Multiplicative Risk Prudence*

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Abstract

We examine the optimal saving decision of individuals who face a multiplicative risk. An individual is defined to be multiplicative risk prudent if multiplying a pure risk to her future wealth raises her optimal savings. We show that convex marginal utility is not sufficient to induce multiplicative risk prudent. Instead, an individual is multiplicative risk prudent if and only if her relative prudence of future consumption uniformly exceeds two. We then study jointly the impact of correlated additive and multiplicative risks on optimal savings decision and demonstrate that the concept of multiplicative risk prudence is stronger than additive-multiplicative risk prudence. Our results suggest one should take the condition of multiplicative risk prudence as a natural restriction on preference. In addition, our findings provide an explanation to the risk-free rate puzzle.

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1. Introduction

It is a natural belief that an individual will save more if she is less certain about the level of her future wealth. The individual who behaves in this way is said to be risk prudent and the uncertainty induced additional savings is called precautionary savings. The uncertainty could be either additive or multiplicative to future wealth.[1] Leland (1968) shows that an individual who faces an additive risk is risk prudent, if and only if, her marginal utility of future wealth is convex.[2] However, one question that has not been addressed so far is: Under what condition will the individual subject to multiplicative risk behave in a similar 'risk prudent' fashion?

There has been a fair amount of literature focusing on the effect of additive background risk. Undiversifiable labour income risk (which is an additive background risk) have been studied in the context of optimal portfolio choice (e.g., see Heaton and Lucas (1997, 2000), Vicera (2001), and Franke, Peterson, and Stapleton (2005)) and equilibrium asset prices (e.g., see Weil (1992)). Moreover, various preference conditions have been established under which the presence of additive background risk increases risk aversion towards other independent risks (e.g., see Pratt and Zeckhauser (1987), Kimball (1993) and Gollier and Pratt (1996)). Surprisingly, there has been little attention given to the case where the background risk is multiplicative. Franke, Schlesinger, and Stapleton (2005) point out that examples with multiplicative risk are at least as prevalent as those with additive background risk and extend some results on optimal bond-stock investment. They establish a set of conditions on preferences under which the presence of multiplicative background risk makes the investor behave in a 'more risk-averse' manner.

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[1] Specifically, for an investor with certain level of wealth w, the impact of an additive risk $\tilde{x}$ on her future wealth can be written as $w + \tilde{x}$, and the effect of a multiplicative risk $\tilde{y}$ is given by the product $w\tilde{y}$.

[2] Note that Leland (1968) examines the impact of future endowment uncertainty on savings where the future endowment is an additive component of future wealth. Under risk aversion, the marginal utility is convex if and only if the degree of *absolute* prudence is uniformly greater than zero. *Absolute* prudence is defined as the negative of the ratio of the third to the second derivative of the utility function, The notion of *absolute* and *relative* prudence were first introduced by Kimball (1990).

We believe that multiplicative risk matters to both corporate and individual savings decisions. For instance, a multinational company may be exposed to exchange rate risk which is multiplicative to its overseas profit, and this additional risk may affect the optimal cash holding decisions of the company. Similarly, an individual savings decision may be influenced by the inflation uncertainty which is multiplicative to her future real wealth.

This paper investigates the impact of multiplicative risk on optimal savings decision. We introduce a concept called multiplicative risk prudence that is analogous to the definition of additive risk prudence. Multiplicative risk prudence is defined as follows.[3]

Definition 1 *An individual is multiplicative risk prudent, if multiplying a pure risk to her future wealth raises her optimal savings.*

Since the seminal paper by Leland (1968), the effect of absolute prudence on savings has became obfuscated by generality. A prudent individual (whose marginal utility is convex) is expected to save more if she is less certain about the level of her future wealth. Our analytical and numerical results indicate that this well-accepted generality does not hold when the uncertainty attached to the future wealth is multiplicative. Using a simple numerical example, we first show that the well-accepted condition of risk prudence (i.e. convex marginal utility) fails to induce multiplicative risk prudence. Assuming additively-seperable (AS) utility, we then derive the necessary and sufficient condition of multiplicative risk prudence, which is stronger than the one suggested by Leland (1968). We then analyze the optimal savings in response to an increase in multiplicative uncertainty following Rothschild

[3] Note that a multiplicative risk is defined to be a pure risk if the expected value of the future wealth is unaffected by the multiplicative risk (i.e. the multiplicative risk has an expected value of one).

and Stiglitz (1971), and find that a multiplicative risk prudent individual will always raise her optimal savings in response to an increase in multiplicative uncertainty.[4]

We further generalize our analysis of multiplicative risk prudence and the sensitivity of optimal responses to the case of non-additively-separable (*NAS*) utility. Utility function is often assumed additively-seperable in the literature for its mathematical convenience in deriving analytical results. Yet, this condition can be restrictive. For example, it does not allow for habit formations, a term for the idea that the marginal utility of future consumption is increasing with the level of past consumption.[5] Our results are unaffected by the introduction of this more general form of utility function.

