(I) - R - \int_t^T \frac{Z_s^P}{\gamma P} dB_s^0
+ \int_t^T \left\{ R_T \pi_s^P \left( \gamma P \frac{Z_s^P}{\gamma P} + g \right) - h(\mu_s) - \frac{1}{2\gamma A}(Z_s^A)^2 \right. \\
\left. + \left( \frac{Z_s^P}{\gamma P} + g \right) \frac{\gamma P}{g} \left( \frac{Z_s^P}{\gamma P} - \frac{Z_s^A}{\gamma A} + g + R_T \pi_s^P \right)^2 \right\} ds. \tag{4.10}
\]

The salary representation (4.9) suggests that \( \frac{Z_t^A}{\gamma A} \) is the sensitivity process of the contract \( C_T \) to the outcome process \( \{D_s\} \). Thus, in the first best, the principal can design \( C_T \) by directly choosing \( \{\mu_s\} \) and \( \{Z_s^A\} \) to maximize her certainty equivalent process (4.10). Applying the BSDE Comparison Theorem to (4.10), the first order conditions (FOCs) for the principal’s maximization problem are as follows.

\[
h_\mu = \left( \frac{Z_s^P}{\gamma P} + g + R_T \pi_s^P \right) \frac{f_\mu}{g}, \tag{4.11}
\]

\[
\theta_s = \gamma P \left( \frac{Z_s^P}{\gamma P} - \frac{Z_s^A}{\gamma A} + g + R_T \pi_s^P \right), \tag{4.12}
\]

\[
Z_s^A = \theta_s. \tag{4.13}
\]

Note that for these FOCs, we have utilized an equilibrium condition \( f^* = f \). The FOCs describe how the principal chooses her decision variables (\( \mu_t, \pi_t^P \)) given market parameters (\( \{\theta_t\}, R_T \)). In the next subsection, we discuss how the parameters are determined in equilibrium through capital market competition.

4.1 Equilibrium

The following Proposition reveals the equilibrium structure of the equity-market parameter process \( \{\theta_s\} \). The proof is omitted, as it is similar to that of the second best.

**Proposition 4.1** Suppose that \( h(\mu) = (\delta/2)\mu^2 \), and \( f(\mu, I) = \mu a(I) \). Then a unique equilibrium exists, and in equilibrium, \( Z_t^P \equiv 0 \), and \( \theta_t \) and \( \mu_t \) are constant over time such that \( \theta_t = \theta \) and \( \mu_t = \mu \), where

\[
\theta = \frac{\gamma P \gamma A}{\gamma P + \gamma A} g(I), \quad \text{and} \quad h_\mu = f_\mu \quad i.e., \quad \mu = \frac{a(I)}{\delta}. \]
Proposition 4.1 tells us that if the drift of the production function is linear and the cost of effort is quadratic, then in equilibrium, \( \theta_s \) and \( \mu_s \) are constant over time. Under this linear-quadratic assumption, the principal’s optimal BSDE (4.10) becomes quadratic. Then, the mathematical result on the existence and uniqueness of solutions to quadratic BSDEs implies that \( \theta_s \) and thus \( \mu_s \) are constant over time.

The equilibrium pair \((\theta, \mu)\) in Proposition 4.1 is based on FOCs (4.11) and (4.13), equilibrium condition \(\pi_s^P \equiv 0\), and \(Z_t^P \equiv 0\). The Proposition, not surprisingly, indicates that the first best optimal effort level is determined by equating the marginal cost of effort \(h\mu\) to the marginal expected product of effort \(f\mu\).

However, even though the agent does not trade, the market price of risk \(\theta\) is affected by the agent’s risk aversion. This implication is consistent with Gorton and He [2006, WP]. The reason is as follows. Recall from classical asset pricing theories that the market price of risk typically depends on the representative investor’s risk aversion and the volatility of the market portfolio. In our agency world, the agent shares the outcome with the principal, and thus the principal’s residual claim on the asset is affected by the sharing rule with the agent. Since the sharing rule depends on the agent’s risk aversion, so does the volatility of the residual claim. Hence, the market price of risk is affected by the agent’s risk aversion.

If the production and cost functions are not linear-quadratic, the principal optimal BSDE may not be quadratic, and thus the existence and uniqueness of solutions to the BSDE may not be guaranteed. As a result, the uniqueness of the equilibrium may not be guaranteed, even though a constant pair of \((\theta, \mu)\) can still be consistent with an equilibrium. For the rest of this section, without losing essence of economics, we just assume the linear-quadratic case and thus \(\theta_s\) is constant over time.

4.1.1 The First-Stage Problem

FOCs (4.11) to (4.13), together with the equilibrium condition \(\pi_s^P \equiv 0\), imply the principal’s BSDE (4.10) has a solution with \(Z_t^P \equiv 0\) as follows: given \(R\), \(c_0^P\) and \(I\),

\[
V_t^P = R_T \left( W_0^P + W_0^A \right) + D_0(I) - R \left[ f - h(\mu_s) - \frac{1}{2\gamma_A} \theta^2 - \frac{\gamma_P}{2} \left( g - \frac{\theta}{\gamma_A} \right) \right] (T - t).
\]

For a complete description of the equilibrium, we now turn to the principal’s first-stage problem, which can be stated in equilibrium as follows. Choose \((c_0^P, I, R, \{\pi_t^P\}; c_0^A, \{\mu_t\})\) to

\[
\text{max } -\exp \left\{ -\gamma_P c_0^P \right\} - \exp \left\{ -\gamma_P V_0^P \right\} ,
\]

\[
\text{s.t. } -\exp \left\{ -\gamma_A c_0^A \right\} - \exp \left\{ -\gamma_A R \right\} \geq L ,
\]

where

\[
V_0^P = R_T \left( M_P + M_A - c_0^P - c_0^A \right) - R + G^{FB}(I; \theta) T ,
\]

\[
G^{FB}(I; \theta) := f - h(\mu_s) - \frac{1}{2\gamma_A} \theta^2 - \frac{\gamma_P}{2} \left( g - \frac{\theta}{\gamma_A} \right) - \frac{R_T I}{T} + \frac{D_0(I)}{T} ,
\]

and \(\mu\) satisfies \(h\mu = f\mu\). Note that the principal’s production decision \(I\) affects \(G^{FB}(I)\) which consists of several parts: expected earnings from production \(\frac{D_0(I)}{T} + f\); minus dollar cost of the
production capital $\frac{R_T I}{T}$; expected compensation for agent’s effort $h(\mu_s) + \frac{\sigma^2}{2\gamma_A}$; and the principal’s aggregate risk premium on both production and portfolio risks $\frac{g^2}{2\gamma_T} \left( \frac{\gamma_T}{2} (g - \frac{\sigma}{\gamma_T})^2 \right)$.

The substitution of the constraint into the principal’s utility function yields

$$\max - \exp \{-\gamma_P c^P_0\} - \exp \left\{-\gamma_P \left[ R_T (M_P + M_A - c^P_0 - c^A_0) + \frac{1}{\gamma_A} \ln \left( -L - e^{-\gamma_A c^A_0} \right) + G^{FB} T \right] \right\}.$$  

The FOCs are as follows:

$$R_T = \exp \{-\gamma_P (c^P_0 - V^P_0)\} \quad (4.14)$$

$$c^A_0 = -\frac{1}{\gamma_A} \ln \left( \frac{-R_T L}{1 + R_T} \right) \quad (4.15)$$

$$R = -\frac{1}{\gamma_A} \ln \left( -\frac{L}{1 + R_T} \right) \quad (4.16)$$

$$f_T + \frac{D^2_0 (I)}{T} = g_I \theta + \frac{R_T}{T} \quad (4.17)$$

The first FOC implies that current consumption decision is made such that the principal’s marginal rate of substitution between current and future certainty-equivalent consumption levels is equal to $R_T$. The second FOC also implies that $R_T = \exp \{-\gamma_A (c^A_0 - V^A_0)\}$. That is, at an equilibrium interest $R_T$, both the principal’s and agent’s intertemporal marginal rates of substitution are equalized. The third FOC indicates that the higher the interest rate, the higher the agent’s certainty equivalent future consumption, $R(= V^A_0)$.

The fourth FOC suggests that the optimal real investment decision $I$ equates the marginal product $f_T + \frac{D^2_0 (I)}{T}$ to the marginal cost of capital $g_I \theta + \frac{R_T}{T}$. This is consistent with the well-known fact that the optimal production/real investment decision is to produce until the marginal NPV (net present value) of production is equal to zero. Furthermore, this FOC also tells us that the investment decision is independent of preferences of both the principal and agent, which is consistent with the Fisher’s separation theorem on the independence of the consumption and investment decisions.

Given the above FOCs, the capital market parameters $(R_T, \theta)$ are determined by the market clearing conditions as follows.

**Proposition 4.2** Let $I(R_T, \theta) \in \arg\max_I G(I)$. Then the capital market parameters $(R_T, \theta)$ in equilibrium are jointly determined by the following two equations: $\theta = \frac{\gamma_A \gamma_P}{\gamma_A + \gamma_P} g(I)$ and

$$M_P + M_A = I (1 + R_T) - \left( \frac{1}{\gamma_P} + \frac{1}{\gamma_A} \right) \ln (R_T) + T G^{FB} (I(R_T, \theta); \theta). \quad (4.18)$$

**Proof:** By the FOCs (4.14) to (4.16), we have

$$(1 + R_T) \left( c^P_0 + c^A_0 \right) = - \left( \frac{1}{\gamma_P} + \frac{1}{\gamma_A} \right) \ln (R_T) + R_T (M_P + M_A) + T G^{FB} (I(R_T, \theta); \theta).$$

Then, the market clearing condition, $M_P + M_A = I + c^P_0 + c^A_0$, implies the statement. □

In Section (6), we use the two equations in Proposition 4.2 to investigate properties of $(R_T, \theta)$ for a special case. Next, we examine the equilibrium stock price.
4.2 Equilibrium Stock Price and Equity Premium

Suppose the first-best equilibrium with \( \mu(I), I(R_T, \theta) \) and \( (\theta, R_T) \) is determined by satisfying various FOCs and market clearing conditions as discussed in the last subsection. Then, one can easily find the first-best equilibrium stock price by computing the present value of the residual claim to the principal, i.e., \( S_0 = E_t^Q [R_T^{-1}(D_T - C_T)] \). As a result, dynamics of the first-best stock price are given as follows.

\[
dS_t = \left\{ rS_t + e^{-r(T-t)} \theta \left( g - \frac{\theta}{\gamma_A} \right) \right\} dt + e^{-r(T-t)} \left( g - \frac{\theta}{\gamma_A} \right) dB_t^p,
\]

with its initial stock price being

\[
S_0 = R_T^{-1} \left\{ D_0(I) + \left( f(\mu, I) - h(\mu_s) - \frac{1}{2\gamma_A} \theta^2 \right) T - \left( g - \frac{\theta}{\gamma_A} \right) \theta T + \frac{1}{\gamma_A} \ln \left( \frac{-L}{1+R_T} \right) + R_T(M_A - c_0^A) \right\}.
\]

(4.19)

The structure of the initial stock price \( S_0 \) suggests that it consists of the following three parts: \( e^{-rT} \left\{ D_0(I) + \left( f(\mu, I) - h(\mu_s) - \frac{1}{2\gamma_A} \theta^2 \right) T \right\} \), a surplus from production, or the present value of future product of effort net of future compensations for the agent’s effort and risk sharing; \( e^{-rT} \left( g - \frac{\theta}{\gamma_A} \right) \theta T \), a market-determined risk premium on the risky dollar-return on the stock; and \( (c_0^A - M_A) - \frac{1}{\gamma_A} e^{-rT} \ln \left( \frac{-L}{1+R_T} \right) \), an effective second-period labor-market opportunity cost to the agent. Note that the second-period agent’s certainty equivalent wealth is \( R \left( = -\frac{1}{\gamma_A} \ln \left( \frac{-L}{1+R_T} \right) \right) \), which is determined by the principal taking into account all future incomes to the agent from both the compensation \( C_T \) and agent’s own savings \( M_A - c_0^A \). Thus, the present value of the agent’s second-period reservation certainty equivalent to be paid by the principal is effectively reduced to \( e^{-rT} R (M_A - c_0^A) \).

