Optimal Investment in a Gordon Model with Externalities

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Abstract
This paper studies the topic of optimal investment in physical capital and R&D in the context of a valuation model, which is a particular version of the Gordon growth model. The proposed valuation model adjusts the Gordon model in two respects. First, it distinguishes between R&D investments and investments in physical capital and shows that the two have distinct optimality conditions. Second, it shows that externalities have implications for a bound from above on the relevant hurdle rate for R&D investments and for the way in which returns on R&D investments are reconciled with returns on financial assets.

Key words: Gordon model, research and development, optimal investment, externalities.

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1. Introduction

Optimal investment in the context of the Gordon (1962) Growth Model was first considered by Lerner and Carlton (1964). Their analysis however was criticised on the grounds of making the assumption that the return on past investments is a variable rather than given (Vickers, 1966; Ben-Shahar and Ascher, 1967; Crocket and Friend, 1967; Nichols, 1968). Derivations that correct this problem are provided by Vickers (1966), Ben-Shahar and Ascher (1967) and Nichols (1968).

The general argument arising from the corrected derivations is that investment in the Gordon model should be increased to a point where the marginal return on investment is equal to the price/earnings ratio, which is lower than the discount rate. Vickers (1966) demonstrates that this value maximization criterion is consistent with the traditional microeconomic view.

This paper extends the above literature in two directions. First, in the above cited literature investments in R&D and investments in physical capital are treated in the same manner with one variable that refers to both of them. There is strong reason to believe however that these two types of investment should be clearly distinguished. Physical capital is a direct input to the production function, while R&D works in a different way; it provides growth shocks to the output function, enabling the firm to produce more with the same amount of input.

To distinguish between the two types of investments it is necessary to reconcile the Gordon model with the traditional production (or profit) function. With this at hand, it is possible to show that the optimality criterion derived in the above cited papers is relevant (with modifications) to investment in R&D, while investments in physical capital should be treated in a way consistent with the profit maximizing approach. More precisely, this paper shows that it is optimal for the firm to increase the investment in R&D to a point where the marginal return on R&D investment is equal to the ratio of the cash flows of the firm (before R&D expenditures) to the value of the
firm. It is shown that this ratio is equal to the average return on distributions the firm makes to its security holders.\(^1\) This provides a straightforward economic interpretation for this optimality criterion; since the firm has two uses for its cash flows, R&D investments and distributions to security holders, it is optimal to equate the benefit of these two.

A second motivation for this paper is to adjust the Gordon model for two externalities that have been considered in the growth literature; the business stealing effect and the research spillover effect. The business stealing effect refers to the observation that if the firm does not make technological progress its rivals might steal business from it (e.g. Aghion and Howitt, 1988, 1992). The knowledge or research spillover effect refers to the fact that a firm cannot fully appropriate the results of innovations and at least part of the reward to new knowledge created by the firm would diffuse to other firms (see, for example, Jaffe, 1986; Steurs, 1995). One implication of these two effects in the context of the Gordon model is that the growth rate of the firm can be different from zero even if the firm does not make any investment at all. This possibility is incorporated into the Gordon model and its implications are examined. The results show that externalities have implications for a bound from above on the hurdle rate for R&D investments and for the way in which returns on R&D investments are synchronized with returns on financial assets.

This paper is organised as follows. Section 2 reconciles the Gordon model with a production/profit function. It shows that the typical profit maximization criterion holds with respect to physical capital investment. Section 3 discusses the topic of returns to capital in the context of the proposed model. Section 4 derives an optimality condition with respect to R&D investment. Section 5 interprets the results of section 4 in terms of a relevant hurdle rate, and

\(^1\) The setting of this paper assumes perfect certainty so debt and equity generate the same return. The firm is either fully financed by equity or offers a debt contract with features identical to those of the firm’s shares.
section 6 derives a bound from above on this hurdle rate. Section 7 shows how returns on R&D investment are reconciled with returns on financial assets. Section 8 provides an example. A summary concludes the paper.

2. The optimization problem of the firm

The setting of this paper considers an infinitely-lived firm with a single product in a world with perfect certainty and no taxes. It is assumed that the firm is fully financed by equity, or alternatively, that it offers a debt contract with features identical to those of the firm’s shares.\(^2\) To keep the framework simple, only three types of inputs are considered: labor, physical capital, and R&D, where labor and physical capital are directly needed for the production function and R&D is an input to a growth function that generates shocks to productivity (e.g. by improving the machines the firm uses).

