A New Paradigm in Asset Pricing

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Abstract

This presentation will show how to fuse two pieces of theory to make a tractable model for asset pricing. The first is the theory of asset pricing using a stochastic discounting function (SDF) processes. This will be reviewed. The second is to use Hidden Markov Models (HMMs) to model uncertainty in an economy.

It turns out that with an elegant representation of HMMs introduced by R.J.Elliott, these models can be calibrated and asset valuation given. We will discuss implications for interest rate models, stock price models, futures pricing, exchange rates. This work represents joint research with R. J. Elliott from University of Calgary, Haskayne School of Business.
Overview

1. Review Stochastic Discounting Functions Processes.


5. Risk Neutral verses Real World Probabilities.

6. Credit Risk.
Stochastic Discounting Function Processes

We will assume that we have a probability space $(\Omega, \mathcal{F}, P)$ on which there is defined a stochastic discounting function process $\{\pi_t \mid t \geq 0\}$ with the property that for any asset price process $\{A_t \mid t \geq 0\}$

$$\pi_t A_t = \mathbb{E}[\pi_s A_s \mid \mathcal{F}_t]$$

where $\mathbb{E}$ is expectation with respect to $P$ and $\mathcal{F}_t$ represents information up to time $t$. Here $s \geq t$ and there are no cash-flows from $A$ over the time interval $(t, s]$. 
Some Motivation

1. From pricing kernels.

\[ \mathcal{A}_t = \mathbb{E}[\zeta(t, s)A_s | \mathcal{F}_t] \]

then as

\[ \zeta(t, s) = \frac{\zeta(0, s)}{\zeta(0, t)} \]

we can use \( \pi_t = \zeta(0, t) \).

2. From utility theory.

\[
\text{maximize } \mathbb{E}\left[ \sum_{s>0} e^{-\beta s} u(c_s) \right]
\]

under suitable constraints leads to

\[ \zeta(t, s) = e^{-\beta(s-t)} \frac{u'(\hat{c}_s)}{u'(\hat{c}_t)} \]

where \( \{\hat{c}_t | t \geq 0\} \) is the optimal consumption process. Then take

\[ \pi_t = \exp(-\beta t) u'(\hat{c}_t). \]
Some Motivation, continued


We say $\{\xi_t | t \geq 0\}$ is a growth optimal portfolio, GOP, if it is a positive process and

$$\mathbb{E}[\log(\xi_T)]$$

is maximal for all $T > 0$. Then

$$\pi_t = \frac{1}{\xi_t}$$

is a stochastic discounting function process. This process and its approximations have been studied extensively in Platen and Heath “A Benchmark Approach to Quantitative Finance” [Springer 2006]. This will be our preferred motivation and indicates that $\{\pi_t\}$ is observed in this sense.
Necessary Properties

The process \( \{\pi_t\} \) must satisfy three conditions:

1. be strictly positive, to prevent arbitrage opportunities.

2. be a super-martingale: this means for \( s \geq t \) that
   \[
   \mathbb{E}[\pi_s|\mathcal{F}_t] \leq \pi_t .
   \]
   This is because
   \[
   \mathbb{E}[\pi_s|\mathcal{F}_t] = \pi_t P(t, s)
   \]
   and \( P(t, s) \) is the value at \( t \) of one dollar at time \( s \).

3. for any \( t \geq 0 \),
   \[
   \mathbb{E}[\pi_s|\mathcal{F}_t] \to 0
   \]
   as \( s \to \infty \), as a dollar at infinity is worth zero now.

In probability theory such a process is called a potential.
Bond Price Formulas

We have zero coupon bond prices:

\[ P(t, T) = \frac{1}{\pi_t} E \left[ \pi_T | \mathcal{F}_t \right] \]

\[ = \exp \left( - \int_t^T f(t, u) du \right) \]

where the forward interest is:

\[ f(t, T) = -\frac{\partial P(t, T)}{\partial T} / P(t, T) \]

The short interest rate is then:

\[ r_t = f(t, t) \]

We set

\[ \beta_t = \exp \left( \int_0^t r_u \, du \right) \]

as the interest rate account process.
Stock Price Formulas

\[ S_t = \frac{1}{\pi_t} \mathbb{E} \left[ \int_t^\infty \pi_s D_s ds | \mathcal{F}_t \right] \]

defines the stock price process for \( S \) in terms of its dividend rate process \( \{D_t\} \) and so

\[ \pi_t S_t + \int_0^t \pi_s D_s ds \]

is an \( \{\mathcal{F}_t\} \) martingale process.