On the other hand, dynamics of the stock price imply that the stock price volatility at time \( t \) is

\[
\sigma_t^S = e^{-r(T-t)} \left( g - \frac{\theta}{\gamma_A} \right) = e^{-r(T-t)} g \frac{\gamma_A}{\gamma_P + \gamma_A}.
\]

Note that the volatility comes from that of the discounted principal’s residual claim on the outcome. In order to compute the equity premium in our first best economy, let us assume \( S_0 > 0 \). Then, since the drift of the (dollar) stock price is \( rS_t + e^{-r(T-t)} \theta \left( g - \frac{\theta}{\gamma_A} \right) \), the drift of the rate of return on the stock at time 0 can be written as \( r + \nu_0 \), where

\[
\nu_0 = \frac{\theta \left( g - \frac{\theta}{\gamma_A} \right)}{e^{rT} S_0}.
\]

(4.20)

Since there is only one stock in our economy, \( \nu_0 \) represents the equity premium of the economy at time 0.

Eq.(4.20) indicates that the equity premium depends on the market price of risk, interest rate, and stock price. However, these three variables interact with each other in equilibrium. Later in Section 6, we compare this first-best equity premium with that of the second best under some simplifying assumptions.

5 The Second best

In the second-best world, the agency problem boils down to two constraints to the principal’s problem: participation and incentive constraints. As seen in the last section, we define \( R \) as the
agent’s second-stage certainty equivalent wealth at time 0, i.e., \(V_0^A = R\). Again, we use Lemma 3.1 to represent the principal’s certainty equivalent wealth process in the form of a BSDE.

**Corollary 5.2** The second-best salary function \(C_T\) satisfying the agent incentive compatibility condition with an agent certainty equivalent wealth level of \(R\) at time 0 can be represented as follows:

\[
C_T = R - R_TW_0^A - \int_0^T \left[ \frac{h\mu}{f\mu} f - h(\mu_s) - \frac{\gamma_A}{2} \left( \frac{h\mu}{f\mu} g \right)^2 \right] ds + \int_0^T \frac{h\mu}{f\mu} g dB_s^0. \tag{5.21}
\]

Moreover, there exists a unique \(\mathcal{F}_t\)-predictable and square integrable processes \(\{Z_t^P\}\) such that the principal’s second-best second-stage problem is equivalent to choosing \((\{\pi_t^P, \mu_t\})\) to maximize, for all \(t \in [0, T]\),

\[
V_t^P = R_T (W_0^P + W_0^A) + D_0(I) - R - \int_t^T Z_t^P dB_s^0
\]

\[
+ \int_t^T \left[ \left( \frac{Z_t^P}{\gamma_P} - \frac{h\mu}{f\mu} g - h(\mu_s) - \frac{\gamma_A}{2} \left( \frac{h\mu}{f\mu} g \right)^2 + R_T \pi_t^P \left( \theta_s - \frac{f^*}{g} + \frac{f}{g} \right) - \frac{\gamma P}{2} \left( \frac{Z_t^P}{\gamma_P} - \frac{h\mu}{f\mu} g + R_T \pi_t^P + g \right) \right] ds. \tag{5.22}
\]

The structure of the second-best salary representation (5.21) is well known since Holmstrom and Milgrom’s [1987] Brownian model.\(^{13}\) Note that as soon as the principal decides on \(C_T\), the capital market infers the agent’s optimal effort choice given \(C_T\), and determines the pricing kernel as explained in Section 2. The principal is aware of capital market responses to her choice of \(C_T\) on the agent’s optimal effort decisions, and thus she treats \(f^* = f\). By applying the Comparison Theorem to her BSDE (5.22) and using \(f^* = f\), the FOCs for the optimality are

\[
\theta_s - \gamma_P \left( \frac{Z_t^P}{\gamma_P} - \frac{h\mu}{f\mu} g + R_T \pi_t^P + g \right) = 0 \tag{5.23}
\]

\[
- h\mu - \gamma_A g^2 \frac{h\mu}{f\mu} \partial_\mu \left( \frac{h\mu}{f\mu} g \right) + \left( \frac{Z_t^P}{\gamma_P} + g \right) \frac{1}{g} f\mu + g \theta_\mu \left( \frac{h\mu}{f\mu} \right) = 0. \tag{5.24}
\]

As seen in the first-best case, these second-best FOCs also depend on the market-price-of-risk process \(\theta_t\). In the next section, we investigate equilibrium \(\theta_t\) and the principal’s first-stage problem.

### 5.1 Equilibrium

First, we let \(I\) fixed, and characterize the structure of capital market parameter process \(\{\theta_t\}\) by using capital market clearing conditions and the principal’s FOCs (5.23) and (5.24).

**Proposition 5.3** Suppose \(f(\mu, I) = \mu a(I)\) and \(h(\mu) = (\delta/2)\mu^2\). Given \((I, c_0^A, c_0^P, R)\), there exists a unique equilibrium. Moreover, in this unique equilibrium, \(Z_t^P \equiv 0\), and \(\theta_t\) and \(\mu_t\) are constant over time such that \(\theta_t = \theta\) and \(\mu_t = \mu\) where

\[
\theta = \frac{\gamma_P \gamma_A \delta g^3}{a^2 + (\gamma_A + \gamma_P) \delta g^2}, \quad \text{and} \quad \mu = \frac{a \left( 1 + \gamma_P \delta \left( \frac{g}{a} \right)^2 \right)}{\delta \left( 1 + (\gamma_A + \gamma_P) \delta \left( \frac{g}{a} \right)^2 \right)}. \]

\(^{13}\)See Schättler and Sung [1993] for general cases.
Recall that the first-best market price of risk $\theta$ depends only on the risk aversion of both the principal and agent, and the riskiness of the production $g$. However, the second-best $\theta$ hinges on not only those two factors but on $\delta$ and $a$, which are parameters for the agent’s effort and production efficiency, respectively. The reason is as follows. In the first best, the sharing rule $C_T$ for the risky part of the outcome $D_T$ solely depends on the risk aversion, $\gamma_A$ and $\gamma_F$, and the production risk $g$, because in the first best, only risk-sharing matters regardless of the agent’s effort efficiency. However, in the second best, the risky part of $C_T$ is necessarily adjusted for the agent’s effort efficiency parameter because of the incentive compatibility condition.

Comparing Propositions 4.1 and 5.3, one can immediately see that the first best market price of risk is greater than that of the second best, i.e., $\theta^F > \theta^S$ if the investment level $I$ is held constant. Given $I$, because of the incentive compatibility condition, the second best contract $C_T$ needs to be more sensitive to the outcome $D_T$ than the first best contract does. As a result, the dollar volatility of the second best principal’s residual claim $D_T - C_T$ becomes lower than that of the first best. Hence, given $I$, we have $\theta_S < \theta_F$. However, as can be seen later, when $I$ is endogenized, the comparison becomes complicated.

5.1.1 The First-Stage Problem

Without losing the essence of economics, let us assume the linear-quadratic case. Then by Proposition 5.3, $\theta$ turns out to be constant. Thus we only focus on cases where $\theta$ is constant, in examining the principal’s decisions in equilibrium. Suppose $\theta$ is given to be an arbitrary constant, where $\theta$, at this moment, may or may not be consistent with equilibrium. Then the unique solution to the principal’s BSDE (5.22) can be found by setting $Z_t^P \equiv 0$, and it turns out to be

$$V_0^P = R_T \left( M_P + M_A - c_0^P - c_0^A \right) + \frac{1}{\gamma_A} \ln \left( -L - e^{-\gamma_A c_0^A} \right) + G^{SB}(I; \theta, f^*) T,$$

where

$$G^{SB}(I; \theta, f) = f - h(\mu_s) - \gamma_A \left( \frac{h^\mu g}{\mu^\mu} \right)^2 - \gamma_F \left( 1 - \frac{h^\mu}{\mu} \right)^2 g^2 - \frac{R_T I}{T} + \frac{D_0(I)}{T}.$$

The principal’s production decision $I$ affects $G^{SB}$, which consists of several components: expected earnings from production $\frac{D_0(I)}{T} + f$; minus the dollar cost of production capital $\frac{R_T I}{T}$; expected compensation for the agent’s effort and compensation risk, $h(\mu_s) + \gamma_F \left( \frac{h^\mu g}{\mu^\mu} \right)^2$; and the principal’s aggregate risk premium on both production and portfolio risks $\frac{\theta^2}{2T} \left( \gamma_F \left( 1 - \frac{h^\mu}{\mu} \right)^2 g^2 \right)$.

With $V_0^P$ given as above, the principal’s first-stage problem can be written as follows.

$$\max_{c_0^P, \theta, I} \left\{ -\gamma_F c_0^P \right\} - \exp \left\{ -\gamma_F V_0^P \right\}.$$

The forms of FOCs with respect to $(c_0^P, c_0^A, R)$ are the same as (4.14) to (4.16) of the first best case. By the Envelop Theorem, the FOC with respect to $I$ is

$$f_I + \frac{D_0(I)}{T} = - \gamma_A \left( \frac{h^\mu}{\mu^\mu} g \right) \partial_I \left( \frac{h^\mu}{\mu^\mu} g \right) = - \gamma_F \partial_I \left( g - \frac{h^\mu}{\mu^\mu} g \right) + \frac{R_T}{T}.$$

The LHS is the marginal product net of the marginal compensation-risk premium, and the RHS is the second best cost of capital consisting of one plus riskfree rate and the marginal residual-claim risk. Given $I$, the LHS is determined in the product market by the agent’s effort, and the
Recall that given risky investment the market price of residual-claim risk related to the compensation risk, based on residual-risk riskiness of the residual claim. As a result, the second-best risk premium on real investment is the best, a change in real investment level can affect the sensitivity of the contract and thus the sensitivity of the contract) is unaffected by the level of real investment. However, in the second best, a change in real investment level can affect the sensitivity of the contract and thus the riskiness of the residual claim. As a result, the second-best risk premium on real investment is based on residual-risk $g \left(1 - \frac{h}{T} \right)$, whereas the first-best risk premium is on the total production risk.