The value of the firm today (time 0) is assumed to be given by

\[
V_0 = \int_{t=0}^{\infty} e^{-rt} \psi_t(K_t, L_t, \theta_{RD}) \, dt
\]

(1)

where \(r\) is the discount rate, \(\psi_t\) is the distribution of the firm to its security holders, \(K_t\) is physical capital, \(L_t\) is labor, \(\theta_{RD}\) is a constant proportion of gross cash flows (defined below) invested in R&D, and \(t\) is notation for time.

Define now

\[
V_0 = (1 - \theta_{RD}) \int_{t=0}^{\infty} e^{-rt} GCF_t(K_t, L_t) \, dt
\]

(2)

\(^2\) This assumption may look odd at first. Note, however, that in a world with perfect certainty bonds and shares will offer identical returns.
where \( GCF \) is the gross cash flow of the firm. This is simply the cash flow before investment in R&D. Note that equations (1) and (2) imply that the distribution of the firm to its security holders is

\[
\psi_t(K_t, L_t, \theta_{RD}) = (1 - \theta_{RD})GCF_t(K_t, L_t)
\]

Equation (3) says that the firm allocates a constant proportion of its cash flows to R&D investment and the rest are distributed to security holders.

To demonstrate the difference between R&D investments and investments in physical capital, consider the following specification of the gross cash flow function

\[
GCF_t(K_t, L_t) = p_t Y_t(K_t, L_t) - z_t K_t - w_t L_t
\]

where \( p_t \) is the price of the firm’s product, \( Y_t \) is a Cobb-Douglass production function, \( z_t \) is the price of one unit of physical capital, \( w_t \) is the cost of labor, and \( K_t \) is both investment and the stock of physical capital (it is assumed that the depreciation rate is 100%, implying that the firm needs to renew its stock of capital every period. This implies that the stock of capital is equal to the investment of the firm in every period).\(^3\)

Substituting (4) into (2), we have

\[
V_0 = (1 - \theta_{RD}) \int_{t=0}^{\infty} e^{-r_t} [p_t Y_t(K_t, L_t) - z_t K_t - w_t L_t]dt
\]

For simplicity, it is assumed that \( p_t, z_t \) and \( L_t \) are all constants through time. Under this assumption (5) can be written as

\[
V_0 = (1 - \theta_{RD}) \int_{t=0}^{\infty} e^{-r_t} [p_0 Y_t(K_t, L_0) - z_0 K_t - w_t L_0]dt
\]

\(^3\) The assumption is made in order to prevent complications arising from distinguishing investment needed for the renewal of existing machines and investment in new machines.
The main purpose of R&D is to improve the machines that the firm uses in the production process. The improved machines enable the firm to produce more with the same amount of physical capital and labor. Following this argument, it is assumed that R&D investments result in a shock to the production function, which implies that at time \( \Delta t \)

\[
Y_{\Delta t}(K_{\Delta t}, L_0) = e^{(1-\alpha)g(0)\Delta t}Y_0(K_{\Delta t}, L_0)
\]

where \( g(\theta _{RD}) \) is assumed to be concave in the amount of R&D investment.

Using the assumption of a Cobb-Douglass production function, (7) can be written as

\[
Y_{\Delta t}(K_{\Delta t}, L_0) = e^{(1-\alpha)g(0)\Delta t}K_{\Delta t}^\alpha L_0^\beta
\]

It is assumed that \( \alpha + \beta < 1 \). This assumption implies diminishing returns to scale in inputs that are directly needed for the production function. The assumption is important as it enables the firm to have a positive gross cash flow from which the firm can allocate resources to R&D.