Finally, we study jointly the impact of correlated additive and multiplicative risks on optimal savings decision. An individual is defined to be additive-multiplicative risk prudent, if the joint presence of additive and multiplicative pure risks raise her optimal savings. Many prior studies on background risk assume that background risks are independent. Tsetlin and Winkler (2005) point out that the independence assumption is often unrealistic, ignoring the correlation of background risk may lead to a poor decision. Their results suggest that the optimal risky choices in the correlated setting can be very different from those that would appear optimal if the correlation were ignored. In practice, additive and multiplicative risks and their correlation can be very important to corporations and individuals in making their cash holding and saving decisions. Additive risk can come in the form of overseas profit uncertainty for a multinational company or future endowment uncertainty for an individual. One may expect overseas profit or future endowment to be correlated with the exchange rate or inflation and jointly affect optimal savings. Our analysis suggests that when additive and multiplicative risks are independent, the concept of multiplicative risk prudence and additive-multiplicative risk prudence are equivalent. However, when additive and multiplicative risks

[4] Kimball (1990) also exploits the observation of Rothschild and Stiglitz (1971) to analyze the sensitivity of an optimal responses, though he focuses on the case of additive risks.
[5] See Constantinides (1990) and Campbell and Cochrane (1999).

are positively correlated, the concept of multiplicative risk prudence is stronger than additive-multiplicative risk prudence.

Our results on the impact of multiplicative risk on aggregate savings also provide an explanation to the risk-free rate puzzle. As stated by Mehra and Prescott (1985), the model of the representative consumer of Lucas (1978) predicts a risk-free rate that is too high given the observed low variability of consumption growth. Intuitively, an increase in inflation uncertainty in an economy should induce aggregate precautionary motive to save. This must be compensated by a reduction of risk-free rate. Fail to take inflation uncertainty into the account may lead to an overestimation of risk-free rate.

The remainder of the paper is organized as follows. We set up our basic model and present a motivating example in Section 2 to demonstrate the insufficiency of convex marginal utility to induce multiplicative risk prudence. Section 3 derives the necessary and sufficient condition for multiplicative risk prudence and studies the effects of an increase in multiplicative uncertainty on optimal savings. We further generalize the concept of multiplicative risk prudence and the sensitivity of optimal responses to case of *NAS* utility. Section 4 studies the impact of correlated additive and multiplicative risks on optimal savings decision and Section 5 concludes.

2. Is a prudent individual really risk prudent?

An individual is called prudent if her marginal utility is convex. As suggested by Leland (1968), a risk prudent individual is expected to save more if she is less certain about the level of her future wealth. Is a prudent individual really risk prudent? We show that, indeed, a prudent individual need not behave in a 'risk prudent' fashion.

2.1 Model

Assume a two-dated model where the dates are indexed 0 and τ. Consider the saving (s) decision of an individual who is endowed with initial wealth w_0. In addition to the initial wealth, the individual is also endowed with wealth w_τ at time τ. The individual chooses optimal savings at time 0 in order to maximize the expected utility function defined over the consumption plan (c_0, c_τ). The utility function $U(c_0, c_\tau)$ is assumed to be additively-separable, so that

$$U(c_0, c_\tau) = u_0(c_0) + u_\tau(c_\tau),$$

where, u_0 and u_τ represent the utility functions defined over the consumption at time 0 and τ, respectively.[6]

Let R_f represent the gross deposit rate of savings. To simplify our analysis, we follow Leland (1968) by assuming a fixed rate of return on savings.[7] The consumptions c_0 and c_τ can be written as $c_0 = w_0 - s$ and $c_\tau = w_\tau + R_f s$. The maximization problem of the individual with certain future wealth is the following

$$\max_s \; u_0(w_0 - s) + u_\tau(w_\tau + R_f s) \tag{1}$$

The *F.O.C.* of problem (1) is

$$u_0'(w_0 - s^*) = R_f u_\tau'(w_\tau + R_f s^*), \tag{2}$$

where s^* represents the optimal savings and u_0' and u_τ' denote the marginal utilities at time 0 and τ, respectively.

Suppose that the individual is exposed to a 'pure' multiplicative risk $\tilde{y}$ at time τ such that $\mathrm{E}\,\tilde{y} = 1$. Since our focus is on the impact of multiplicative risk on savings decision, the future endowment w_τ, which is additive to future wealth, is assumed to be constant for the

[6] The case of nonseparable utility is addressed in Section 3.
[7] Leland (1968) argues that most savings, including bank deposits and government bonds, offer constant rates of return.

moment.[8] The maximization problem and its corresponding *F.O.C.* are written, respectively, as

$$\max_s f(s) = u_0(w_0 - s) + Eu_\tau((w_\tau + R_f s)\tilde{y}), \tag{3}$$

and

$$u_0'(w_0 - \hat{s}) = R_f E\tilde{y}u_\tau'((w_\tau + R_f \hat{s})\tilde{y}), \tag{4}$$

where $\hat{s}$ represents the optimal savings for an individual facing a multiplicative risk. An individual is said to be multiplicative risk prudent if the existence of multiplicative risk $\tilde{y}$ results in precautionary savings (i.e. $\hat{s} \geq s^*$).

It is worth pointing out that in contrast to problem (3), Leland (1968) examines the optimal savings problem of an individual whose future endowment is stochastic,

$$\max_s f(s) = u_0(w_0 - s) + Eu_\tau(\tilde{w}_\tau + R_f s), \tag{5}$$

where $\tilde{w}_\tau$ is the stochastic future endowment.

The *F.O.C.* of problem (5) can be written as

$$u_0'(w_0 - \bar{s}) = R_f Eu_\tau'(\tilde{w}_\tau + R_f \bar{s}), \tag{6}$$

where $\bar{s}$ represents the optimal savings for an individual with an additive risk.