In order to shed light on consequences of the differences in the first and second best costs of capital on real investment decisions, let us rearrange the FOC (5.26) as follows.

$$f_I + \frac{D'(I)}{T} = g_I \theta + \frac{R_T}{T} + \left( \gamma_A \frac{h}{T} - \theta \right) \partial_I \left( \frac{h}{T} \right).$$  \hspace{1cm} (5.26)

The net marginal agency cost to the principal for a marginal increase in real investment is equal to a marginal change in the compensation-risk premium, $\gamma_A \left( \frac{h}{T} \right)^2$, minus marginal change in the market price of residual-claim risk related to the compensation risk, $\frac{h}{T} \theta$. (Note that as the compensation volatility increases, the residual claim volatility decreases.) Note that since $\gamma_A \frac{h}{T} \theta > 0$ in equilibrium, the sign of the net agency cost is the same as the sign of $\partial_I \left( \frac{h}{T} \right)$ which can be positive or negative in equilibrium.

In order to obtain further insight into the second-best cost of capital, let us look at a special case: Suppose $f(\mu, I) = \mu I^\alpha$, $g(\mu, I) = I^\beta$ and $h(\mu) = (\delta/2) \mu^2$. Then (5.26) becomes

$$f_I + \frac{D'(I)}{T} = g_I \theta + \frac{R_T}{T} + \gamma_A \left( \frac{h}{T} \right)^2 I^{2(\beta - \alpha)} \left( 1 + (\gamma_A + \gamma_p) \delta \sigma_D^2 \right)^{\alpha - 1} - (\beta - \alpha).$$  \hspace{1cm} (5.27)

Thus, given $(R_T, \theta, I)$, the second-best cost of capital is greater than that of the first best if and only if $\beta - \alpha > 0$, i.e., the volatility of $D_T$, $g$, grows with the size of the firm faster than the marginal effort productivity, $f_\mu$, does. In other words, given $(R_T, \theta, I)$, the second-best cost of capital can be smaller than that of the first-best when the growth rate of the agent’s marginal effort productivity is higher than that of the volatility, because then the compensation risk decreases in $I$.

Now, let us assume $\alpha = \beta$, and focus on the classical form of the cost of capital $g_I \theta + R_T/T$. Recall that given risky investment $I$, the second best $\theta$ is lower than that of the first best. Thus, given $I$, the second-best risk premium, $g_I \theta$, on the risky investment is lower than that of the first best. This result is in stark contrast with the popular belief that the second-best cost of capital is greater than that of the first best. See, for example, Stulz [1995] and Romer [1996].

Note that our foregoing discussion after Eq.(5.25) on the cost of capital is about partial equilibria, because both are based on the assumption that the capital market variables such as...
\((R_T, \theta)\) are held constant. Even in partial equilibrium, implications of the FOC (5.26) do not support the traditional belief that agency costs increase the cost of capital.

In general equilibrium, the variables \((R_T, \theta)\) in second best may not be the same as those of the first best. In Section 6.3.2, we use a special case to show that, indeed, the second-best equilibrium cost of capital can be lower or higher than that of the first best, and as a result, equilibrium real investment levels in the second best can be higher or lower than those of the first best.

On the other hand, as already seen in the first-best case, the market clearing conditions imply the following Proposition.

**Proposition 5.4** Let \(I(R_T, \theta) \in \arg \max IG^{SB}(I)\). Then, equilibrium capital market parameters \((R_T, \theta)\) are jointly determined by the following two equations:

\[
M_P + M_A = (1 + R_T)I - \left( \frac{1}{\gamma_P} + \frac{1}{\gamma_A} \right) \ln(R_T) + TG^{SB}(I(R_T, \theta); \theta).
\]

In Proposition 5.4, the equation for \(\theta\) is from the equity market clearing condition \(\hat{\pi}_P^P \equiv 0\) and the second equation is from the initial financial market clearing condition, \(W_P^0 + W_P^0 = 0\). In Section 6, we use these two equations to examine properties of \((R_T, \theta)\).

### 5.2 Equilibrium Stock Price and Equity Premium

In order to examine the equity premium in the second-best economy, it is necessary for us to understand dynamics of the stock price. Utilizing the fact that discounted stock price is the \(Q\)-expectation of future cash flows, one can easily show that the equilibrium stock price evolves over time as follows.

\[
dS_t = \left\{ r_t S_t + e^{-r(T-t)} \left(1 - \frac{h_{\mu}}{f_{\mu}}\right) g\theta \right\} dt + e^{-r(T-t)} \left(1 - \frac{h_{\mu}}{f_{\mu}}\right) gdB_t^\mu,
\]

with its initial stock price given by

\[
S_0 = R_T^{-1} \left\{ D_0(I) - \left(1 - \frac{h_{\mu}}{f_{\mu}}\right) g\theta T + \frac{1}{\gamma_A} \ln \left(\frac{-L}{1 + R_T}\right) + R_T(M_A - c_A^A) \right. \\
\left. + \left( f(\mu, I) - h(\mu_s) - \frac{\gamma_A}{2} \left(\frac{h_{\mu}}{f_{\mu}} g\theta - h(\mu_s) \right)^2 \right) T \right\}.
\]

Note that the initial second-best stock price shares the same structure as that of the first best. Thus interpretations of individual terms composing the initial second-best stock price are the same as those for the first best case.

On the other hand, the dollar volatility of the stock price at time \(t\) is

\[
\sigma_t^S = e^{-r(T-t)} \left(1 - \frac{h_{\mu}}{f_{\mu}}\right) g = e^{-r(T-t)} \frac{\gamma_A \delta \frac{\sigma^2}{\alpha}}{1 + (\gamma_A + \gamma_P) \delta \frac{\sigma^2}{\alpha}} g.
\]

\(^{14}\)That is, \(R_T^{-1} S_t = E_t^Q[R_T^{-1}(D_T - C_T)]\) where

\[
C_T = R - R_T(M_A - c_A^A) - \left[ \frac{h_{\mu}}{f_{\mu}} g\theta - h(\mu_s) - \frac{\gamma_A}{2} \left(\frac{h_{\mu}}{f_{\mu}} g\theta - h(\mu_s) \right)^2 \right] T + \frac{h_{\mu}}{f_{\mu}} gB_T^\mu.
\]
This equation implies that the higher the agent’s risk aversion, the higher the stock price volatility. Holding $I$ constant, because of both risk-sharing and incentive motivations, the sensitivity of the contract decreases in the agent’s risk aversion, and thus the market value of the residual claim to the principal, i.e., the stock price, becomes riskier as the agent’s risk aversion increases. Furthermore, the stock price volatility depends not only on $g$ the outcome volatility, but on the ratio $g/a$ measuring the dollar volatility to generate one expected dollar. The larger the ratio, the higher the stock price volatility.

In order to compute the equity premium, note that the drift of the stock price process is composed of two parts: risk-free dollar return and excess dollar return. Since the equity premium is customarily measured in terms of rate of return, we define it as excess dollar return divided by the stock price. However, since the stock price in this paper can be zero, we simply compute the equity premium at time zero under the assumption that initial stock price is greater than zero. Then, the equity premium, denoted by $\nu_0$, is

$$\nu_0 = \frac{\left(1 - \frac{b_n}{T}\right) g \theta}{e^{\gamma T} S_0}.$$  \hfill (5.29)

This structure simply tells us that holding other things constant, the equity premium is positively related to the dollar-risk premium of the stock price and inversely related to the current stock price and interest rate, where the dollar-risk premium is a dollar-risk premium on the principal’s residual claim $D_T - C_T$. However, these are only partial relationships, because one variable can affect the others: for example, a change in the interest rate affects both the initial stock price and the residual claim. We examine more detailed properties of first- and second-best equity premia in the next section.

### 6 A Special Case

In order to obtain further insight into our economy, we examine a special case where the firm has a set of project opportunities exhibiting decreasing returns to scale in both expected productivity and dollar volatility. In particular, we assume $h = \frac{\gamma}{2} \mu^2$, $f = \mu I^\frac{1}{2}$, $g = \sigma_D I^\frac{1}{2}$ and $D_0(I) = AI^\frac{1}{2}$.

Thus, in this special case,

$$dD_t = I^\frac{1}{2} \left(\mu dt + \sigma_D dB_t^u\right).$$

We interpret $\sigma_D$ as a risk measure of the production opportunity set: high $\sigma_D$ implies a set of high-risk production opportunities. Given $\sigma_D$, the principal chooses a project $(I; \sigma_D)$ from project/capital-budgeting opportunities $\{(I; \sigma_D) : I \in \mathbb{R}_+\}$. Once the principal’s choice is made, $(I; \sigma_D)$ completely determines the production function of the firm with a dollar volatility of $\sigma_DI^\frac{1}{2}$.

#### 6.1 The first best

We assume that $R_T + \kappa_F > 0$, in order to ensure an interior solution for $I$. Then, substituting $(\mu, \theta)$ in Proposition 4.1 into FOC (4.17), we have

$$I^\frac{1}{2} = \frac{A/2}{R_T + \kappa_F},$$  \hfill (6.30)

where

$$\kappa_F := \frac{1}{2} \left\{ \frac{\gamma_P \gamma A}{\gamma_P + \gamma_A} \sigma_D^2 - \frac{1}{\delta} \right\} T.$$  \hfill (6.31)
One may view $\kappa_F$ as marginal risk premium adjusted for the growth rate of the outcome, or the agent’s effort efficiency/productivity, and $R_T + \kappa_F$ as the growth-adjusted first-best cost of capital for the production asset. Eq.(6.30) implies that equilibrium $I$ directly depends on $(R_T, \kappa_F, A)$. Let $\tau$ be the aggregate risk tolerance, i.e., $\tau = \frac{1}{\gamma P} + \frac{1}{\gamma A}$. Then, $\kappa_F$ depends on $(\tau, \sigma_D, \delta)$.

Substituting Eq.(6.30) into the bond market clearing condition given in Proposition 4.2, we have

$$M = \frac{A^2(1 + \kappa_F + 2R_T)}{4(R_T + \kappa_F)^2} - \tau \ln(R_T). \quad (6.32)$$

Condition (6.32) implies that the interest rate $R_T$ directly depends on $\kappa_F$, $M$, $A$ and $\tau$, i.e., $R_T(\kappa_F, M, A, \tau)$. Note that the interest rate in equilibrium can also depend on the risk tolerance. The reason is that given limited endowment/budget, the risk tolerance affects demand for capital for risky assets, which in turn affects demand for riskfree assets.

In this paper, interest rate $R_T$ and the market price of risk $\theta$ are a pair of important capital market variables. Comparative static results for $R_T$ and $\theta$ are provided in Propositions 6.5 and 6.6, respectively.

**Proposition 6.5** Assume an interior solution to the special case. Then, comparative statics for the first-best economy are as follows.

1. The larger the aggregate wealth $M$, the lower the interest rate $R_T$, the lower the growth-adjusted cost of capital $R_T + \kappa_F$, and the higher the real investment $I$.