Using (8), (6) can be written as

\[
V_0 = (1 - \theta _{R&D}) \int_0^\infty e^{-nt} \left[ p_0 e^{(1-\alpha)g(0)\Delta t}K_{t}^\alpha L_0^\beta - K_t z_0 - w_t L_0 \right] dt
\]

It is assumed now that wages are determined exogenously to the firm and grow at a rate proportional to \( e^{g(0)\Delta t} \). This means that \( w_t \) in (9) can be replaced with \( e^{g(0)\Delta t}w_0 \), implying that

\[
V_0 = (1 - \theta _{R&D}) \int_0^\infty e^{-nt} \left[ p_0 e^{(1-\alpha)g(0)\Delta t}K_{t}^\alpha L_0^\beta - K_t z_0 - e^{g(0)\Delta t}w_0 L_0 \right] dt
\]

It is now shown that wages, investment in physical capital and output, all grow at an identical rate proportional to \( e^{g(0)\Delta t} \). This feature of identical growth rates for the three variables is compatible with the common modeling of economic growth (see for example Aghion and Howitt, 1998). To show that investment in physical capital grows at a rate \( e^{g(0)\Delta t} \) per one unit of time, (10) is
derivated with respect to $K_0$ and $K_{\Delta t}$. Appendix A shows that the derivatives with respect to $K_0$ and $K_{\Delta t}$ yield

\[ K_0 = \left( \frac{z_0}{\alpha p_0 L_0^0} \right)^{\frac{1}{\alpha-1}} \quad (11) \]

\[ K_{\Delta t} = e^{g(0_{RD})\Delta t} \left( \frac{z_0}{\alpha p_0 L_0^0} \right)^{\frac{1}{\alpha-1}} \quad (12) \]

Equations (11) and (12) imply that

\[ K_{\Delta t} = e^{g(0_{RD})\Delta t} K_0 \quad (13) \]

Equation (13) demonstrates that the capital stock grows at a rate of $e^{g(0_{RD})\Delta t}$ per unit of time. The fact that capital stock grows in time implies that there is another source of output growth besides the one generated by investments in R&D. To find the total change of output at time $\Delta t$, one needs to use equations (8), and (13), which yield

\[ Y_\Delta (K_{\Delta t}, L_0) = e^{(1-\alpha)g(0_{RD})\Delta t} K_{\Delta t} L_0^\beta = e^{(1-\alpha)g(0_{RD})\Delta t} \left( e^{g(0_{RD})\Delta t} K_0 \right)^\alpha L_0^\beta = e^{g(0_{RD})\Delta t} K_0^\alpha L_0^\beta \quad (14) \]

Since $Y_0(K_0, L_0) = K_0^\alpha L_0^\beta$, equation (14) implies that the initial shock to output generated by R&D investments together with the secondary shock generated by the increase in the stock of capital yield a total growth rate of $e^{g(0_{RD})\Delta t}$ per unit of time. Because the combined effect of investments in R&D and physical capital sum to $g(\theta_{RD})$, it follows that the setting displays diminishing returns to capital due to the concavity of $g(\theta_{RD})$. This result is different from the standard assumption of constant or increasing returns to capital of the endogenous growth models. This property is discussed in more detail in the following section.
The above results indicate that, in this setting, output, physical capital and wages, all grow at exactly the same rate of $e^{\theta (0) \Delta t}$, implying that the GCF of the firm at time $\Delta t$ can be written as

$$GCF_{\Delta t} = e^{\theta (0) \Delta t} GCF_0$$

Equation (15) suggests thus that (6) can be written as

$$V_0 = (1 - \theta_{R&D}) \int_{t=0}^{\infty} e^{-\eta} e^{\theta (0) \Delta t} \left[ p_0 K_0^\beta L_0^\beta - z_0 K_0 - w_0 L_0 \right] dt$$

Note that (16) can be also written as

$$V_0 = \left[ p_0 K_0^\alpha L_0^\alpha - z_0 K_0 - w_0 L_0 \right] \int_{t=0}^{\infty} (1 - \theta_{R&D}) e^{-\eta} e^{\theta (0) \Delta t} dt$$

The expression in (17) suggests that the value of the firm can be decomposed into two independent parts; the first is the gross cash flow of the firm at time 0 and the second relates to the impact of R&D investments. Equation (17) shows more clearly why the firm has positive gross cash flows as this is a direct result of the assumption $\alpha + \beta < 1$ which implies decreasing returns to scale in inputs that are directly needed for the production process.

It is assumed that the objective function of the firm is to maximize its value as given by (17). This objective function indicates that the value of the firm at time 0 is a function of three decision variables; the allocation to R&D, investment in physical capital at time 0, and the amount of production labor at time 0. Optimal investment in physical capital is given by equation (11). The optimal usage of labor is found from differentiation of (17) (or (10)) with respect to $L_0$ and is given by

$$L_0 = \left( \frac{w_0}{p_0 K_0^\alpha} \right)^{\frac{1}{\beta - 1}}$$
Note that the expression inside the integral in (17) has the same general look as the Gordon model, but the variables are different. Instead of using earnings and a retention ratio, the model in (17) uses gross cash flows and allocation to R&D investments.