**Forward prices** are given as usual by

\[ F(t, T) = \frac{S_t}{P(t, T)} \]

and **futures prices** by

\[ G(t, T) = \frac{\mathbb{E} [\pi_T \beta_T S_T | \mathcal{F}_t]}{\pi_t \beta_t} \]
Exchange Rates

For each country there is a similar framework though eventually the noise process which drives the model can be global.

Let $X_{t}^{a,b}$ denote the value at time $t$ of 1 unit of currency in country $a$ expressed in terms of the currency of country $b$. So if $a = \text{AUS}$ and $b = \text{USA}$ then $X_{t}^{a,b}$ is about 0.80 if $t$ is today.

Then, with $\xi$ the GOP

$$X_{t}^{a,b} = \frac{\pi_{t}^{a}}{\pi_{t}^{b}} = \frac{\xi_{t}^{b}}{\xi_{t}^{a}}$$

This is because

$$\pi_{t}^{a}A_{t} = E[\pi_{s}^{a}A_{s}|F_{t}]$$

and

$$\pi_{t}^{b}X_{t}^{a,b}A_{t} = E[\pi_{s}^{b}X_{s}^{a,b}A_{s}|F_{t}]$$

holds for any asset $A$ in country $a$ and for all $s \geq t$. 
In summary so far:

The stochastic discounting function is a process which determines many financial instruments.

For stock prices, the dividend process should be modelled.

For currencies a “correct” form of purchase power parity is implied.

The task is to provide a tractable model for the stochastic discounting function process. We propose to model uncertainty in the world in terms of the evolution of a $N$-state hidden Markov chain model.

In terms of this model, all quantities above are computable and there seem to be good ways to calibrate the model.

We hope that this approach will generate some empirical studies.
Hidden Markov Models

This will be a process $X = \{X_t\}$ with a finite number of states which we will take as $\{e_1, e_2, ..., e_N\} \subset R^N$. Here $e_i = (0, 0, ..., 1, ..., 0)'$ where the 1 is in the i-th spot. This is a crucial parametrization of the state-space of $X$. Then

$$X_t = X_0 + \int_0^t A_u X_u du + M_t$$

where

$$A_{ij}^t \Delta t \approx P[X_{t+\Delta t} = e_i | X_t = e_j]$$

if $i \neq j$ and otherwise

$$1 + A_{ii}^t \Delta t \approx P[X_{t+\Delta t} = e_i | X_t = e_i]$$

$A$ is the intensity matrix process for $X$ and $M$ defined by (*) is an $\{\mathcal{F}_t\}$ martingale, where we now specify $\mathcal{F}_t = \sigma\{X_u|0 \leq u \leq t\}$ (subject to some usual technical requirements).
A typical Calculation

Suppose $A_t \equiv A$ (constant with respect to $t$).

Let $f : \mathbb{R}^N \to \mathbb{R}$, then

$$f(X_t) = \langle \tilde{f}, X_t \rangle$$

where $\tilde{f} \in \mathbb{R}^N$ is defined by

$$\tilde{f}^i = f(e_i)$$

so if $s \geq t$

$$E[f(X_s)|\mathcal{F}_t] = E[f(X_s)|X_t]$$

$$= \langle \tilde{f}, E[X_s|X_t] \rangle$$

$$= \langle \tilde{f}, \exp((s - t)A)X_t \rangle$$

$$= \langle \exp((s - t)A^*)\tilde{f}, X_t \rangle$$

where $A^*$ is the transpose of $A$. This can be generalized to time varying $A_t$. 
A final note

The matrix $A_t$ is an **irreducible** Q-matrix for each $t$ and has the properties:

$$A_{ij}^t \geq 0$$

when $i \neq j$, and

$$\sum_{i=1}^{N} A_{ij}^t = 0.$$

This implies that

$$A_{jj}^t = - \sum_{i \neq j} A_{ij}^t < 0.$$

We prefer to work in continuous time, but everything also works in discrete time.