2.2 A numerical example

Assume the additively-seperable constant relative risk aversion (*CRRA*) utility function,

$$U(c_0, c_\tau) = u_0(c_0) + u_\tau(c_\tau),$$

where,

$$u_\tau(c_\tau) = \beta u_0(c_\tau), \tag{7}$$

[8] It is noteworthy that this assumption is merely for simplicity. The case of the stochastic and correlated future endowment is discussed in Section 4.

$$u_0(c_0) = \frac{c_0^\gamma}{\gamma} \beta u_0(c_\tau), \tag{8}$$

and $\gamma < 1$ under risk aversion. $0 \leq \beta \leq 1$ is the discount factor for the utility on consumption at time τ.

For simplicity, we isolate the wealth effect from the consumption smoothing effect in our analysis by assuming $R_f = 1$, $\beta = 1$ and $w_0 = w_\tau = 1$. Given these assumptions, there is no incentive for any individual to save under certainty (i.e. $s^* = 0$).[9]

Define a tri-variate random variable, $\tilde{y}_0 = (1.3, 1, 0.8)$, with probability distribution, $P(\tilde{y}_0) = (0.32, 0.2, 0.48)$. $\tilde{y}_0$ is a pure risk since $E\tilde{y}_0 = 1$ and it is a multiplicative (additive) risk if it is multiplicative (additive) to the future wealth.[10]

It is widely believed that a prudent individual will save more if she is less certain about the level of her future wealth. This is the notion of precautionary savings. While the motive of precautionary savings is natural, the convex marginal utility is not sufficient to induce precautionary savings. This can be illustrated by considering the optimal savings of a prudent individual with $\gamma = 0.5$.[11]

Table 1 shows the optimal savings of the individual with $\gamma = 0.5$. Optimal savings are computed under three situations - where her future wealth is exposed to no uncertainty, additive uncertainty and multiplicative uncertainty, respectively.[12] The individual will save 1.6773% (-0.585%) of her wealth when $\tilde{y}_0$ is additive (multiplicative). Contrary to the common belief, the prudent individual saves less when her future wealth is exposed to multiplicative uncertainty.

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[9] One can check that $s^* = 0$ by solving *F.O.C.* (2) for any value of γ.

[10] Specifically, $\tilde{y}_0$ is a multiplicative (or additive) risk if the future wealth is given by $(w_\tau + R_f s)\ \tilde{y}_0$ (or $w_\tau\ \tilde{y}_0 + R_f s$).

[11] Given the *CRRA* utility in equation (8), the degree of absolute prudence is given by $(2 - \gamma) / w$. It is positive for all $\gamma < 2$ (i.e. all individuals with $\gamma < 2$ are prudent).

[12] The optimal savings under multiplicative (additive) uncertainty is computed by solving equation (4) (equation (6)), where u_τ and u_0 are given by equation (7) and (8).

Figure 1 plots the degree of relative prudence on the horizontal axis with the optimal savings on the vertical axis.[13] The solid line (dashed line) represents the optimal savings for the case where the uncertainty is multiplicative (additive). Note that the optimal savings are increasing with the degree of relative prudence (i.e. an individual will save more when she is more prudent). More importantly, Figure 1 shows that the convex marginal utility is insufficient to induce precautionary savings for the case where uncertainty is multiplicative (i.e. optimal savings are negative when the relative prudence is below 2). This implies that a prudent individual need not behaves in a 'risk prudent' fashion.

Like risk aversion, risk prudence is generally viewed as a natural human behaviour. Given that the uncertainty of future wealth can take different forms, if one believes that additive risk prudence is a natural assumption, then there is no reason why we should ignore the assumption of its multiplicative counterpart. Our numerical results suggest that a stronger condition than the traditional convex marginal utility is required for an individual to be risk prudent.

3. The condition of multiplicative risk prudence

In this section, we derive the condition on preferences for an individual to be multiplicative risk prudent. We first look at the case where the utility function is additively-separable (*AS*), followed by the derivation for the non-additively-separable (*NAS*) utility.

3.1 Additively-separable (*AS*) utility

Recall that an individual is said to be multiplicative risk prudent, if $\hat{s} \geq s^*$. This implies that

$$f'(s^*) = -u_0'(w_0 - s^*) + R_f E\tilde{y}u_\tau'((w_\tau + R_f s^*)\tilde{y}) \geq 0. \tag{9}$$

Rearranging inequality (9) and using equation (2) yields

[13] The *relative* prudence is defined as wealth multiplied by the absolute prudence (i.e. $-wu'''(w)/u''(w)$). Given the *CRRA* utility in equation (8), the degree of the relative prudence is given by 2 - γ.

$$E\tilde{y}u'_\tau((w_\tau + R_f s^*)\tilde{y}) \geq u'_\tau(w_\tau + R_f s^*). \tag{10}$$

Thus, the equivalent condition to multiplicative risk prudence is

$$\forall \tilde{y}, w : E\tilde{y} = 1 \Rightarrow Eg_w(\tilde{y}) \geq g_w(1), \tag{11}$$

where $g_w(y) = yu'_\tau(wy)$, and $w = w_\tau + R_f s^*$.

We find that condition (11) holds, if and only if, $p_\tau \geq 2$, where $p_\tau(w) = -w\dfrac{u'''_\tau(w)}{u''_\tau(w)}$ represents the degree of relative prudence at time τ. The sufficiency is immediate from Jensen's inequality and simple algebra. That is, $Eg_w(\tilde{y}) \geq g_w(1)$ if $g_w(y)$ is convex in y (i.e. $p_\tau \geq 2$).