3. Returns to capital: a discussion

The previous section demonstrates that the endogenous growth model discussed in this paper displays diminishing returns to capital. This is in contrast to existing models of endogenous growth that typically assume constant or increasing returns to capital. This section discusses this property of the model.

The neoclassical growth model (Solow, 1956) assumes a Cobb-Douglass production function with two types of inputs, physical capital and labor. The setting of the neoclassical growth model is characterized by constant returns to scale and diminishing returns to capital. Attempts to incorporate endogenous growth into the Solow model is problematic, because constant returns to scale imply that all the revenue of the firm is rewarded to physical capital and labor, leaving no place to compensate a third type of input, such as R&D.

Two lines of literature have developed to take account of this problem. Both, however, have substantial weaknesses. The first, originally suggested by Arrow (1962), is typically referred to as “learning by doing.” Learning by doing implies that technological progress results as an unintended by-product of production or investment. Since there is no purposive investment in knowledge creation, there is no need to compensate such an activity and therefore one can get a model with sustained growth while maintaining the assumption of constant returns to scale. The main weakness of this argument is that it cannot explain why firms in the real world engage in purposive knowledge enhancing investments, such as R&D.
Another line of argument that tries to explain the existence of purposive investment in knowledge belongs to the endogenous growth literature. This literature incorporates a third type of capital into the Cobb-Douglass model, typically human capital (e.g., Lucas, 1988) or R&D (e.g. Grossman and Helpman, 1991) and assumes constant or increasing returns to capital. To compensate all three factors, the endogenous growth literature typically incorporates a degree of monopoly power into the model (see for example Romer, 1987; Romer, 1990a; Grossman and Helpman, 1991; Aghion and Howitt, 1992.) The main weakness of the endogenous growth models is not theoretical but empirical. Studies that estimate the degree of returns to capital typically find decreasing returns to capital and not constant or increasing returns to capital (see, for example, Romer, 1990b; Benhabib and Jovanovic, 1991; Benhabib and Spiegel, 1994; King and Levin, 1994; Mankiew, Romer and Weil, 1992). Empirical studies also reject another prediction of the endogenous growth models; that output per capita across different countries converges through time (e.g., Parente, 2001).

Consistent with the empirical evidence, the framework analysed in this paper is characterized by diminishing returns to capital (which includes both R&D and physical capital). This demonstrates that endogenous growth models do not have to display constant or increasing returns to capital. As indicated in the previous section, the property of diminishing returns to capital is obtained by assuming that the growth function is concave in $\theta_{RD}$.

4. Optimal investment in R&D

Several standard assumptions regarding the properties of $g(\theta_{RD})$ are made. First, the growth rate is assumed to be monotonically increasing in $\theta_{RD}$, implying that $g'(\theta_{RD}) > 0$. Second, it is
assumed that investment in R&D displays diminishing returns to scale. This implies that \( g''(\theta_{RD}) < 0 \). Third, it is assumed that \( g'(0) = \infty \). This assumption implies that the marginal return on investment in R&D goes to infinity as investment in R&D approaches zero.

Another characteristic of the growth function refers to the growth rate of the firm when it makes no investment in R&D. In general, the growth of the firm generated by R&D is not only determined by its own R&D but also by R&D activities of other firms. Two well known externalities in this respect are the business stealing effect and the research spillover effect. The first externality relates to the fact that if a firm does not make enough research, competing firms can steal business from it (see, for example, Aghion and Howitt, 1988; Aghion and Howitt, 1992). The second externality, the research spillover effect, relates to the fact that the firm can benefit from the research efforts of other firms by using some of the knowledge or new products generated by other firms (see, for example, Romer, 1990; Aghion and Howitt, 1988; Aghion and Howitt, 1992). These effects suggest that the growth rate of the firm can be different from zero even if it does not make any investment at all. The growth rate of the firm when it does not make investments at all is denoted below by \( g(0) \).