2. The higher the initial production efficiency $A$, the higher the interest rate, the higher the growth-adjusted cost of capital, and the lower the real investment $I$.

3. The riskier the production opportunities (high $\sigma_D$), the lower the interest rate. Moreover, as $\sigma_D$ increases, the growth-adjusted cost of capital increases (decreases) and thus the real investment level decreases (increases), if and only if $R_T I < (>) \tau$.

4. The higher the managerial effort efficiency ($1/\delta$), the higher the interest rate. Moreover as $1/\delta$ increases, the growth-adjusted cost of capital decrease (increases) and thus the real investment level increases (decreases), if and only if $R_T I < (>) \tau$.

We shall see later all comparative statics results stated in Proposition 6.5 also hold for the second best case. The first claim of Proposition 6.5 is well known and its economic intuitions is almost obvious. As the income increases, the investor increases both current consumption and real investment. Furthermore, the interest rate decreases as the supply of money is increased.

Note that for the second to fourth claims, investors’ aggregate income/endowment $M$ is held constant, but productivity parameters are changed. In the real-business-cycle (RBC) literature, the definition of productivity varies depending on models. In this paper, improvement in productivity/profitability of investment opportunities can occur as $A$ increases or as either $\sigma_D$ or $\delta$ decreases. As well noted in the literature, a productivity shock brings about both income and substitution effects on investors consumption/investment decisions.\(^{15}\)

In order to see the second claim, note that an increase in $A$ can be roughly viewed as an upward parallel shift of the frontier of real investment opportunities in the dollar risk-return

\(^{15}\)See for example Romer [2005] for a textbook explanation of those effects.
space with the dollar return scaled in the vertical axis. When there is productivity improvement due to an increase in $A$, demand for capital rises and so does the interest rate. However, since the parallel shift does not change the riskiness of each real investment opportunity, the growth-adjusted marginal risk premium $\kappa_F$ remains unaffected. This means the growth-adjusted cost of capital increases due to the increase of the interest rate, and thus the real investment level decreases. In other words, when $A$ increases, the income effect dominates the substitution effect and thus the real investment level decreases.

In the third claim, a productivity improvement means a decrease in $\sigma_D$, a reduction in the riskiness of each real investment opportunity that, roughly speaking, moves the real-investment opportunity frontier to the left in the dollar risk-return space. Note that the improvement can affect small real-investment projects more than large ones, in terms of the marginal cost of capital which is the RHS of (4.17). Note that holding $(R_T, \theta)$ constant, the marginal risk reduction reduces $g_I = \frac{\sigma_D}{2} I^{-\frac{1}{2}}$ more when $I$ is smaller. Thus, upon a productivity shock of a decrease in $\sigma_D$, the principal has stronger incentive to increase her real investment level when $I$ is small than when $I$ is already large. In other words, the substitution effect is stronger when $I$ is smaller. In particular, if $R_T I < \tau$, then the substitution effect dominates the income effect, and the principal invests more as $\sigma_D$ increases.

Alternatively, one may understand the third claim as follows. In general equilibrium, as $\sigma_D$ changes, not only the growth-adjusted risk premium but the interest rate changes. As $\sigma_D$ increases, the growth-adjusted risk premium on the risky real asset $\kappa_F$ increases. Then, holding $R_T$ constant, (6.30) implies the demand for capital initially decreases, which in turn decreases $R_T$ and makes real investment recover. If $R_T I < (>) \tau$, as a response to a initial decrease in real investment demand, the interest rate $R_T$ decreases so fast (slowly) that $R_T + \kappa_F$ decreases (increases) and the demand recovery more (less) than offsets its initial decrease. Hence, the riskier the production opportunities, the higher (lower) the real investment level.

In the fourth claim, a productivity improvement can also occur when the managerial effort efficiency improves, or when $1/\delta$ increases. Recall that the optimal effort $\mu = \frac{1}{\delta} I^{\frac{1}{2}}$ and $f_I = \frac{1}{2} \frac{1}{\delta} I^{-\frac{1}{2}}$. Thus, the benefit of improved managerial effort efficiency is larger to a smaller (real-investment) project, in terms of the marginal product of real investment as in the LHS of (4.17). Note that both changes in $\frac{1}{\delta}$ and $\sigma_D$ influence the real investment level, by affecting, respectively, the LHS (marginal product) and RHS (marginal cost) of the same equation. Thus, in the fourth case, we also have a conclusion similar to that of the third claim: that is, if $R_T I < \tau$, then the substitution effect dominates the income effect, and the principal invests more, as $1/\delta$ increases.

Furthermore, all the three claims, the second to the fourth, uniformly suggest that in spite of the above-mentioned different effects on equilibrium real investment levels, the productivity improvement always increases the equilibrium interest rate. When $R_T I < \tau$, a high interest rate stemming from productivity improvement can be easier to understand, because the improvement induces high demand for capital which can result in a high interest rate. Even when $R_T I > \tau$, productivity improvement can result in a high interest rate, because although the real investment and thus demand for capital decreases, demand for current consumption increases so strongly that the aggregate demand for money increases to push up the equilibrium interest rate.

In sum, Proposition 6.5 indicates that productivity improvement results in high interest rate, but can lead to an increase or decrease in real investment. Thus, it can happen that one observes interest rates increasing when there is a decrease in real investment or a decrease in demand for capital. In particular, the third and fourth claims tells us that such a decrease can occur when the existing investment level is sufficiently high. These results are consistent with
those in the RBC literature.

The next Proposition describes how the market price of risk \( \theta \) can be affected as model primitives \((M, A, \sigma_D, \delta)\) change.

**Proposition 6.6** Assume an interior solution to the special case. Then, comparative statics for the first-best economy are as follows.

1. The larger the aggregate wealth \( M \), the higher the market price of risk.

2. The higher the initial production efficiency \( A \), the higher (lower) the market price of risk if and only if \( 2R_T I > (<) 0 \).

3. The riskier the production opportunities (high \( \sigma_D \)), the higher (lower) the market price of risk if and only if \( (R_T + \kappa_F)\Phi_{R_T} - \frac{\sigma_D^2}{\delta^2} \left( I - \frac{\tau}{R_T} \right) < (>) 0 \).

4. The higher the managerial effort efficiency \((1/\delta)\), the higher the market price of risk, if and only if \( R_T I < (> \tau \).

When the aggregate wealth increases, the real investment increases, increasing the overall risk of the investment, \( g \). Thus the market price of risk increases, as it is proportional to \( g \). However, the productivity improvement can result in either an increase or decrease in the market price of risk. The main reason is that the productivity improvement can increase or decrease the real investment level as seen in Proposition 6.5.

### 6.2 The second best

Note first that since \( \alpha = \beta = \rho = \frac{1}{2} \), the marginal cost of contract in Eq.(5.26) turns out to be zero. That is, in this special case, the marginal agent’s risk premium happens to be equal to the marginal market price of compensation risk. As a consequence, the principal decision on \( I \) is simplified. Substituting \((\mu, \theta)\) in Proposition 5.3 into (5.26), we have

\[
I^\frac{1}{2} = \frac{A/2}{R_T + \kappa_S},
\]

where

\[
\kappa_S := T \left\{ \frac{\gamma_A (1 + \gamma_P \delta \sigma_D^2)}{1 + (\gamma_A + \gamma_P) \delta \sigma_D^2} \sigma_D^2 \frac{1 - \delta}{\delta} \right\}.
\]

We assume \( R_T + \kappa_S > 0 \), in order to ensure an interior solution to the principal’s problem. As seen in the first best case, \( \kappa_S \) can be viewed as the second-best marginal risk premium on the risky production asset adjusted for the marginal capital growth rate which depends on both the agent’s effort and the size of the production investment.

Substituting (6.33) into the bond market clearing condition in Proposition 5.4, we have

\[
M = \frac{A^2 (1 + \kappa_S + 2R_T)}{4(R_T + \kappa_S)^2} - \tau \ln(R_T)
\]

(6.34)

**Proposition 6.7** Assume interior solutions to the special case. Then, comparative statics for the second-best economy can be stated in the same way as Parts 1 through 6 of Proposition 6.5.\(^\ast\)

That is, moral hazard problems do not alter qualitative implications of the comparative statics resulted from the first-best economy.

\(^\ast\)As in the first best case, one can immediately see from (6.34) that \[ \frac{\partial R_T}{\partial M} < 0, \frac{\partial R_T}{\partial A} > 0, \text{ and } \frac{\partial R_T}{\partial \kappa_S} < 0. \text{ Since }
6.3 Comparing the first- and second-best solutions

In this section, we try to compare interest rates, real investment levels, initial stock prices and equity premia between the first- and second-best economies.

6.3.1 Interest rates

Recall that the first- and second-best market clearing conditions in (6.32) and (6.34), respectively, can be written as follows. For \( \kappa = \kappa_F, \kappa_S \),

\[
M \equiv \Phi(R_T, \kappa) := \frac{A^2(1 + \kappa + 2R_T)}{4(R_T + \kappa)^2} - \tau \ln(R_T).
\]

However,

\[
\kappa_S - \kappa_F = \frac{T}{2} \left\{ \frac{\gamma_A^2 \sigma^2_D}{1 + (\gamma_A + \gamma_P)\delta \sigma^2_D} \right\} (\gamma_P + \gamma_A) > 0,
\]

and

\[
\Phi_R = -\frac{A^2(1 + R_T)}{2(R_T + \kappa)^2} - \frac{\tau}{R_T} < 0,
\]

\[
\Phi_\kappa = -\frac{A^2}{4(R_T + \kappa)^3} (2 + \kappa + 3R_T).
\]

Note that \( \kappa_S > \kappa_F \), mainly because the first best marginal capital growth rate is higher than the second best. Since \( R_T + \kappa = \frac{A}{2}I^{-\frac{1}{3}} > 0 \), we have \( 2 + \kappa + 3R_T > 2(1 + R_T) > 0, \Phi_\kappa < 0, \) and

\[
\frac{\partial R_T}{\partial \kappa} = -\frac{\Phi_\kappa}{\Phi_R} < 0.
\]

Therefore, we have the following proposition.

**Proposition 6.8** Assume interior solutions to the special case. The second best interest rate, \( R_T^S \), is lower than the first best, \( R_T^F \).

This result provides a moral-hazard explanation of Weil’s [1989] riskfree rate puzzle. Figure 1 graphically illustrates Proposition 6.8, showing that the second best interest rates are lower than those of the first best. The Figure is also consistent with the fourth statement of Proposition 6.5: the higher the managerial effort efficiency (\( 1/\delta \)), the higher the interest rate.