HMM models are often related to regime switching models, and unfortunately economists and engineers are often oblivious of the other’s existence.
Representation of the SDF process

We propose:

\[ \pi_t = \exp \left[ - \int_0^t \langle C_u, dX_u \rangle - \int_0^t \langle D_u, X_u \rangle du \right] \]

This is motivated by the optimal consumption framework in economics which we described earlier:

\[ \frac{\pi_{t+1}}{\pi_t} = \beta \frac{u'(\hat{c}_{t+1})}{u'(\hat{c}_t)} \]

\[ = \beta \exp \left[ \log u'(\hat{c}_{t+1}) - \log u'(\hat{c}_t) \right] \]

If we assume \( \hat{c}_t = \Phi(t, X_t) = \langle \phi_t, X_t \rangle \), then

\[ \log u'(\hat{c}_t) = \langle f_t, X_t \rangle \]

where \( f_t^i = \log u'(\phi_t^i) \). Then

\[ \langle f_{t+1}X_{t+1} \rangle - \langle f_t, X_t \rangle = \langle f_{t+1}, X_{t+1} - X_t \rangle + \langle f_{t+1} - f_t, X_t \rangle \]

\[ \approx - \int_t^{t+1} \langle C_u, dX_u \rangle - \int_t^{t+1} \langle D_u, X_u \rangle du \]
Remark

The form of the SDF can be generalized to

$$\pi_t = \exp \left[ - \int_0^t X_u'C_u \, dX_u - \int_0^t D'_u X_u \, du \right]$$

where now $C_u$ is $N \times N$ matrix valued for each $u$. We have similar results for this more general model.

$$X'_{u-}C_u \, dX_u = \langle C'_u X_u, dX_u \rangle$$
$$D'_u X_u = \langle D_u, X_u \rangle$$
Size of the model.

For $A$ there are $N^2 - N$ components.

For $C$ there are $N - 1$ components as only $C^i - C^j$ can be estimated, since

$$
\int_0^t \langle C_u, dX_u \rangle = \sum_{i,j=1}^{N} \sum_{0<u\leq t} \langle X_u-, e_i \rangle \langle X_u, e_j \rangle (C^j_u - C^i_u).
$$

For $D$ there are $N$ components. We also need some restrictions on $D$ so that our SDF process is a potential. The components of $D$ need to be large enough.
Useful Lemma

Let

\[ \Lambda_{t,T} = \frac{\pi_T}{\pi_t} \]

\[ Z_{t,T} := \Lambda_{t,T} X_T \]

then

\[ \hat{Z}_{t,T} = E \left[ \Lambda_{t,T} X_T | F_t \right] \]
\[ = E \left[ \Lambda_{t,T} X_T | X_t \right] \]
\[ = \Psi(t, T) X_t \]

where

\[ \frac{\partial \Psi}{\partial T}(t, T) = \Gamma_T \Psi(t, T) \]
\[ \Psi(t, t) = I_N \]

and

\[ \Gamma_{ij}^u = A_{ij}^u e^{C_{ij}^u - C_{ij}^u} \quad \text{if} \quad i \neq j \]
\[ = A_{ij}^j - D_{ij}^j \quad \text{if} \quad i = j. \]
Bond prices

\[ P(t, T) = \mathbb{E} \left[ \Lambda_{t, T} \mid \mathcal{F}_t \right] \]
\[ = \mathbb{E} \left[ \langle 1, Z_{t, T} \rangle \mid \mathcal{F}_t \right] \]
\[ = \langle 1, \hat{Z}_{t, T} \rangle \]
\[ = \langle 1, \Psi(t, T)X_t \rangle \]
\[ = \langle \Psi(t, T)^* 1, X_t \rangle \]

and as

\[ \hat{Z}_{t, T} = X_t + \int_t^T \Gamma_u \Psi_{t, u} X_t du \]