To show the necessity, consider the risk $\tilde{y}_\varepsilon$ with the probability distribution, $P(\tilde{\varepsilon} = 1 + \varepsilon) = P(\tilde{\varepsilon} = 1 - \varepsilon) = 1/2$. Using a Taylor series expansion, one can show that for sufficiently small ε,

$$Eg_w(\tilde{y}_\varepsilon) - g_w(1) = \frac{1}{2} g''_w(1) E\tilde{\varepsilon}^2, \tag{12}$$

where $\tilde{\varepsilon}$ is a risk with the probability distribution, $P(\tilde{\varepsilon} = \varepsilon) = P(\tilde{\varepsilon} = -\varepsilon) = 1/2$. Equation (12) is positive, if and only if, $g_w(1) \geq 0$, or equivalently, $p_\tau \geq 2$. Thus, if $p_\tau < 2$ for some w, there will exist some risk $\tilde{y}_\varepsilon$ with small ε such that the individual is not multiplicative risk prudence. This leads directly to the following proposition.

Proposition 1 *An individual is multiplicative risk prudent, if and only if,*

$$g_w(y) = yu'_\tau(wy) \text{ is convex in } y,$$

or equivalently,

$$p_\tau(w) \geq 2 \ \forall w,$$

where $p_\tau(w) = -w\dfrac{u'''_\tau(w)}{u''_\tau(w)}$ *represents the degree of relative prudence at time τ.*

Leland (1968) demonstrates that for an individual facing an additive pure risk to raise savings, the degree of the absolute risk prudence, $-\frac{u_\tau'''(w)}{u_\tau''(w)}$, must be greater than zero. In contrast, our Proposition 1 states that the uncertainty of future wealth, caused by a pure multiplicative risk, raises optimal savings if and only if the relative prudence uniformly exceeds two. Given that an individual cannot be a net borrower when she retires (i.e. $w \geq 0$ at time τ), the concept of the multiplicative risk prudence is stronger than that of the additive risk prudence. Thus our Proposition 1 suggests a new necessary and sufficient condition that needs to be imposed on preferences for individuals to be risk prudent.

Our result of the multiplicative risk prudence may have implications for the risk-free rate puzzle. Specifically, the risk-free rate puzzle means that the representative consumer model of Lucas (1978) predicts a risk-free rate that is too high given the observed low variability of consumption growth.

Consider the representative consumer whose maximization problem is defined in equation (3). Using equation (4) and assuming that the market clears at $\hat{s}$ = 0, one can have

$$R_{f,\tilde{y}} = \frac{u_0'(w_0)}{E\tilde{y}u_\tau'(w_\tau \tilde{y})}, \tag{13}$$

where $R_{f,\tilde{y}}$ represents the risk-free rate in the presence of inflation uncertainty $\tilde{y}$.

If the representative consumer is risk prudent, condition (11) holds and the risk-free rate is lower in the presence of inflation uncertainty. Intuitively, the presence of inflation uncertainty gives rise to an aggregate precautionary motive to save. This must be accompanied by a reduction in risk-free rate. The failure to take inflation uncertainty into account may lead to an overestimation of risk-free rate.[14]

[14]Note that the model in equation (13) is used only to illustrate that the implication of multiplicative risk prudence for the risk-free rate puzzle. It assumes a deterministic consumption growth and the only way of shifting consumption through time is via a nominal bond. The deterministic growth assumption is merely for

A natural extension of Proposition 1 is to examine the effect of an increase in multiplicative risk on optimal savings. Consider a pair of multiplicative risks, $\tilde{y}_1$ and $\tilde{y}_2$, such that $\tilde{y}_1$ is an increase in risk of $\tilde{y}_2$ in the sense of Rothschild and Stiglitz (1970) (i.e. $\tilde{y}_2$ second order stochastically dominates $\tilde{y}_1$). Intuition suggests that the individual will save more under $\tilde{y}_1$. We show that, however, the individual will save more under $\tilde{y}_1$, if and only if, the individual is multiplicative risk prudent.

Consider an individual with utility as a function of some control parameter α and a random variable $\tilde{\theta}$. The individual chooses α in order to maximize her expected utility

$$\max_{\alpha} \mathrm{E}U(\tilde{\theta},\alpha).$$

Rothschild and Stiglitz (1971) show that if $\partial_\alpha U(\theta,\alpha)$ is a convex function of θ and decreasing in α, an increase in risk of $\tilde{\theta}$ will raise the optimal α.[15] Let $U(\tilde{y},s)=u_0(w_0-s)+Eu_\tau((w_\tau+R_f s)\tilde{y})$, this gives rise to an immediate generalization of Proposition 1.

Proposition 2 *For any pair of multiplicative risks $\tilde{y}_1$ and $\tilde{y}_2$, such that $\tilde{y}_2$ second order stochastically dominates $\tilde{y}_1$, $\tilde{y}_1$ induces greater optimal savings, if and only if, $p_\tau(w)\geq 2\ \forall w$.*

Taken together, our Proposition 2 suggests that an increase in inflation uncertainty will encourage optimal savings if the individual is risk prudent. By the same token, if the

simplicity and will be relaxed in Section 4. The equilibrium model which incorporates inflation uncertainty and allows individuals to shift consumption through time via nominal bond or stock can be founded in Chang, Grundy, and Wong (2006).

[15] Kimball (1990) also exploits the observation of Rothschild and Stiglitz (1971) to analyze the sensitivity of an optimal response, though he focuses on the case of the additive risk.

representative consumer is risk prudent, then an increase in inflation uncertainty should lead to a reduction in risk-free rate.

3.2 Non-additively-separable (*NAS*) utility

We extend our analysis of multiplicative risk prudence to the case of non-additively-separable (*NAS*) utility. Since *AS* utility is only a special case of *NAS* utility, the conditions of multiplicative risk prudence in the case of *NAS* utility will be more restrictive than (or, at least, as restrictive as) $p_\tau \geq 2$.