In order to intuitively see the result of low second-best interest rates as in Proposition 6.8, note that the second-best market clearing condition (6.34) would have been identical to that of the first best if \( \kappa_S = \kappa_F \). Therefore, the difference between \( R_T^F \) and \( R_T^S \) can be related to a difference in \( \kappa \), or \( \kappa_S - \kappa_F (> 0) \). When \( \kappa \) is high, the value of marginal real investment is low, which in turn induces the principal to decrease the demand for investment capital. As a result, the second best equilibrium interest rate is lower than that of the first best.

\[
\frac{\partial \theta}{\partial \sigma_D} > 0 \quad \text{and} \quad \frac{\partial \kappa^S}{\partial \sigma_D} < 0, \quad \text{we have} \quad \frac{\partial R_F}{\partial \sigma_D} < 0, \quad \text{and} \quad \frac{\partial R_T^S}{\partial \sigma_D} > 0. \quad \text{Moreover,}
\]

\[
\frac{\partial \theta}{\partial M} = \frac{\gamma_P \gamma_A \delta \sigma^2_D}{1 + (\gamma_A + \gamma_P)\delta \sigma^2_D} \frac{-A/2}{(R_T + \kappa_S)^2} \frac{\partial R_T}{\partial M} > 0,
\]

and \( \text{sign} \left( \frac{\partial I}{\partial \sigma_D} \right) = -\text{sign} \left( \frac{\partial R_T}{\partial M} \right) (R_T + \kappa_S) = -\text{sign} \left( \frac{2R_T}{\kappa + 1} \right) \). Therefore, if \( R_T + \kappa_S > 0 \), then the comparative statics seen for the first best in Proposition 6.5 hold exactly for the second best.
6.3.2 Real Investment Levels

The inequality (6.35) can also be used to examine equilibrium investment levels. Recall that for $\kappa = \kappa_F, \kappa_S$,

$$I^2 = \frac{A}{2(R_T + \kappa)}.$$ 

Thus, one can immediately see that the second best investment level is higher than the first best, i.e., $I^S > I^F$ if and only if $R^S_T + \kappa^S - R^F_T - \kappa^F < 0$. Recall that $R^S_T + \kappa^S$ is a growth-adjusted cost of capital or a growth-adjusted discount rate for the production project in the second best world. If the second-best growth-adjusted cost of capital is lower than that of the first best, then the second best investment will be higher.

What is striking is that as can be seen shortly, the second-best growth-adjusted cost of capital can actually turn out to be lower than that of the first best, and thus the second best investment level can sometimes be higher than that of the first best. In the next Proposition, we provide a sufficient condition under which the second-best cost of capital is lower than that of the first best.

**Proposition 6.9** Assume interior solutions to the special case.

1. If $\frac{\gamma_P \gamma_A A^2}{3(\gamma_P + \gamma_A)} - 4\kappa_F < 0$, then $I_F > I_S$,

$$\mu_F > \mu_S \frac{(1 + \gamma_A \delta \sigma_D^2 + \gamma_P \delta \sigma_D^2)}{(1 + \gamma_P \delta \sigma_D^2)}, \quad \text{and} \quad \theta_F > \left( \frac{1}{\delta \sigma_D^2 (\gamma_A + \gamma_P)} + 1 \right) \theta_S.$$

2. If $\frac{\gamma_P \gamma_A A^2}{3(\gamma_P + \gamma_A)} - 4\kappa_S > 0$ and the aggregate wealth $M$ is given such that $M \in \Upsilon$, then $I_F < I_S$,

$$\mu_F < \mu_S \frac{(1 + \gamma_A \delta \sigma_D^2 + \gamma_P \delta \sigma_D^2)}{(1 + \gamma_P \delta \sigma_D^2)}, \quad \text{and} \quad \theta_F < \left( \frac{1}{\delta \sigma_D^2 (\gamma_A + \gamma_P)} + 1 \right) \theta_S.$$
where

\[ \Upsilon := \left\{ M : \max_{\kappa \in \Lambda} \Phi(\bar{R}_1^T(\kappa), \kappa) < M < \min_{\kappa \in \Lambda} \Phi(\bar{R}_2^T(\kappa), \kappa) \right\}, \]  

(6.36)

\( \bar{R}_1^T \) and \( \bar{R}_2^T \) are roots of the quadratic equation \( G(R_T(\kappa), \kappa) = 0 \) such that \( \bar{R}_1^T \leq \bar{R}_2^T \), and

\[ G(R_T(\kappa), \kappa) := R_T^2 + \left( 2\kappa - \frac{\gamma_A\gamma P A^2}{4(\gamma_A + \gamma_P)} \right) R_T + \kappa^2. \]

Remark: By Lemma A.1 in the Appendix, the set \( \Upsilon \) is nonempty.

For an intuitive understanding of the case \( I_F < I_S \) appearing in Proposition 6.9, recall that the second-best growth-adjusted risk premium \( \kappa \) on risky investment is higher than that of the first best. The high risk premium decreases the demand for investment capital and thus the interest rate. However as the interest rate decreases, the demand for capital increases. Thus, if the interest rate decreases faster (slower) than \( \kappa \) increases, or if the interest rate differential between the first and the second best is larger than the risk premium differential, then the second best production investment level becomes higher (lower) than that of the first best.

Both Figures 2 and 3 illustrate cases for \( I_F < I_S \), in which the interest rate differential between the first and the second best is larger than the real-investment risk premium differential. Figure 4 demonstrates, in one graph, a case where both \( I_F < I_S \) and \( I_F > I_S \) can occur. In Figure 4, the second-best real investment becomes higher (lower) than the first best, because the interest rate differential between the first and the second best is smaller (larger) than the real-investment risk premium differential when the risk of production opportunities is not too high (low).

6.3.3 Stock Prices and Equity Premia

The stock price is the present value or the discounted Q-expectation of future cash flow, \( D_T - C_T \). In equilibrium, the principal would be willing to invest \( I \) only when \( S_0 \geq I \), or when the NPV (net present value) of the investment is nonnegative.

For our linear-quadratic case, we examine equilibrium stock prices in the first- and second-best economies where the NPV of the principal’s investment is equal to zero. That is, in equilibrium, \( S_0 = I \). Given the interest rate, the agent’s reservation utility level can affect the NPV of the principal’s investment. Our zero NPV condition implies that the agent’s reservation utility is determined in such a way to make the principal’s NPV equal to 0.

In the literature, Kahn [1990] argues that the second-best equity premium is higher than the first best, whereas Kocherlakota [1998] claims the opposite. In our linear-quadratic case, the comparison depends relative sizes of the first- and second-best equilibrium interest rates as in the following proposition.

Proposition 6.10 (i) The second best equity premium is higher than the first best if and only if

\[ \frac{R_T^S}{R_T^F} \leq \frac{(\gamma_A + \gamma P)^2}{\left( \frac{1}{\gamma_A} + (\gamma_A + \gamma_P) \right)^2}. \]  

(6.37)

(ii) In both first- and second-best cases, interest rates are negatively correlated with equity premia.
Figure 2: Equilibrium equity premia and real investment levels with $M_P = 784$, $M_A = 0$, $\gamma_A = 3$, $\gamma_P = 1$, $\delta = 1$, $A = 50$, and $T = 1$.

Figure 3: Equilibrium equity premia and real investment levels with $M_P = 1500$, $M_A = 0$, $\gamma_A = \gamma_P = 1$, $\delta = 5$, $A = 50$, and $T = 1$.  

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In order to obtain a general insight into relative sizes of the first and second best equity premia, recall from (4.20) and (5.29) that the equity premium is the discounted dollar risk premium of the principal’s residual claim divided by the stock price, and that the residual claim depends on the sharing rule between the principal and agent. Thus, there are three major components affecting the equity premium: interest rate, outcome sharing rule and stock price. However, even these three factors, in equilibrium, interact with each other in a highly complicated manner. As a result, the second best equity premium can sometimes be higher or lower than that of the first best.

Let us first suppose both the first and second best economies share the same \( I \), and thus the same \( S_0 \). Then, it can be shown that the second-best dollar risk premium on the residual claim is always lower than the first best, i.e., \((g - \frac{\delta}{\gamma} g) \theta S < \left(g - \frac{\delta}{\gamma} F\right) \theta F\). Thus, with \( I \) held constant, (4.20) and (5.29) imply that the second-best equity premium can be larger (smaller) than the first-best equity premium if \( R^{S} \) is (not) sufficiently smaller than \( R^{F} \). Condition (6.37) quantifies how sufficiently small the second best interest rate has to be in order for the second-best equity premium to be larger than that of the first best.

In Figure 2, the second-best equity premium is higher than the first best. In this case, the second-best interest rate is far lower than the first best, and thus the second-best discounted dollar risk premium turns out to be higher than the first-best premium. In Figure 3, the second-best equity premium is lower than that of the first best, as the second-best interest rate is not sufficiently lower than the first best.

Figure 4 provides a numerical case where the second-best real investment level (thus the second-best stock price) can be higher than that of the first best until \( \sigma_D \) reaches a threshold. However, beyond the threshold, the second best real investment level is lower than that of the first best.
first best. When $\sigma_D$ becomes higher than the threshold, although the second-best interest rate is much lower than that of the first best, the second-best equity premium is so high that the real investment becomes less attractive in the second best than it is in the first best. That is, in our equilibrium agency model, both over- and underinvestment can occur. This result as well as Proposition 6.9 is in contrast with Albuquerque and Wang [2008] who argue that agency conflicts between controlling shareholders (similar to the manager) and noncontrolling shareholders can lead to overinvestment problems.

7 Conclusion

We have presented an integrated equilibrium model of real investment, production and portfolio management with a closed form solution. The model enables us to examine effects of moral hazard on capital/financial markets.

We have shown that moral hazard problems can result in low cost of capital for risky real assets. The main reason is that given each real investment level, the principal’s residual-claim risk is a just fraction of the total production risk, and both the second-best interest rate and market price of risk are lower than those of the first best. This result of low second-best cost of capital is in contrast with a popular belief that agency costs increase the cost of capital.

Moreover, we show that even when the real investment is endogenized in full equilibrium, the second best interest rate is lower than that of the first best. We believe this result serves as an explanation for Weil’s [1989] low (riskfree) interest rate puzzle. However, Mehra-Prescott’s [1985] high equity premium puzzle may or may not be explainable using moral hazard problems, because we find that the second best equity premium can sometimes be higher or lower than that of the first best. We have provided sufficient conditions for high second-best equity premia. In other words, under the sufficient conditions, moral hazard problems can be used to simultaneously explain both Weil’s and Mehra-Prescott’s puzzles.

Thanks to our closed form solutions, we have provided a number of striking comparative statics results, most of which can be empirically testable. In particular, the higher the current social wealth, the lower the interest rate and the higher the market price of risk. High social wealth means high money supply, thus decreasing the interest rate. However, high social wealth also means high real investment, increasing the total risk of production and thus the market price of risk.

Interest rates can also be positively related to the agent’s effort efficiency and inversely related to the riskiness of project/production opportunities. Intuitively, given monetary endowment, the interest rate is determined by the demand for capital, and the high effort efficiency increases and the riskiness decreases the demand for capital for real investment.

Our comparative statics also provide insight into some conceptual issues. We have shown that in equilibrium, the second best production investment and stock prices can sometimes be higher than those of the first best. Roughly, the growth-adjusted cost of capital (which is the sum of the interest rate and growth-adjusted risk premium) can be sometimes lower in the second best than it is in the first best, as moral hazard problems decrease the interest rate more than they increase the growth-adjusted risk premium on the risky real asset. The second-best stock price can be higher than that of the first best for various reasons, among which a major reason is that the first-best interest rate is sometimes too much higher than the second best.