it also follows that

\[ P(t, T) = 1 + \int_t^T 1^* \Gamma_u \Psi_{t, u} X_t du \]
\[ \leq 1 \]

if and only if for all \( u \)

\[ 1^* \Gamma_u \leq 0 \]
A few technicalities

Writing $1 = (1, 1, 1, ..., 1)'$, we have

$$[1^* \Gamma_u]^j = \sum_{i=1}^{N} A_{ui}^{ij} e^{C_u^i - C_u^j} - D_u^j$$

$$= A_{uj}^{jj} - D_u^j + \sum_{i \neq j} A_{ui}^{ij} e^{C_u^i - C_u^j}$$

$$\leq A_{uj}^{jj} - D_u^j + M \sum_{i \neq j} A_{ui}^{ij}$$

$$= (1 - M) A_{uj}^{jj} - D_u^j$$

$$\leq 0$$

if the components of $D_u$ are large enough [we assume the $C_u$ are bounded here]. Then $D_u$ a little larger ensures $P(t, T) \to 0$ as $T \to \infty$. 
More formulas

\[ f(t, T) = -\frac{\langle \psi(t, T)^* \Gamma_T^* 1, X_t \rangle}{\langle \psi(t, T)^* 1, X_t \rangle} = \langle \text{diag}(\psi(t, T)^* 1)^{-1} \psi(t, T)^* \Gamma_T^* 1, X_t \rangle \]

and

\[ r_t = f(t, t) = -\frac{\langle \Gamma_t^* 1, X_t \rangle}{\langle 1, X_t \rangle} = -\langle \Gamma_t^* 1, X_t \rangle \geq 0 \equiv \langle \mu_t, X_t \rangle \]
Some more economics

Our model supports constant interest rates, [but not zero interest rates, else \( P(t, T) \equiv 1 \)]. This is \( \mu_t = r_0 \mathbf{1} \). We need to use the Perron-Frobenius Theorem to get a non-trivial model.

The model also supports deterministic interest rates. This is \( \mu_t = r(t) \mathbf{1} \) for a deterministic function \( r \) of \( t \).

We have

\[
(diag(D_t) - A^*_t) e^{-C_t} = diag(\mu_t) e^{-C_t}
\]

and we note that \( C \equiv 0 \) implies \( \mu_t = D_t \) and so \( \pi_t = 1/\beta_t \) which corresponds to risk neutrality.
Stock models

Assume that $D_s = \langle \delta_s, X_s \rangle$. We could assume an increasing process, but one which depends on the state of the world one is in. Then we have

$$S_t = \frac{1}{\pi_t} \mathbb{E} \left[ \int_t^\infty \pi_s D_s ds | X_t \right]$$

$$= \frac{1}{\pi_t} \mathbb{E} \left[ \int_t^\infty \pi_s \langle \delta_s, X_s \rangle ds | X_t \right]$$

$$= \langle \sigma_t, X_t \rangle$$

but

$$S_t = \int_t^\infty \mathbb{E} \left[ \Lambda_{t,s} \langle \delta_s, X_s \rangle | X_t \right] ds$$

$$= \int_t^\infty \langle \delta_s, \hat{Z}_{t,s} \rangle ds$$

$$= \int_t^\infty \langle \delta_s, \Psi(t,s)X_t \rangle ds$$

$$= \int_t^\infty \langle \psi(t,s)^* \delta_s, X_t \rangle ds$$

$$= \langle \int_t^\infty \psi(t,s)^* \delta ds, X_t \rangle$$
so
\[ \sigma_t = \int_t^\infty \psi(t, s)^* \delta_s \, ds \]

In fact \( \sigma_t \) satisfies the ordinary differential equation:
\[
\frac{d\sigma_t}{dt} + \Gamma^*_t \sigma_t = -\delta_t
\]
\[
\sigma_t \to 0 \quad \text{as} \quad t \to \infty
\]
Applications to European Stock Options

We have $C_T = G(S_T)$ and wish to find $C_t$ the value of a claim at time $t$.