Without the loss of generality, we assume $w_\tau = 0$ and $R_f = 1$. The individual chooses optimal savings at time 0 in order to maximize the expected utility function defined over the consumption plan (c_0, c_τ),

$$\max_s \ \mathrm{E}U(w_0 - s, s\tilde{y}). \tag{14}$$

The optimal solution of problem (14) exists if U is concave in s, i.e.,

$$\frac{d^2U}{ds^2} = U_{11} - U_{12} - U_{12} + U_{22} \leq 0. \tag{15}$$

For the case of *AS* utility, $U_{12} = 0$, condition (15) holds naturally. For the case of *NAS* utility, condition (15) holds if $U_{12} \geq 0$, i.e., a condition which allows for internal habit formation - a term for the situation where the marginal utility of future consumption is increasing with the level of past consumption (e.g., see Constantinides (1990) and Campbell and Cochrane (1999)).

Let $s^* = \arg\max_s U(w_0 - s, s),$ the individual is multiplicative risk prudent, if and only if,

$$E\tilde{y}U_2(w_0 - s^*, s^*\tilde{y}) \geq EU_1(w_0 - s^*, s^*\tilde{y}). \tag{16}$$

Note that, $U_1(w_0 - s^*, s^*) \geq EU_1(w_0 - s^*, s^*\tilde{y}),$ if and only if, U_1 is concave in y. This implies that

$$U_2(w_0 - s^*, s^*) = U_1(w_0 - s^*, s^*) \geq EU_1(w_0 - s^*, s^*\tilde{y}), \tag{17}$$

Thus, condition (16) holds if the function $g(y) = yU_2(w_0 - s^*, s^* y)$ is convex in y (i.e. $-c_\tau \frac{U_{222}(c_0, c_\tau)}{U_{22}(c_0, c_\tau)} \geq 2$ by algebra). Given that *AS* utility is only a special case of *NAS* utility, we have the following proposition which is an extension of Proposition 1 to the case of *NAS* utility.

Proposition 3 *A sufficient condition of multiplicative risk prudence for NAS utility is*

$$-c_\tau \frac{U_{222}(c_0, c_\tau)}{U_{22}(c_0, c_\tau)} \geq 2 \text{ and } U_{122} \leq 0. \tag{18}$$

A necessary condition is

$$-c_\tau \frac{U_{222}(c_0, c_\tau)}{U_{22}(c_0, c_\tau)} \geq 2.$$

As in the case of *AS* utility, the term $-c_\tau \frac{U_{222}(c_0, c_\tau)}{U_{22}(c_0, c_\tau)}$ is a measure of relative prudence of the time τ consumption. In particular, if the utility is additively-separable, $U_{122} = 0$, and $-c_\tau \frac{U_{222}(c_0, c_\tau)}{U_{22}(c_0, c_\tau)} = -c_\tau \frac{u_\tau'''(c_\tau)}{u_\tau''(c_\tau)}$, Proposition 3 is reduced to Proposition 1.

Campbell and Cochrane (1999) show that habit formation provides an explanation to the risk-free rate puzzle, i.e.,

$$R_{f\,\text{with habit formation}} \leq R_{f\,\text{without habit formation}}.$$

In contrast, our Proposition 3 suggests that the presence of multiplicative risk and habit formation together may provide an even stronger explanation to the risk-free rate puzzle, i.e.,

$$R_{f\,\text{with habit formation and } \tilde{y}} \leq R_{f\,\text{without habit formation}}.$$

According to Proposition 3, equation (19) holds if the relative prudence of time τ consumption uniformly exceeds two (the necessary condition of multiplicative risk prudence), and the marginal utility of past consumption is increasing (the necessary condition of internal habit formation) and concave in the level of future consumption. Following the observation of Rothschild and Stiglitz (1971), one can further generalize Proposition 3 as follows.

Proposition 4 *For any pair of multiplicative risk* $\tilde{y}_1$ *and* $\tilde{y}_2$*, such that* $\tilde{y}_2$ *second order stochastically dominates* $\tilde{y}_1$*,* $\tilde{y}_1$ *induces greater amount of optimal savings, if*

$$-c_\tau \frac{U_{222}(c_0,c_\tau)}{U_{22}(c_0,c_\tau)} \geq 2 \text{ and } U_{122} \leq 0.$$

The proof is immediate given that $\frac{\partial U(w_0 - s, sy)}{\partial s}$ is convex in *y*, if and only if,

$$-sU_{122} + 2U_{22} + syU_{222} \geq 0. \tag{20}$$

Equation (20) holds if $-c_\tau \frac{U_{222}(c_0,c_\tau)}{U_{22}(c_0,c_\tau)} \geq 2$ and $U_{122} \leq 0$.

Note that Proposition 4 is simply a generalization of Proposition 2 to case of NAS utility. Similar to Proposition 2, Proposition 4 also predicts that an increase in inflation uncertainty will encourage optimal savings if the individual with NAS utility is risk prudent.

4. Additive-multiplicative risk prudence

So far, the future endowment w_τ has assumed to be constant. To ensure that our results are robust to the stochastic future endowment, we assume that in addition to multiplicative risk $\tilde{y}$, the individual is exposed to future endowment uncertainty, $\tilde{w}_\tau = \overline{w}(1+\tilde{w})$, where $E\tilde{w} = 0$ and $\tilde{w} > -1$.