In this paper, we have assumed CARA preferences for both the principal and agent, and
an outcome process driven by an arithmetic Brownian motion. As a result, the stock price can sometimes be negative. It would be interesting to have our model recast with a general class of preferences and outcome processes which produce only positive stock prices. On the other hand, our model is intended to serve as a benchmark for moral-hazard economies driven by conglomerates as in Far Eastern countries like China, Japan and Korea. The other extreme is a moral-hazard economy like the U.S.A, consisting of many firms where no individual firms may significantly affect systematic risks of the whole economy. We leave these topics for future research.
Appendix

A Incentives When the Agent Is Allowed to Trade the Stock

A folklore in the profession is that if the agent is allowed to trade the stock of his own firm, he can undo his incentive scheme to receive a constant salary, and consequently he has no incentives to work. This kind of implicit argument has been giving a justification for partial equilibrium agency models to prohibit the agent from trading the stock of his own firm and to focus only on contracting problems in product markets. Nevertheless, it is not completely clear how allowing the agent to trade can affect the principal’s equilibrium expected utility. Also immediately unclear is whether the agent’s incentives would improve or worsen incentives as the agent tries to hedge his risky position in securities markets, because the agent hedging may make the performance measure less noisy. Moreover, it may be the case that the agent may not want to completely hedge away his compensation risks in the process of optimally managing his own financial/asset portfolio, and as a result, he may still have incentives to work. The latter case can be particularly true when the risk of the firm can affect systematic risk of the whole economy as is the case in this paper.

In this section, we provide a condition under which the folklore holds in equilibrium, and also argue that in general, the principal is better off with forcing the agent not to trade the asset which the agent manages. It shall be seen that if the agent is allowed to trade the stock privately, then his optimal effort level is determined independently of incentives provided by the contract but is not zero in general, and the principal is worse off than she is with agent being prohibited from stock trading.

In order to examine an economy where the principal allows the agent to trade, we revise the condition of equity-market equilibrium from (2.7) to

\[ \tilde{\pi}_P + \tilde{\pi}_A \equiv 0. \]

We still assume the linear-quadratic case, where the outcome process is linear and the cost of effort is quadratic in \( \mu \). When the agent is fully allowed to trade the stock as well as bonds, the principal’s problem can be stated as follow.

\[ \text{An alternative justification can be based on the following argument. Since the agent’s trading is observable, the contract can be contingent on the agent’s transaction history in such a way that the agent’s incentives remains unchanged. Thus, allowing the agent to trade does not affect the principal’s utility. This kind of reasoning, however, may not be reasonable for continuous-time models, because contracts contingent on the agent’s private transactions history may not be realistically possible.}\]
Problem 3 Choose $C_T$ and $(\mu_t, \pi_t)$ to

$$\max E^\mu \left[ -\exp \left\{ -\gamma P \left( W_T^P + D_T - C_T \right) \right\} \right]$$

s.t. $W_T^P = R_T W_0^P + \int_0^T \pi_t^P dB_t^0$

$$(\mu_t, \pi_A) \in \arg \max \hat{\mu}, \hat{\pi}_A E^{\hat{\mu}} \left[ -\exp \left\{ -\gamma_A \left( W_T^A + C_T - \int_0^T h(\mu_t) dt \right) \right\} \right]$$

s.t. $W_T^A = R_T W_0^A + \int_0^T \pi_A^t dB_t^0$

$$E^\mu \left[ -\exp \left\{ -\gamma_A c_A \right\} - \exp \left\{ -\gamma_A \left( W_T^A + C_T - \int_0^T h(\mu_t) dt \right) \right\} \right] \geq -L.$$ 

By the incentive compatibility condition, the agent’s expected utility is

$$\max E^\mu \left[ -\exp \left\{ -\gamma_A \left( C_T + W_T^A - \int_0^T h(\mu_s) ds \right) \right\} \right]$$

$$= E^\mu \left[ -\exp \left\{ -\gamma_A \left( C_T + R_T W_0^A - \int_0^T \left( R_T^A \left( \theta_s - \frac{f^*}{g} \right) - h(\mu_s) \right) ds + \int_0^T R_T^A \pi_A^s dB_s^0 \right) \right\} \right].$$

Thus, by Lemma 3.1, the BSDE representation of the agent’s certainty equivalent wealth process $\{V_t^A\}$ is as follows.

$$V_t^A = C_T + R_T W_0^A - \int_t^T Z_s^A dB_s^0$$

$$+ \int_0^T \left( R_T^A \left( \theta_s - \frac{f^*}{g} \right) - h(\mu_s) \right) ds + \int_0^T R_T^A \pi_A^s dB_s^0.$$ (A.1)

In this BSDE, note that $f^*$ is a state variable to the agent whereas $f$ is a control variable. The reason is that the agent’s effort choice $\mu$ is not observable to investors, and $f^*$ is not based on the agent’s actual effort choices, but it simply represents investors’ expectations about the agent’s effort choices.

By applying the comparison theorem to the agent BSDE (A.1), we have the FOCs with respect to $\mu$ and $\pi_A$ for the agent’s maximization of his certainty equivalent as follows.

$$\begin{align*}
\frac{h^\mu}{f^\mu} g &= Z_s^A + R_T^A \theta_s, \\
\theta_s - \frac{f^*}{g} - \gamma_A \left( Z_s^A + R_T^A \right) &= 0.
\end{align*}$$

Substituting the first FOC into the second, we have

$$\theta_s - \frac{f^*}{g} - \gamma_A \frac{h^\mu}{f^\mu} g = 0.$$ 

This condition implies that when he is allowed to trade, the agent’s effort decision is determined independently of the contract incentives. When $f^* \neq f$, the market is in disequilibrium, because investors’ expectations about the agent’s effort levels are not fulfilled. If $f^* > (\leq) f$, public investors overvalue (undervalue) the stock, and the agent is motivated to decrease (increase) his share holdings, which means low (high) incentives to work.
When the market is in equilibrium, we have $f^* = f$ and the agent’s optimal effort is determined by

$$\theta_s = \gamma_A \frac{h_{\mu}}{f_{\mu}} g.$$  \hspace{1cm} (A.2)

In other words, in equilibrium, the agent’s effort levels are simply determined by the market-price-of-risk process $\{\theta_s\}$, not by the contract sensitivity. It also means that given a contract with arbitrary incentives, the agent can simply undo them and construct his own incentives through his own capital market transactions. It is striking, however, that the agent does not completely undo the contract incentives, and let himself exposed to some incentives to work. The reason is that as long as $\theta$ is positive, the agent can be rewarded for taking risks. 

However, this implication can also be used to defend the folklore discussed in the introduction. Since compensation risks are mostly idiosyncratic, $\theta$ for idiosyncratic risks is zero and the agent will not have incentives to work given any incentive contract. The reason is that exposure to idiosyncratic risks is not rewarded in capital markets, and thus he simply unloads the compensation risk through capital market transactions. As a result, he will end up with receiving a risk-free compensation and he has no incentives to work.

Thus, the above results are summarized as follows.

**Proposition A.11** When he is allowed to trade the stock and bonds, the agent’s optimal level of effort in equilibrium is determined such that $\theta_s = \gamma_A \frac{h_{\mu}}{f_{\mu}} g$, regardless of $C_T$.

**Corollary A.3** (Folklore) If the outcome risk $\{B_t^0\}$ is idiosyncratic such that $\theta_t \equiv 0$, then the agent’s optimal effort in equilibrium is zero, i.e., $\mu_t \equiv 0$, regardless of $C_T$.

Nevertheless, we allow $\theta$ to be greater than or equal to zero. From (A.1), the contract $C_T$ can be represented as follows:

$$-C_T = -R + R_T W_0^A - \int_0^T Z_t^A dB_t^0$$

$$+ \int_0^T \left( R_T \pi_s^A \left( \theta_s - \frac{f_s}{g} \right) - h(\mu_s) + (Z_s^A + R_T \pi_s^A) \frac{f}{g} \right) \frac{\gamma_A}{2} (Z_s^A + R_T \pi_s^A)^2 ds.$$  \hspace{1cm} (A.3)

Then the principal’s terminal wealth is

$$W_T^P + D_T - C_T$$

$$= D_0 - R + R_T W_0^A - \int_0^T \left( \frac{h_{\mu}}{f_{\mu}} g - R_T \pi_s^A - R_T \pi_s^P - g \right) dB_s^0$$

$$+ \int_0^T \left( (R_T \pi_s^A + R_T \pi_s^P) \left( \theta_s - \frac{f_s}{g} \right) - h(\mu_s) + \frac{h_{\mu}}{f_{\mu}} g \right) \frac{\gamma_A}{2} (\frac{h_{\mu}}{f_{\mu}} g)^2 ds$$

Define the principal’s certainty equivalent wealth $V_T^P$ as follows:

$$-e^{-\gamma_P V_T^P} = E^\mu \left[ -\exp \left\{ -\gamma_P \left( R_T (W_0^A + W_0^P) + D_0 - R - \int_0^T \left( \frac{h_{\mu}}{f_{\mu}} g - R_T \pi_s^A - R_T \pi_s^P - g \right) dB_s^0 \right. \right. \right.$$ 

$$\left. \left. + \int_0^T \left( (R_T \pi_s^A + R_T \pi_s^P) \left( \theta_s - \frac{f_s}{g} \right) - h(\mu_s) + \frac{h_{\mu}}{f_{\mu}} g \right. \right. \right.$$ 

$$\left. \left. - \frac{\gamma_A}{2} (\frac{h_{\mu}}{f_{\mu}} g)^2 \right) ds \right] \left| \mathcal{F}_t \right].$$

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Recall that the agent’s effort is uniquely determined by the agent’s incentive compatibility condition as in Proposition A.11. Thus, when the agent is allowed to trade, the principal has no room to choose effort levels for the agent. Thus, the principal only chooses $\pi^P_t$.

By Lemma 3.1, $V^P_t$ can be represented as follows.

\[
V^P_t = R_T (W^A_0 + W^P_0) + D_0 - V^A_0 - \int_t^T Z^P_s dB^0_s
\]
\[
+ \int_0^T \left( R_T \pi^A_s + R_T \pi^P_s \right) \left( \theta_s - \frac{f^*}{g} \right) - h(\mu_s) + \frac{h_k}{f_k} \left( \frac{h_k}{f_k} g - \frac{\gamma_A}{2} \right) \frac{f^2}{g} ds
\]
\[
+ \int_0^T \left\{ \left( Z^P_s - \frac{h_k}{f_k} g + R_T \pi^A_s + R_T \pi^P_s + g \right) \frac{f}{g} - \frac{\gamma_P}{2} \left( Z^P_s - \frac{h_k}{f_k} g + R_T \pi^A_s + R_T \pi^P_s + g \right)^2 \right\} ds.
\]

(A.3)

Since the principal treats $f^* = f$, the Comparison Theorem implies that the FOC is

\[
\theta_s - \gamma_P \left( Z^P_s - \frac{h_k}{f_k} g + R_T \pi^A_s + R_T \pi^P_s + g \right) = 0.
\]

Since $\mu$ is linear in $\theta$ and $\pi^P_s + \pi^A_s = 0$ in equilibrium, we have

\[
\frac{\theta_s}{\gamma_P} = \left( 1 - \frac{h_k}{f_k} \right) g.
\]

Therefore, the substitution of (A.2), implies that $\theta$ is constant over time at $\theta$, where

\[
\theta = \frac{\gamma_A \gamma_P}{\gamma_A + \gamma_P} g.
\]

That is, when the agent is allowed to trade freely, the second-best market price of risk becomes the same as that of the first best. The net second-best contract sensitivity $Z^A_t + R_T \pi^A_t$, which is equal to the sum of the contract sensitivity $Z^A_t$ and the agent share position in the stock, is equal to that of the first best risk-sharing contract, i.e., $Z^A_t + R_T \pi^A_t = \frac{\gamma_A}{\gamma_A + \gamma_P} g$, for all $t \in [0, T]$.