Note that $C_T = \langle g, X_T \rangle$ with $g^i = G(\langle \sigma_T, e_i \rangle)$

$$C_t = \mathbb{E} \left[ \Lambda_{t,T} C_T | \mathcal{F}_t \right]$$
$$= \mathbb{E} \left[ \Lambda_{t,T} \langle g, X_T \rangle | \mathcal{F}_t \right]$$
$$= \langle g, \hat{Z}_{t,T} \rangle$$
$$= \langle g, \psi(t,T) X_t \rangle$$
$$= \langle \psi(t,T)^* g, X_t \rangle$$
$$= \langle c_t, X_t \rangle$$

with

$$c_t = \psi(t,T)^* g$$

This solves the backward equation:

$$\frac{dc_t}{dt} + \Gamma_t^* c_t = 0$$
$$c_T = g$$
Remark

We have corresponding algorithms for computing American Style Options using techniques from Variational Inequalities and Numerical Algorithms have been developed which use special features of the model and Saigal’s algorithm to solve all linear complementarity problems in $N$ iterations.
Applications to Portfolio Theory

\[ J_t = \mathbb{E} [U(Y_T)|\mathcal{F}_t] \]

where

\[ Y_T = \sum_{i=1}^{k} \alpha_i S_{iT} \]

given

\[ Y_t = \sum_{i=1}^{k} \alpha_i S_{it} \]

is known. Now

\[ U(Y_T) = U \left( \sum_{i=1}^{k} \alpha_i \langle \sigma_{iT}, X_T \rangle \right) \]

\[ = \langle u(\alpha_1, \alpha_2, ..., \alpha_k), X_T \rangle \]

where

\[ w_j = U \left( \sum_{i=1}^{k} \alpha_i \sigma_{iT}^j \right) \]

\[ S_{iT} = \langle \sigma_{iT}, X_T \rangle \]
so if \( A_u \equiv A \)

\[
J_t = \langle u(\alpha_1, \alpha_2, \ldots, \alpha_k), e^{(T-t)A}X_t \rangle \\
= \langle e^{(T-t)A^*}u(\alpha_1, \alpha_2, \ldots, \alpha_k), X_t \rangle
\]

We are then interested in maximizing

\[
\langle e^{(T-t)A^*}u(\alpha_1, \alpha_2, \ldots, \alpha_k), e_j \rangle
\]

for each \( j \) subject to wealth at \( t \) being:

\[
\sum_{i=1}^{k} \alpha_i \langle \sigma_{it}, e_j \rangle
\]
Futures Prices

\[ G(t, T) \] is computed from

\[ dM_t = r_t M_t dt + d_t G(t, T) \]

\[ G(T, T) = S_T \]

where \( \{M_t\} \) is the margin account process.

One can show that \( \{\beta_t \pi_t G(t, T) | 0 \leq t \leq T\} \) is an \( \{\mathcal{F}_t\} \) martingale and so

\[ G(t, T) = \frac{E[\pi_T \beta_T S_T | \mathcal{F}_t]}{\pi_t \beta_t} \]

This can be computed

\[ G(t, T) = \langle \sigma_T, \Phi(t, T) X_t \rangle \]

\[ = \langle \Phi(t, T)^* \sigma_T, X_t \rangle \]
where

\[
\frac{\partial \Phi(t, T)}{\partial T} = L_T \Phi(t, T)
\]

\[
\Phi(t, t) = I_N
\]

\[
L_{ij}^T = A_{ij}^T \exp(C_j^T - C_i^T) \quad \text{if} \quad i \neq j
\]

\[
L_{jj}^T = \sum_{m \neq j} A_{mj}^T \exp(C_j^T - C_m^T)
\]
Example of Estimations

We assume that $A_u \equiv A$, $C_u \equiv C$, $D_u \equiv D$, then

$$r_t = \langle \mu, X_t \rangle$$

and the states of the world could be identified with different levels of interest rates in some country. We make observations over $[0, T]$. The MLE estimator for $A$ is

$$\hat{A}^{ij} = \frac{\sum_{0<s\leq T} \langle X_{s-}, e_j \rangle \Delta N^i_s}{\int_0^T \langle X_s, e_j \rangle ds}$$

where

$$N^i_t = \int_0^t (I - \text{diag}(X_{s-})) \, dX_s$$

$$= \sum_{j \neq i} \sum_{0<u\leq t} \langle X_{u-}, e_j \rangle \langle X_u, e_i \rangle$$

counts the number of times on $[0,t]$ that $X$ jumps to state $i$, and $\int_0^T \langle X_s, e_j \rangle ds$ counts the amount of time that $X$ spends in state $e_j$ over the period $[0,t]$. 