We study the concept of additive-multiplicative risk prudence. An individual is said to be additive-multiplicative risk prudent, if the joint presence of additive and multiplicative pure risks raises her optimal savings. We show that the concept of multiplicative risk prudence is stronger than additive-multiplicative risk prudence. This implies that the condition of multiplicative risk prudence is always sufficient to induce risk prudent behaviour.

Most research on background risk assumes that background risks are independent of each other. As pointed out by Tsetlin and Winkler (2005), the independence assumption is often unrealistic and ignoring the correlation of background risks may lead to a poor decision. Accordingly, we allow $\tilde{w}$ and $\tilde{y}$ to be correlated. We only consider the case where the correlation between $\tilde{w}$ and $\tilde{y}$ is positive, simply because a positive correlation between $\tilde{w}$ and $\tilde{y}$ will make future wealth more uncertain and should further induce precautionary motive to save. Whereas a negative correlation between $\tilde{w}$ and $\tilde{y}$ gives rise to a hedging effect which can be an ally and need not induce precautionary savings. In Appendix we will demonstrate how a negative correlation between $\tilde{w}$ and $\tilde{y}$ gives rise to a hedging effect.

To allow for the correlation between $\tilde{w}$ and $\tilde{y}$ while isolating the change in endowment risk effect, we let

$$\tilde{w}(a) = \sqrt{1-a}\tilde{x} + \sqrt{a}\tilde{z},$$

where $0 \le a \le 1$, $\tilde{z} = \tilde{y} - 1$, and $\tilde{x}$ and $\tilde{z}$ are i.i.d. a is the correlation coefficient between $\tilde{w}$ and $\tilde{y}$. $\tilde{w}$ and $\tilde{y}$ are independent when $a = 0$ and perfectly positively correlated when $a = 1$.[16] Note that the variance of $\tilde{w}(a)$ is constant in a, i.e., the change in the correlation does not affect the endowment risk.

The utility is assumed to be additively-seperable. By the convex marginal utility, we have

$$E\tilde{y}u'_\tau((\bar{w}(1+\sqrt{1-a}\tilde{x}+\sqrt{a}\tilde{z})+s^*R_f)\tilde{y}) \ge E\tilde{y}u'_\tau((\bar{w}(1+\sqrt{a}\tilde{z})+s^*R_f)\tilde{y}).$$

[16] If $\tilde{y}$ is the rate of inflation which is random, the independence of $\tilde{w}_\tau$ and $\tilde{y}$ implies the neutrality of money. While the positive correlation between $\tilde{w}_\tau$ and $\tilde{y}$ leads to Tobin's portfolio shift effect (namely, the inflation increases capital per capita which in turn increases output per capita along the growth path).

The presence of the correlated multiplicative and additive risks induces precautionary savings, if and only if,

$$E\tilde{y}u_\tau'((\overline{w}(1+\sqrt{a}\tilde{z})+s^*R_f)\tilde{y}) \geq u_\tau'(\overline{w}+s^*R_f). \tag{21}$$

By applying the technique similar to the one employed for proving Proposition 1, one can show that condition (21) holds if, $\forall \overline{w}, y: g_{\overline{w}}''(y) \geq 0$, and only if, $g_{\overline{w}}''(1) \geq 0 \ \forall \overline{w}$, where $g_{\overline{w}}(y) = yu_\tau'((\overline{w}(1+\sqrt{a}(y-1))+s^*R_f)y)$. By simple calculus and algebra, we have $\forall \overline{w}, y: g_{\overline{w}}''(y) \geq 0$, if and only if,[17]

$$p_\tau \geq 2 - \frac{2}{(2+(1-\sqrt{a}+s^*R_f/\overline{w})/\sqrt{a}y)^2} \quad \forall \overline{w}, y, \tag{22}$$

and $g_{\overline{w}}''(1) \geq 0 \ \forall \overline{w}$, if and only if,

$$p_\tau \geq 2 - \frac{2}{(2+(1-\sqrt{a}+s^*R_f/\overline{w})/\sqrt{a})^2} \quad \forall \overline{w}. \tag{23}$$

Given that $(1-\sqrt{a}+s^*R_f/\overline{w})/\sqrt{a} \geq 0$, $p_\tau \geq \frac{3}{2}$ (a condition stronger than convex marginal utility) is necessary to induce precautionary savings. When $\tilde{w}$ and $\tilde{y}$ are independent (i.e. a = 0), conditions (22) and (23) become $p_\tau \geq 2$ (the equivalent condition of multiplicative risk prudence). For all $0 \leq a \leq 1$, the right hand side of condition (22) always lies between $\frac{3}{2}$ and 2. This leads to the following proposition.

Proposition 5 *If $\tilde{w}$ and $\tilde{y}$ are independent, the concepts of additive-multiplicative risk prudence and multiplicative risk prudence are equivalent; If $\tilde{w}$ and $\tilde{y}$ are positively correlated, the concept of multiplicative risk prudence is stronger than additive-multiplicative risk prudence. The sufficient and the necessary conditions of additive-multiplicative risk prudence are given by conditions (22) and (23), respectively.*

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[17] The detailed proof can be found in Appendix.

In other words, Proposition 5 suggests that when $\tilde{w}$ and $\tilde{y}$ are independent, the equivalent condition of additive-multiplicative risk prudence is $p_\tau \geq 2$. Whereas, when $\tilde{w}$ and $\tilde{y}$ are positively correlated, the relative prudence must be at least greater than 3/2 in order to induce precautionary savings. To see why the condition of multiplicative risk prudence is stronger than that of additive-multiplicative risk prudence, note that in comparing to the presence of an independent additive risk, the presence of a positively correlated additive risk makes the future wealth "more uncertain". Therefore, a lower relative risk prudence is required to induce precautionary savings.