This result is intuitive in the sense that when the agent is allowed to freely trade, there is no way for the principal to induce her desired level of the agent’s effort, because the agent can always undo the contract, and re-balance his risky position in securities markets according to his own preference. Then the agent’s net position in the risky asset will be determined as if both the principal and agent competitively trade without restrictions. As a result, both the principal and agent’s positions in the risky asset coincide with the first-best competitive equilibrium allocations.

Since the principal’s FOC implies $\theta$ is linear in $Z^P_t$ in equilibrium, the principal’s BSDE is quadratic in $Z^P_t$. Note that the principal’s quadratic BSDE has a solution with $Z^P_t \equiv 0$. The solution is

\[
V^P_t = R_T (W^A_0 + W^P_0) + D_0 - R
\]
\[
+ \left\{ f - h(\mu_s) - \frac{\gamma_A}{2} \left( \frac{h_k}{f_k} g \right)^2 - \frac{\gamma_P}{2} \left( 1 - \frac{h_k}{f_k} \right)^2 g^2 \right\} (T - t)
\]

(A.4)

where $\mu$ is constant over time. Therefore, by the uniqueness of solutions to quadratic BSDEs, (A.4) is the unique solution to the principal’s BSDE (A.3) in equilibrium.

Comparing the two expected utilities (A.5) and (A.4), which are, respectively, the principal’s expected utilities with and without agent trading restrictions, one can see that both utilities are
are given in identical forms, and that in the latter case, the principal only chooses \( \pi \), whereas in the former case, she can choose \( \mu \) as well as \( \pi \). Therefore, the principal is better off with restricting the agent from trading the stock, as stated in the following proposition.

**Proposition A.12** The principal’s expected utility with restriction on the agent’s stock trading is always greater than or equal to that without the restriction.

This proposition provides a justification for our principal to forbid the agent from his stock trading.

**B Proof of Lemma 3.1**

Let

\[
J_t := -\exp \left\{-\gamma \left( V_t + \int_0^t H(\mu_s, \pi_s)ds + \int_0^t v(\mu_s, \pi_s)dB_s^0 \right) \right\}.
\]

Since \( J_t \) is a \( P^\mu \)-martingale, by the martingale representation theorem, there exists a unique square-integrable and predictable process \( \{Z_s\} \) such that

\[
J_t = -\exp \left\{-\gamma \left( F(B_T^0) + \int_0^T H(\mu_s, \pi_s)ds + \int_0^T v(\mu_s, \pi_s)dB_s^0 \right) \right\} + \int_t^T J_s \tilde{Z}_s dB_s^\mu.
\]

Thus,

\[
dJ_t = -J_t \tilde{Z}_t dB_t^\mu = -J_t \tilde{Z}_t \left( dB_t^0 - \frac{f}{g} dt \right).
\]

On the other hand, the definition of \( J_t \) implies

\[
dJ_t = -\gamma J_t \langle dV_t + H(\mu_t, \pi_t)dt + v(\mu_t, \pi_t)dB_t^0 \rangle + \frac{\gamma^2}{2} J_t \left( \langle dV_t, dV_t \rangle + 2 \langle dV_t, v_t dB_t^0 \rangle + v^2 dt \right).
\]

Hence,

\[
dV_t + H(\mu_t, \pi_t)dt + v(\mu_t, \pi_t)dB_t^0 - \frac{\gamma}{2} \left( \langle dV_t, dV_t \rangle + 2 \langle dV_t, v_t dB_t^0 \rangle + v^2 dt \right) = \tilde{Z}_t \left( dB_t^0 - \frac{f}{g} dt \right),
\]

and

\[
\langle dV_t, dV_t \rangle = \left( \frac{\tilde{Z}_t}{\gamma} - v(\mu_t, \pi_t) \right)^2 dt, \quad \text{and} \quad \langle dV_t, v_t dB_t^0 \rangle = \left( \frac{\tilde{Z}_t}{\gamma} - v(\mu_t, \pi_t) \right) v_t dt.
\]

That is

\[
dV_t = -\left( H(\mu_t, \pi_t) + \frac{\tilde{Z}_t f}{g} - \frac{\gamma}{2} \left( \frac{\tilde{Z}_t}{\gamma} \right)^2 \right) dt + \left( \frac{\tilde{Z}_t}{\gamma} - v(\mu_t, \pi_t) \right) dB_t^0.
\]

Let

\[
\frac{1}{\gamma} Z_t := \frac{1}{\gamma} \tilde{Z}_t - v.
\]

Then, by substitution, the assertion follows.
C Proof of Corollary 4.1

Lemma 3.1 implies that given admissible \( \mu \), the second-stage agent’s expected utility with the certainty equivalent wealth being \( R \) can be represented as follows.

\[
V^A_0 = R = R_T W^A_0 + C_T + \int_0^T \left[ \frac{Z^A_s f}{\gamma_A g} - h(\mu_s) - \frac{1}{2\gamma_A} (Z^A_s)^2 \right] ds - \int_0^T \frac{Z^A_s}{\gamma_A} dB^0_s.
\]

This relationship immediately reveals the structure of the compensation function \( C_T \) as stated in the Lemma.

Substituting the above salary representation (4.9) for \( C_T \) into the principal’s problem, we define the principal’s certainty equivalent wealth process \( V^P_t \) as follows:

\[
- \exp \left\{ -\gamma P V^P_t \right\} = E^{\mu} \left[ -\exp \left\{ -\gamma P (W^P_t + D_T - C_T) \right\} \right| \mathcal{F}_t
\]

\[
= E^{\mu} \left[ -\exp \left\{ -\gamma P \left( R_T (W^P_0 + W^A_0) + D_0(I) - R_T \right)
+ \int_t^T \left[ \frac{Z^A_s f}{\gamma_A g} - h(\mu_s) - \frac{1}{2\gamma_A} (Z^A_s)^2 + R_T \pi^P_s (\theta_s - f^*_s) \right] ds
- \int_t^T \left( \frac{Z^A_s}{\gamma_A} - g - R_T \pi^P_s \right) dB^0_s \right\} \right| \mathcal{F}_t
\].

Then, given \((c^P_0, I)\), the principal’s second-stage problem is to choose \( (\pi^P_t, \mu_t, Z^A_t) \) to maximize

\[
- \exp \left\{ -\gamma P V^P_0 \right\}.
\]

By Lemma 3.1, the certainty equivalent process for the principal’s future unrealized wealth can be represented as as stated in (4.10). That is, at each time \( t > 0 \), the principal maximizes her conditional certainty equivalent wealth \( V^P_t \) by choosing \((\pi^P_t, \mu_t, Z^A_t)\).

D Proof of Corollary 5.2

The incentive compatibility condition tells us that given \( C_T \), the agent chooses his effort levels \( \{\mu_s\} \) to maximize his own certainty equivalent wealth, \( V^A_0 \) where

\[
V^A_t = R_T W^A_0 + C_T + \int_t^T \left[ \frac{Z^A_s f}{\gamma_A g} - h(\mu_s) - \frac{1}{2\gamma_A} (Z^A_s)^2 \right] ds - \int_t^T \frac{Z^A_s}{\gamma_A} dB^0_s.
\]

By the comparison theorem, the FOC for the maximization is

\[
\frac{Z^A_s}{\gamma_A} = h(\mu_s). \frac{f}{g}.
\]

Thus, we have the representation of the second-best salary function as stated in (5.21).

Substituting the salary representation into the principal’s second-stage certainty equivalent
wealth $V_t^P$, we have
\[
\max - \exp\{-\gamma_P V_0^P\}
= E^\mu \left[ - \exp \left\{ -\gamma_P (W_t^P + D_T - C_T) \right\} \right] \\
= E^\mu \left[ - \exp \left\{ -\gamma_P \left( R_T (W_0^P + W_0^A) + D_0(I) - R \right) \right\} \\
+ \int_0^T \left[ \frac{h}{f} f - h(\mu_s) - \frac{\gamma_A}{2} \left( \frac{h}{f} g \right)^2 + R_T \pi_s^P \left( \theta - \frac{f^*}{g} \right) \right] ds \right. \\
- \left. \int_0^T \left[ \frac{h}{f} g - R_T \pi_s^P - g \right] dB_s^0 \right] \}
\]
Thus, by Lemma 3.1, the principal’s certainty equivalent process can be represented by a BSDE as stated in (5.22).

\[ \square \]

E Proof of Proposition 5.3

Under the stated assumptions, by FOCs (5.23) and (5.24) and the equilibrium condition $\pi_s^P \equiv 0$, we have
\[ \mu_s = \frac{\theta}{\gamma_A} \left( \frac{a}{\delta g} \right)^2 \left\{ \frac{\delta g}{a} + \frac{1}{\gamma_P a} \right\}. \]
This implies $\mu_s$ is linear in $\theta_s$. Then, the equilibrium condition $\pi_s^P \equiv 0$, the linearity of $\mu_s$ and FOC (5.23) imply $\theta_s$ is linear in $Z_s^P$, and thus $\mu_s$ is linear in $Z_s^P$ as well.

On the other hand, since $\mu_s$ and $\theta_s$ are linear in $Z_s^P$, the principal’s BSDE (5.22) is quadratic in $Z_s^P$. Furthermore, if $Z_s^P \equiv 0$, FOCs (5.23) and (5.24) and the equilibrium condition $\pi_s^P \equiv 0$ imply that both $\mu_s$ and $\theta_s$ are constant over time such that $\mu_t = \mu$, and $\theta_t = \theta$ for all $t$ where $(\mu, \theta)$ are given as stated in the proposition. Moreover, with the pair of constants $(\mu_s, \theta)$, BSDE (5.22) has a solution as follows.
\[
V_t^P = R_T (W_0^P + W_0^A) + D_0(I) - R \\
+ \left[ f - h(\mu_s) - \frac{\gamma_A}{2} \left( \frac{h}{f} g \right)^2 - \frac{1}{2\gamma_P} \theta^2 \right] (T-t).
\]
Thus, by the uniqueness of the solution to the quadratic BSDE, this solution is unique. Therefore, in capital market equilibrium, there exists a unique equilibrium with $\mu_t = \mu$, such that $\theta_t = \theta$ for all $t$.

\[ \square \]

F Proof of Proposition 6.5

Let the RHS of condition (6.32) denoted by $\Phi(\kappa, R_T, \tau, A)$. Note that
\[
\Phi_{R_T} = - \frac{A^2 (1 + R_T)}{2(R_T + \kappa_F)^3} - \frac{\tau}{R_T} < 0, \quad \Phi_A = \frac{A(1 + \kappa_F + 2R_T)}{2(R_T + \kappa_F)^2} > 0 \\
\Phi_\kappa = - \frac{A^2}{4(R_T + \kappa_F)^3} (2 + \kappa_F + 3R_T) < 0.
\]
Then, the condition implies that
\[
\frac{\partial R_T}{\partial M} = \frac{1}{\Phi_{R_T}} < 0, \quad \text{and} \quad \frac{\partial R_T}{\partial A} = -\frac{\Phi_A}{\Phi_{R_T}} > 0.
\]
Therefore, these inequalities together with (6.30) imply the first and second claims.