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Example of Estimations continued

If $\xi_t = \frac{1}{\pi_t}$ is observed,

$$\log \xi_t = \int_0^t \langle C, dX_u \rangle + \int_0^t \langle D, X_u \rangle du$$

and so

$$\Delta \log \xi_t = C^* \Delta X_t$$

so

$$\widehat{C^i} - \widehat{C^j} = \frac{\sum_{0<u\leq T} (\Delta \log \xi_u) \langle X_u, e_j \rangle \langle X_{u-}, e_i \rangle}{\sum_{0<u\leq T} \langle X_u, e_j \rangle \langle X_{u-}, e_i \rangle}$$

This deals with the jump parts.
Example of Estimations continued

Let \( y_t = \log \xi_t - \int_0^t \langle C, dX_u \rangle = \int_0^t \langle D, X_u \rangle du \). Then

\[
\hat{D}^j = \frac{\int_0^T \frac{dy_u}{du} \langle X_u, e_j \rangle du}{\int_0^T \langle X_u, e_j \rangle du}
\]

The denominator measures the amount of time \( X \) spends in state \( e_j \) over \([0,T]\) and if \( \tau_1 \) is a time of jump to \( e_j \) and \( \tau_2 \) the next time of jump away from \( e_j \), then

\[
\int_{\tau_1}^{\tau_2} \frac{dy_u}{du} \langle X_u, e_j \rangle du = y_{\tau_2} - y_{\tau_1}
\]

If we wish to assume that \( A, C, D \) are time dependent, then we can use parametric dependence on time, and adapt some of the techniques suggested. The quality of the results will depend on the quality of the observed process \( \{\xi_t\} \). Financial institutions will use their own constructed indices for the SDF process.
Real World versus Risk Neutral World

So far everything has been expressed in real world probabilities, the probabilities used by econometrics. It is possible to express all the dynamics and various statistics in terms of risk neutral probabilities. This means that market prices of risk are provided by the model and do not have to be specified exogenously. If $P^*$ represents the risk neutral probability and $E^*$ risk neutral expectations, then we can recover some well known formulas in our model for stocks, bonds, futures, currencies and so on.

The measure change is given by:

$$\frac{dP^*}{dP} \bigg|_{\mathcal{F}_T} = \frac{\beta_T \pi_T}{E[\beta_T \pi_T]}$$
1. **Stocks prices**

\[ dS(t) = [r(t)S(t) - D_t] \, dt + dM_t^1 \]

where \( M^1 \) is a \( P^* \) martingale process.

2. **Bonds prices**

\[ dP(t, T) = r(t)P(t, T)\, dt + dM_t^2 \]

where \( M^2 \) is a \( P^* \) martingale process.

3. **Exchange rate**

\[ dX^{a,b}(t) = \left( r^b(t) - r^a(t) \right) X^{a,b}(t) \, dt + dM_t^3 \]

where \( M^3 \) is a \( P^* \) martingale process.

4. **Futures prices**

\[ G(t, T) = E^* [S(T)|\mathcal{F}_t] \]

are \( P^* \) martingale processes.
5. **Forward rates**

\[ df(t, T) = \left( r_t - \frac{P_t(t, T)}{P(t, T)} \right) f(t, T) \, dt + dM_t^4 \]

where \( M^4 \) is a \( P^* \) martingale process.