5. Conclusion

Leland (1968) shows that an individual who faces an additive risk is risk prudent, if and only if, her marginal utility of future wealth is convex. Our analysis suggests that convex marginal utility is not sufficient to induce risk prudent behaviour and a stronger condition on preference is required for an individual to be risk prudent.

The stronger concept called multiplicative risk prudence is introduced and its corresponding condition on preference is derived. An individual is multiplicative risk prudent if and only if her relative prudence of future consumption uniformly exceeds two. A multiplicative risk prudent individual will always raise her optimal savings when facing a multiplicative uncertainty or in response to an increase in multiplicative uncertainty.

We generalize the concept of multiplicative risk prudence to the case of non-additively-separable utility. An individual with *NAS* utility is multiplicative risk prudent if her relative prudence of future consumption uniformly exceeds two and her marginal utility of current consumption is concave in future consumption.

The concept of multiplicative risk prudence provides an explanation to the risk-free rate puzzle. Intuitively, if the representative consumer is multiplicative risk prudent, the presence of inflation uncertainty will induce aggregate precautionary motive to save. This must be compensated by a reduction of risk-free rate. Fail to take inflation uncertainty into the account may lead to an overestimation of risk-free rate.

Finally, we study jointly the impact of correlated additive and multiplicative risks on optimal savings decision and find that our result is robust and is unaffected by the presence of correlated additive background risk. An individual is said to be additive-multiplicative risk prudent, if the presence of positively correlated additive and multiplicative pure risks raise her optimal savings. We show that the concept of multiplicative risk prudence is stronger than additive-multiplicative risk prudence.

Like risk aversion, risk prudence has been regarded as a natural human behaviour. As uncertainty of future wealth can take different forms, if one believes that additive risk prudence is a natural assumption, then there is no reason why we should ignore the assumption of its multiplicative counterpart. Our results suggest that the condition of multiplicative risk prudence is always sufficient to induce risk prudent behaviour in different situations. Accordingly, one should take the condition of multiplicative risk prudence as a natural assumption.

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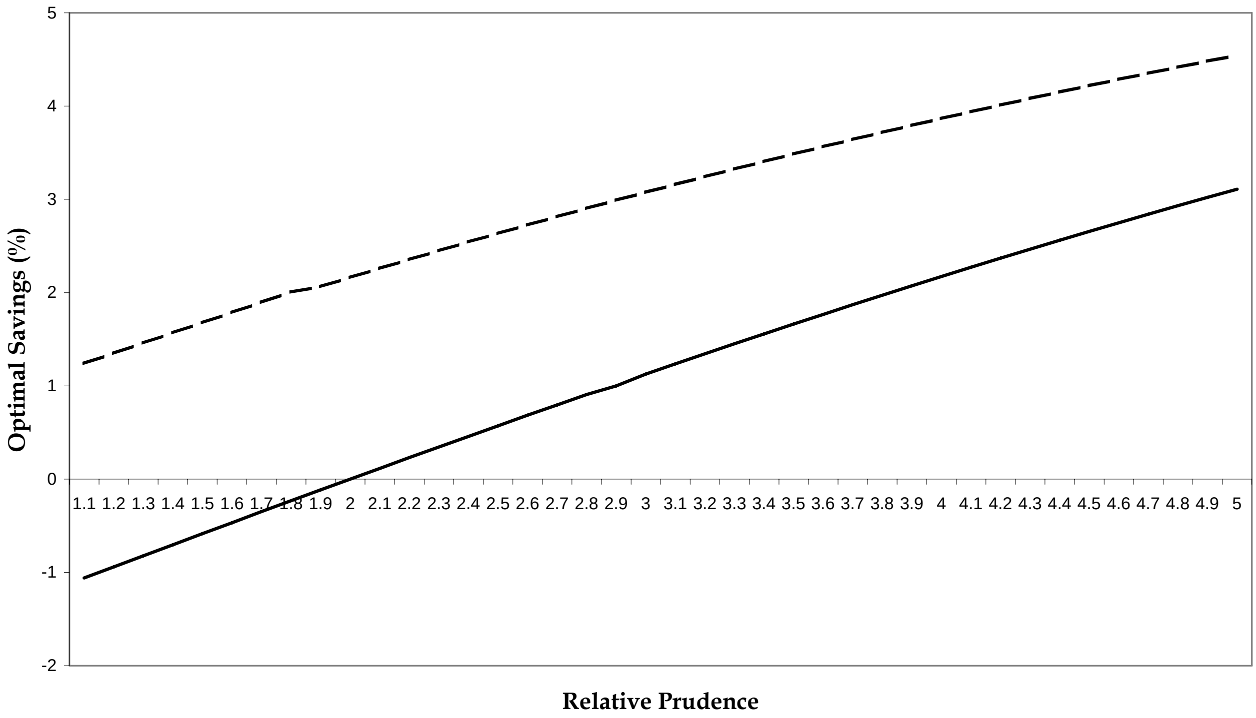
Table 1: The optimal savings of a prudent individual with $\gamma = 0.5$.

The optimal savings is expressed in percentage wealth. NU, MU and AU represent the case where the individual is exposed to no uncertainty, multiplicative uncertainty and additive uncertainty, respectively.

	NU	MU	AU
Optimal savings (%)	0	-0.5850	1.6773

Figure 1: The optimal savings under uncertainty vs relative prudence.

The optimal savings (in percentage) is plotted on vertical axis and the degree of relative prudence is plotted on the horizontal axis. The solid line (dashed line) represents the optimal savings for the case where the uncertainty is multiplicative (additive).