Moreover,
\[
\frac{\partial R_T}{\partial \sigma_D} = \frac{1}{\Phi_{R_T}} \left( I - \frac{\tau}{R_T} \right) \frac{\partial \kappa_F}{\partial \sigma_D},
\]
\[
\frac{\partial}{\partial (1/\delta)}(R_T + \kappa_F) = \frac{1}{\Phi_{R_T}} \left( I - \frac{\tau}{R_T} \right) \frac{\partial \kappa_F}{\partial (1/\delta)}.
\]
These relationships together with (6.30) suggest the third and fourth claims.

\[\blacksquare\]

\section*{G Proof of Proposition 6.6}

Note that \( \theta = \frac{1}{\tau} g(I) = \frac{\varphi}{\tau} I^{\frac{1}{2}} \). We have
\[
\frac{\partial \theta}{\partial M} = \frac{\sigma_D}{2\tau} \frac{\partial I}{\partial M} > 0,
\]
\[
\frac{\partial \theta}{\partial A} = \frac{\sigma_D}{\tau} \frac{1}{2(R_T + \kappa_F)^2} \Phi_{R_T} \left( 2I - \frac{\tau}{R_T} \right) \frac{\partial \kappa_F}{\partial \sigma_D},
\]
\[
\frac{\partial \theta}{\partial \sigma_D} = \frac{1}{\tau} \frac{A}{2(R_T + \kappa_F)^2} \Phi_{R_T} \left( R_T + \kappa_F \right) \frac{\partial \kappa_F}{\partial \sigma_D} - \frac{\sigma_D^2 I}{\tau} \left( I - \frac{\tau}{R_T} \right) \frac{\partial \kappa_F}{\partial \sigma_D},
\]
\[
\frac{\partial \theta}{\partial (1/\delta)} = -\frac{\sigma_D}{\tau} \frac{A}{2(R_T + \kappa_F)^2} \Phi_{R_T} \left( I - \frac{\tau}{R_T} \right) \frac{\partial \kappa_F}{\partial (1/\delta)}.
\]
Therefore, the assertions follow.

\[\blacksquare\]

\section*{H Proof of Proposition 6.9}

Our sufficient condition relies on the set \( \Upsilon \). Thus, we first need to establish the nonemptiness of \( \Upsilon \).

\begin{lemma}
Suppose \( g(a, \kappa_S) < 0, g(b, \kappa_S) < 0, \) and \( \Phi(b, \kappa_F) < \Phi(a, \kappa_S) \) for \( a < b \). If \( M \) is given such that \( \Phi(b, \kappa_F) < M < \Phi(a, \kappa_S) \), then \( M \in \Upsilon \) and \( R_T \in [a, b] \).
\end{lemma}

\begin{proof}
Note that \( \forall \kappa \in \Lambda, g(R^1_T(\kappa), \kappa) = g(R^2_T(\kappa), \kappa) = 0, \) and \( g(R_T(\kappa), \kappa) < 0 \) for \( R_T(\kappa) \in [R^2_T(\kappa), R^1_T(\kappa)] \). However, since \( g_\kappa > 0, \forall \kappa \in \Lambda, \) we have
\[
g(a, \kappa) \leq g(a, \kappa_S) < 0,
\]
\[
g(b, \kappa) \leq g(b, \kappa_S) < 0.
\]
Thus, \( \forall \kappa \in \Lambda, \)
\[
R^2_T(\kappa) \leq a < b \leq R^1_T(\kappa).
\]

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Therefore, if \( R_T(\kappa) \in [a, b] \), then \( g(R_T(\kappa), \kappa) < 0 \). On the other hand, recall that \( \Phi_{R_T} < 0 \). Thus the following inequalities should hold: \( \forall \kappa \in \Lambda \),

\[
\Phi(R_T^1(\kappa), \kappa) \leq \Phi(b, \kappa) < \Phi(a, \kappa) \leq \Phi(R_T^2(\kappa), \kappa).
\]

However, since \( \Phi(b, \kappa_F) < \Phi(a, \kappa_S) \) by assumption, we have, \( \forall \kappa \in \Lambda \),

\[
\Phi(R_T^1(\kappa), \kappa) \leq \Phi(b, \kappa) < \Phi(b, \kappa_F) < \Phi(a, \kappa_S) < \Phi(a, \kappa) \leq \Phi(R_T^2(\kappa), \kappa).
\]

Therefore, \( M \in \Upsilon \). Moreover since in equilibrium \( M = \Phi(R_T(\kappa), \kappa) \) and

\[
\Phi(b, \kappa_S) < \Phi(b, \kappa_F) < M = \Phi(R_T(\kappa), \kappa) < \Phi(a, \kappa_S) < \Phi(a, \kappa_F),
\]

we must have \( a < R_T(\kappa) < b \), for \( \kappa \in [\kappa_F, \kappa_S] \).

Now we are ready to prove Proposition 6.9. Note that

\[
R_T^S - R_T^F + \kappa^S - \kappa^F = -\int_{\Lambda} \frac{\Phi(R_T(\kappa), \kappa)}{\Phi_{R_T}(R_T(\kappa), \kappa)} - 1 \, d\kappa.
\]

Since \( \Phi_{R_T} < 0 \), this equation implies that if \( \Phi_e(R_T(\kappa), \kappa) - \Phi_{R_T}(R_T(\kappa), \kappa) < 0 \) for all \( \kappa \in \Lambda = [\kappa_F, \kappa_S] \), then \( I^S > I^F \), i.e., the second best investment level is higher than the first best. Note that

\[
\Phi_e(R_T(\kappa), \kappa) - \Phi_{R_T}(R_T(\kappa), \kappa) = \frac{(\gamma_A + \gamma_p)}{\gamma_p \gamma_A R_T(\kappa + \kappa)^2} G(R_T(\kappa), \kappa)
\]

where

\[
G(R_T(\kappa), \kappa) := \left( 2\kappa - \frac{\gamma_p \gamma_A A^2}{4(\gamma_p + \gamma_A)} \right) R_T + \kappa^2.
\]

Let

\[
\Delta := \left( 2\kappa - \frac{\gamma_p \gamma_A A^2}{4(\gamma_p + \gamma_A)} \right)^2 - 4\kappa^2 = \left( \frac{\gamma_p \gamma_A A^2}{4(\gamma_p + \gamma_A)} \right) \left( \frac{\gamma_p \gamma_A A^2}{4(\gamma_p + \gamma_A)} - 4\kappa \right)
\]

If \( \Delta < 0 \) for all \( \kappa \in \Lambda \), then \( g(R_T, \kappa) > 0 \) for all \( \kappa \), and thus \( \Phi_e(R_T(\kappa), \kappa) - \Phi_{R_T}(R_T(\kappa), \kappa) > 0 \) for all \( \kappa \in \Lambda \), i.e., the first-best production investment level is greater than the second-best.

By assumption \( \Delta > 0 \) for all \( \kappa \) and \( \kappa > 0 \). Note that the roots of equation \( g(R_T, \kappa) = 0 \) are as follows:

\[
R_T^1(\kappa) = \frac{1}{2} \left\{ \frac{\gamma_p \gamma_A A^2}{4(\gamma_p + \gamma_A)} - 2\kappa + \sqrt{\Delta} \right\},
\]

\[
R_T^2(\kappa) = \frac{1}{2} \left\{ \frac{\gamma_p \gamma_A A^2}{4(\gamma_p + \gamma_A)} - 2\kappa - \sqrt{\Delta} \right\}.
\]

Then, \( 0 < R_T^S(\kappa) < R_T^F(\kappa) \). Thus, if \( g(R_T, \kappa) < 0 \) for all \( \kappa \in \Lambda \), then \( I^S > I^F \). This can be achieved if for each \( \kappa \in \Lambda \), \( R_T^S(\kappa) \) lies in between \( R_T^2(\kappa) \) and \( R_T^1(\kappa) \).

However, recall \( \Phi_{R_T} < 0 \), and the market-clearing identity, \( \Phi(R_T(\kappa), \kappa) = M \) for all \( \kappa \). Since \( M \in \Upsilon \) by assumption, we have \( \Phi(R_T^1(\kappa), \kappa) < \Phi(R_T(\kappa), \kappa) = M < \Phi(R_T^2(\kappa), \kappa) \), which implies that \( R_T(\kappa) \), for all \( \kappa \in \Lambda \), lies in between \( R_T^2(\kappa) \) and \( R_T^1(\kappa) \).

\[\text{I Proof of Proposition 6.10}\]

By applying Propositions (4.1) and (5.3) to our linear-quadratic case, we have the first- and second-best market prices of risk are given as follows.

\[
\theta^F = \frac{\gamma_A \gamma_p \sigma_D A}{2(\gamma_A + \gamma_p) (R_T^2 + \kappa^F)} = \frac{\gamma_A \gamma_p \sigma_D}{(\gamma_A + \gamma_p)} I^{\frac{1}{2}}
\]

\[
\theta^S = \frac{\gamma_p \gamma_A \sigma_D A}{2 \left[ \frac{1}{\sigma_D^2} + (\gamma_A + \gamma_p) \right] (R_T^2 + \kappa^S)} = \frac{\gamma_p \gamma_A \sigma_D}{\left[ \frac{1}{\sigma_D^2} + (\gamma_A + \gamma_p) \right]} I^{\frac{1}{2}}
\]

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However, by (4.20) and (5.29),

\[
\begin{align*}
\nu_0^F &= \frac{\theta^2}{\gamma P R_T S_0} = \frac{(\gamma_A \gamma P \sigma_D)^2}{\gamma P R_T^F (\gamma_A + \gamma P)^2} = \frac{\gamma P \gamma_A^2 \sigma_D^2}{R_T^F (\gamma_A + \gamma P)^2} \\
\nu_0^S &= \frac{\theta^2}{\gamma P R_T S_0} = \frac{(\gamma_A \gamma P \sigma_D)^2}{\gamma P R_T^S \left( \frac{1}{\delta \sigma_D} + (\gamma_A + \gamma P) \right)^2} = \frac{\gamma P \gamma_A^2 \sigma_D^2}{R_T^S \left( \frac{1}{\delta \sigma_D} + (\gamma_A + \gamma P) \right)^2}
\end{align*}
\]

Thus, the assertion follows.
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