Solving for the integral in (17), we have

\[
V_0(\theta_{RD}) = \frac{(1 - \theta_{RD})GCF_0^*(K_0^*, L_0^*)}{r - g(\theta_{RD})}
\]  
(19)

where \( GCF_0^*(K_0^*, L_0^*) \) is the optimal level of gross cash flows given optimal levels of physical capital and labor. (Below \( GCF_0^*(K_0^*, L_0^*) \) will be denoted by \( GCF_0^* \).) Note that since the optimal gross cash flow can be solved independently of the optimal levels of labor and physical capital, it follows that, at time zero, (3) can be written as

\[
\psi_0(\theta_{RD}) = (1 - \theta_{RD})GCF_0^*
\]  
(20)
where (20) assumes that the firm has already optimized its labor and physical capital inputs; this assumption is maintained in the analysis below.

Appendix B demonstrates that the first-order condition with respect to optimal R&D investment is given by

\[ g'(\theta_{RD}) = \frac{r - g(\theta_{RD})}{1 - \theta_{RD}} \]  

(21)

5. Optimal investment in R&D in terms of a hurdle rate

For the purpose of deriving the relevant hurdle rate for optimal investment in R&D, the following standard decomposition of the firm’s cost of capital is used

\[ r = dy(\theta_{RD}) + g(\theta_{RD}) \]  

(22)

where \( g(\theta_{RD}) \) is the growth rate of the firm, and \( dy(\theta_{RD}) \) is the firm’s distribution yield, defined as

\[ dy(\theta_{RD}) = \frac{\psi(\theta_{RD})}{V_0(\theta_{RD})} \]  

(23)

(Note that the distribution yield is assumed to be constant through time and therefore one can use the time-zero variables to compute it.)

It is assumed that as a result of externalities, the growth rate of the firm can be decomposed into two components as suggested by the following equation

\[ g(\theta_{RD}) = g(0) + f(\theta_{RD}) \]  

(24)

Equation (24) suggests that the growth function can be decomposed into \( g(0) \), the effect of the externalities, and a function of \( \theta_{RD} \), \( f(\theta_{RD}) \), that intersects the origin.
From (22) and (24)

\[ r = \psi(\theta_{RD}) + f(\theta_{RD}) + g(0) \]  

(25)

Two other variables are now defined; the average return on R&D and the average return on distributions. The average return on R&D is defined as the growth rate that results from R&D, \( g(\theta_{RD}) - g(0) \), divided by the portion of GCF\(^*\) allocated to investments in R&D

\[ r_g(\theta_{RD}) = \frac{g(\theta_{RD}) - g(0)}{\theta_{RD}} = \frac{f(\theta_{RD})}{\theta_{RD}} \]  

(26)

(Note that an equation similar to (26) is typically used in the Gordon model. (See, for example, Bodie, Kane and Marcus p. 616.) The definition in (26) is different in the sense that it takes into account the possibility that \( g(0) \) can be different from zero.)

In a similar manner, the average return on distributions is defined as the distribution yield divided by the portion of GCF allocated to distributions

\[ r_\psi(\theta_{RD}) = \frac{\psi(\theta_{RD})}{1 - \theta_{RD}} \]  

(27)

Using (27) and (22), the optimality condition in (21) becomes

\[ g'(\theta_{RD}) = r_\psi(\theta_{RD}) \]  

(28)

Equation (28) indicates that the hurdle rate for optimal investment in R&D is the above defined average return on distributions. Note now that using (27) and (23), (28) can be also written as

\[ g'(\theta_{RD}) = \frac{\psi(\theta_{RD})}{(1 - \theta_{RD})V_0(\theta_{RD})} \]  

(29)

And using (20), (29) becomes

\[ g'(\theta_{RD}) = \frac{GCF^*}{V_0(\theta_{RD})} \]  

(30)

Equation (30) suggests that the investment in R&D should be increased until the marginal return on R&D is equal to the ratio of GCF\(^*\) to the value of the firm. This result is consistent with the
results shown in Vickers (1966), Ben-Shahar and Ascher (1967) and Nichols (1968), but instead of using earnings and a retention ratio it uses gross cash flows and allocation to R&D. The explanation for the results in (28) and (30) is straightforward. The firm can use its gross cash flow for two purposes, either to invest it in R&D or to distribute it. The conditions in (28) and (30) merely say that the firm should equate the marginal output of these two outflows. Note that while there is an explicit growth function, there is not an explicit function for distributions (although distributions are a function of $\theta_{\text{RD}}$ indirectly). This is why the marginal return on the growth function is equated with the average return on distributions (and not with the marginal return on distributions).