6. **Martingales**

\[ M^*_t = M_t + \int_0^t (A_u - L_u) X_u \, du \]

7. **Forward Measures**

\[ \frac{dP^T}{dP} \bigg|_{\mathcal{F}_T} = \frac{\pi_T}{\mathbb{E}[\pi_T]} \]

8. **Forward rates again**

\[ f(t, T) = \mathbb{E}^T [r_T | \mathcal{F}_t] \]
Credit Risk

We present some initial explorations:

Let $\{a_t\}$ be a given deterministic process and

$$V_t := a'_t X_t \equiv \langle a_t, X_t \rangle$$

which could represent a "firm value" and a credit default event occurs if $V_t \leq \epsilon$. Given a time horizon $T > 0$ we would like to compute:

$$\text{Prob} \left[ V_s \leq \epsilon \text{ for some } s \in [t, T], \text{ given } V_t = V \right]$$

Further we would like to calculate this probability both under real world probabilities (for risk management) and under risk neutral probabilities (for derivative valuation). This can be achieved in our framework.
Equations

Let \( p(t, V) \) denote this \textbf{real world} probability. Then

\[
\begin{align*}
    p(t, a^i_t) &= 1 \text{ if } a^i_t \leq \epsilon \\
    -\frac{\partial p}{\partial t}(t, a^i_t) &= \sum_{j=1}^{N} A^{ji}_t \left[ p(t, a^j_t) I[a^j_t > \epsilon] + I[a^j_t \leq \epsilon] \right] \\
        &\text{otherwise}
\end{align*}
\]

\[
p(T, a^i_T) = I[a^i_T \leq \epsilon]
\]

For \textbf{risk neutral} calculation replace \( A \) by \( L \) in these equations.
Some Proofs - Some Magic!

1. The dynamics of \( \{\pi_t\} \)

\[
\pi_t = \exp(\zeta_t) \quad \text{(see slide 14)}
\]

\[
= \pi_0 + \int_{0}^{t} \exp(\zeta_{u-})d\zeta_u
+ \sum_{0<u\leq t} \left[ \exp(\zeta_u) - \exp(\zeta_{u-}) - \exp(\zeta_{u-})\Delta \zeta_u \right]
\]

\[
= \pi_0 - \int_{0}^{t} \exp(\zeta_{u-})\langle C_{u-}, dX_u \rangle
- \int_{0}^{t} \exp(\zeta_u)\langle D_u, X_u \rangle du
+ \sum_{0<u\leq t} \left[ \exp(\zeta_u) - \exp(\zeta_{u-}) + \exp(\zeta_{u-})\langle C_{u-}, \Delta X_u \rangle \right]
\]

\[
= \pi_0 - \int_{0}^{t} \exp(\zeta_u)\langle D_u, X_u \rangle du
+ \sum_{0<u\leq t} \left[ \exp(\zeta_u) - \exp(\zeta_{u-}) \right]
\]

\[
= \pi_0 - \int_{0}^{t} \exp(\zeta_u)\langle D_u, X_u \rangle du
+ \sum_{0<u\leq t} \exp(\zeta_{u-}) \left[ \exp(\Delta \zeta_u) - 1 \right]
\]
Some Proofs - continued

Now

\[ \exp(\Delta \zeta_u) - 1 = \exp(-\langle C_u^-, \Delta X_u \rangle) - 1 \]
\[ = \sum_{i,j} \langle X_{u-}, e_i \rangle \langle X_u, e_j \rangle \left[ \exp(C_u^i - C_u^j) - 1 \right] \]
\[ = \sum_{i,j} \langle X_{u-}, e_i \rangle \langle \Delta X_u, e_j \rangle \left[ \exp(C_u^i - C_u^j) - 1 \right] \]
\[ = X_u^* G_u - \Delta X_u \]

where

\[ G_u^{ij} = \exp(C_u^i - C_u^j) - 1 \]

and so

\[ \pi_t = \pi_0 - \int_0^t \pi_u \langle D_u, X_u \rangle du + \int_0^t \pi_u X_u^* G_u dX_u \]
\[ = \pi_0 - \int_0^t \pi_u \langle D_u, X_u \rangle du + \int_0^t \pi_u X_u^* G_u A_u X_u du \]
\[ + \int_0^t \pi_u X_u^* G_u dM_u \]
Some Proofs - continued

2. The dynamics of $Z_t := \frac{\pi_t}{\pi_0} X_t$

Assume without loss of generality that $\pi_0 = 1$.