Appendix

The case where $\tilde{w}$ and $\tilde{y}$ are negatively correlated:

Since $\tilde{y} \geq 0$ and an individual cannot be a net borrower when she retires, $\tilde{w}$ and $\tilde{y}$ cannot be negatively correlated. If $\tilde{w}$ and $\tilde{y}$ are negatively correlated, i.e.,

$$\tilde{w}(a) = \sqrt{1-a}\tilde{x} - \sqrt{a}\tilde{z},$$

then w_τ can be negative for some $y > 0$.

Therefore, to examine the hedging effect of negatively related background risks, we let

$$\tilde{w}(a) = \sqrt{1-a}\tilde{x} - \sqrt{a}\tilde{z}', \tag{A1}$$

where $\tilde{z}' = \tilde{y}^{-1} - \mu$ and $\mu = E\tilde{y}^{-1}$. $\tilde{x}$ and $\tilde{z}'$ are i.i.d. a measures the degree of negative 'relation' between $\tilde{w}$ and $\tilde{y}$. $\tilde{w}$ and $\tilde{y}$ are independent when $a = 0$ and perfectly negatively 'related' when $a = 1$. Given equation (A1), it is obvious that $w_\tau \geq 0$ for all $y \geq 0$.

By convex marginal utility we have

$$E\tilde{y}u'_\tau((\overline{w}(1+\sqrt{1-a}\tilde{x}+\sqrt{a}\tilde{z}')+R_f s^*)\tilde{y}) \geq E\tilde{y}u'_\tau((\overline{w}(1+\sqrt{a}\tilde{z}')+R_f s^*)\tilde{y}).$$

Let $g_{\overline{w}}(y) = yu'_\tau((\overline{w}(1+\sqrt{a}(y^{-1}-\mu)+R_f s^*)y),$ then

$$E\tilde{y}u'_\tau((\overline{w}(1+\sqrt{a}\tilde{z}')+R_f s^*)\tilde{y}) \geq u'_\tau(\overline{w}(1+\sqrt{a}(1-\mu))+R_f s^*), \tag{A2}$$

if $\forall \overline{w}, y : g''_{\overline{w}}(y) \geq 0,$ and only if, $g''_{\overline{w}}(1) \geq 0 \ \forall \overline{w}.$ Since $\mu > 1$ and by risk aversion, the following condition holds naturally,

$$u'_\tau(\overline{w}(1+\sqrt{a}(1-\mu))+R_f s^*) \geq u'_\tau(\overline{w}+R_f s^*).$$

By simple calculus and algebra, condition (A2) holds if

$$p_\tau \geq 2 + \frac{2\sqrt{a}\overline{w}y^{-1}}{\overline{w}(1-\sqrt{a}\mu)+R_f s^*} \forall \overline{w}, y \tag{A3}$$

and only if,

$$p_\tau \geq 2 + \frac{2\sqrt{a}\overline{w}}{\overline{w}(1-\sqrt{a}\mu)+R_f s^*} \forall \overline{w}. \tag{A4}$$

Again, when $\tilde{w}$ and $\tilde{y}$ are independent (i.e. $a = 0$), conditions (A3) and (A4) reduce to $p_\tau \geq 2$ (the equivalent condition of multiplicative risk prudence). Note that if condition (A4) fails to hold, then it is possible to exist some $\tilde{y}$ such that

$$u_\tau'(\overline{w}(1+\sqrt{a}(1-\mu))+R_f s^*) \geq u_\tau'(\overline{w}+R_f s^*) \geq E\tilde{y}u_\tau'((\overline{w}(1+\sqrt{a}(\tilde{y}^{-1}-\mu))+s^* R_f)\tilde{y}).$$

Given that $y \geq 0$ and an individual cannot be a net borrower when she retires, the right hand side of conditions (A3) and (A4) are greater than two (i.e. condition which is stronger than multiplicative risk prudence). This demonstrates how a negative relation between $\tilde{w}$ and $\tilde{y}$ gives rise to a hedging effect. The hedging effect is stronger when $\tilde{w}$ and $\tilde{y}$ are more negatively related. As a increases, the second term in conditions (A3) and (A4) increases. This is because a higher a value gives rise to a stronger hedging effects, thus a higher degree of prudence is required in order to induce precautionary savings.

Proof of condition (22):

$$g_{\overline{w}}''(y) = (6\sqrt{a}y + 2((1-\sqrt{a}) + s^* R_f / \overline{w}))u_\tau'' + y((1-\sqrt{a}) + s^* R_f / \overline{w} + 2\sqrt{a}y)^2 u''' \geq 0$$

By algebra, the equivalent condition of $g_{\overline{w}}''(y) \geq 0$ is

$$p_\tau \geq \frac{(6\sqrt{a}y + 2((1-\sqrt{a}) + s^* R_f / \overline{w}))(1-\sqrt{a} + s^* R_f / \overline{w} + \sqrt{a}y)}{(1-\sqrt{a} + s^* R_f / \overline{w} + 2\sqrt{a}y)^2}.$$

Substituting into the above inequality

$$\begin{aligned}&(6\sqrt{a}y + 2((1-\sqrt{a}) + s^* R_f / \overline{w}))u_\tau'' + y((1-\sqrt{a}) + s^* R_f / \overline{w} + 2\sqrt{a}y)^2 u''' \\ &= 2((1-\sqrt{a}) + s^* R_f / \overline{w} + 2\sqrt{a}y)^2 - 2ay^2.\end{aligned}$$

and rearranging terms yield conditions (22) and (23).