$$Z_t = Z_0 + \int_{0+}^{t} X_u - d\pi_u + \int_{0+}^{t} \pi_u - dX_u + \sum_{0 < u \leq t} \Delta \pi_u \Delta X_u$$

where

$$\sum_{0 < u \leq t} \Delta \pi_u \Delta X_u = \sum_{0 < u \leq t} \pi_u - X_u - G_u - \Delta X_u \Delta X_u$$
\[ X_u - G_u - \Delta X_u \Delta X_u \]
\[ = \sum_{ij} \langle X_u, e_i \rangle \langle X_u, e_j \rangle e_i^* G_u - (e_j - e_i)(e_j - e_i) \]
\[ = \sum_{ij} \langle X_u, e_i \rangle \langle \Delta X_u, e_j \rangle e_i^* G_u - (e_j - e_i)(e_j - e_i) \]
\[ = \sum_{ij} \langle X_u, e_i \rangle \langle A_u X_u, e_j \rangle e_i^* G_u - (e_j - e_i)(e_j - e_i) \]
\[ + \sum_{ij} \langle X_u, e_i \rangle \langle \Delta M_u, e_j \rangle e_i^* G_u - (e_j - e_i)(e_j - e_i) \]
\[ = \sum_{ij} \langle X_u, e_i \rangle \langle A_u e_i, e_j \rangle e_i^* G_u - (e_j - e_i)(e_j - e_i) \]
\[ + \sum_{ij} \langle X_u, e_i \rangle \langle \Delta M_u, e_j \rangle e_i^* G_u - (e_j - e_i)(e_j - e_i) \]
\[ = H_u X_u du \]
\[ + \sum_{ij} \langle X_u, e_i \rangle \langle \Delta M_u, e_j \rangle e_i^* G_u - (e_j - e_i)(e_j - e_i) \]

\[ H_u^{ij} = e_i^* \sum_k \langle A_u e_j, e_k \rangle e_j^* G_u (e_k - e_j)(e_k - e_j) \]
\[ = \sum_k A_{u}^{kj} [G_u^{jk} - G_u^{jj}] [\delta_{ik} - \delta_{ij}] \]
\[ = \sum_k A_{u}^{kj} G_u^{jk} [\delta_{ik} - \delta_{ij}] = G_u^{ji} A_u^{ij} - \delta_{ij} [G_u A_u]^{jj} \]
Some Proofs - continued and the dust settles

\[ Z_t = Z_0 + \int_{0+}^{t} X_u - d\pi_u + \int_{0+}^{t} \pi_u - dX_u \]
\[ + \sum_{0 < u \leq t} \Delta \pi_u \Delta X_u \]
\[ = Z_0 - \int_{0}^{t} \pi_u \langle D_u X_u \rangle X_u du \]
\[ + \int_{0}^{t} \pi_u X_u^* G_u A_u X_u X_u du \]
\[ + \int_{0}^{t} \pi_u A_u X_u du + \int_{0}^{t} \pi_u H_u X_u du \]
\[ + \text{martingale} \]
\[ = Z_0 + \int_{0}^{t} \Gamma_u Z_u du + \text{martingale} \]

\[ \Gamma_u = -\text{diag}(D) + \text{diag}([G_u A_u]^{jj}) + A_u + H_u \]
In fact

\[ \Gamma_{ij}^u = A_{ij}^u \exp\left[ C_i^j - C_i^i \right] \quad \text{if} \quad i \neq j \]

and

\[ \Gamma_{jj}^u = -D_{ij}^j + A_{ij}^{jj} \quad \text{if} \quad i = j \]

so

\[ \hat{Z}_t \equiv \mathbb{E}[Z_t|\mathcal{F}_0] = \int_0^t \Gamma_u \hat{Z}_u du \]

and so

\[ \hat{Z}_t = \Psi(0, t)X_0 \]
We can clearly replace

0 by \( t \),

t by \( T \),

\( Z_t \) by \( Z_{t,T} = \frac{\pi_T}{\pi_t} X_T \) generalizing \( Z_t = Z_{0,t} \)

and then obtain in the same way:

\[
\hat{Z}_{t,T} = \mathbb{E}[Z_{t,T} | \mathcal{F}_t] = \Psi(t, T) X_t
\]