6. A bound from above on the hurdle rate

This section demonstrates a bound from above on the hurdle rate, $r_{\psi}(\theta_{\text{RD}})$. To find this bound, (25), (26) and (27) are used to get

$$r = (1 - \theta_{\text{RD}})r_{\psi}(\theta_{\text{RD}}) + \theta_{\text{RD}}r_{\rho}(\theta_{\text{RD}}) + g(0)$$  \hspace{1cm} (31)

which can be written as

$$r - g(0) = (1 - \theta_{\text{RD}})r_{\psi}(\theta_{\text{RD}}) + \theta_{\text{RD}}r_{\rho}(\theta_{\text{RD}})$$  \hspace{1cm} (32)

Equation (32) suggests that the firm’s cost of capital minus $g(0)$ can be represented as a weighted average of two components, the average return on R&D and the average return on distributions.

It is demonstrated now that $r_{\psi}^*(\theta_{\text{RD}})$ and $r_{\rho}^*(\theta_{\text{RD}})$ satisfy

$$r_{\psi}^*(\theta_{\text{RD}}) < r - g(0) < r_{\rho}^*(\theta_{\text{RD}})$$  \hspace{1cm} (33)
where $r^*_w(\theta^*_\text{RD})$ and $r^*_g(\theta^*_\text{RD})$ are the optimal average returns on investments and distributions, respectively, given optimal investment in R&D.

To demonstrate, consider a firm that does not exploit its growth opportunities. The value in this case is found by substituting $\theta^*_\text{RD} = 0$ and $g(0)$ into equation (19), resulting in

$$V_0(\theta^*_\text{RD} = 0) = \frac{\text{GCF}^*_0}{r - g(0)}$$

(34)

By the definition of $\theta^*_\text{RD}$ as the optimal allocation to R&D

$$V^*_0(\theta^*_\text{RD}) > V_0(\theta^*_\text{RD} = 0)$$

(35)

where $V^*_0(\theta^*_\text{RD})$ is the optimal value of the firm given an optimal investment in R&D.

From (28), (30), (34) and (35) it follows that

$$V_0(\theta^*_\text{RD}) - V_0(\theta^*_\text{RD} = 0) = \frac{\text{GCF}^*_0}{r^*_w(\theta^*_\text{RD})} - \frac{\text{GCF}^*_0}{r - g(0)} > 0$$

(36)

Equation (36) implies that $r^*_w(\theta^*_\text{RD}) < r - g(0)$, and it follows immediately from (32) that

$$r^*_w(\theta^*_\text{RD}) < r - g(0) < r^*_g(\theta^*_\text{RD})$$

(37)

Equation (37) suggests that the hurdle rate $r^*_w(\theta^*_\text{RD})$ is bounded from above by $r - g(0)$. The right hand side inequality in (37) suggests that in order for an investment in R&D to be worthwhile for the firm to pursue, the average return on this investment should be higher than the cost of capital minus the impact of externalities. This is easier to explain, if one considers the case $g(0) = 0$ for which the right hand side inequality suggests that the average return on R&D should be higher than the discount rate.
7. The Synchronization of financial markets and R&D investment

This section shows how returns in financial markets are synchronized with returns on R&D investments. To see the exact relation between the hurdle rate for R&D investment and the discount rate, start with equation (30), which implies that

\[ V^*_0(\theta^*_\text{RD}) = \frac{GCF^*_0}{g'(\theta^*_\text{RD})} \]  

(38)

Isolating \( GCF^*_0 \) in (38) and (35) and equating the two results in

\[ r - g(0) = \frac{V^*_0(\theta^*_\text{RD})}{V_0(\theta_{RD} = 0)} g'(\theta^*_\text{RD}) \]  

(39)

Equation (39) can be written as

\[ r - g(0) = \frac{V_0(\theta_{RD} = 0) + AV^*_0(\theta^*_\text{RD})}{V_0(\theta_{RD} = 0)} g'(\theta^*_\text{RD}) \]  

(40)

Where \( AV^*_0(\theta^*_\text{RD}) \) is the optimal added value created by R&D investments. Equation (40) shows that the value of future R&D investments, \( AV^*_0(\theta^*_\text{RD}) \), is an important determinant of the marginal return on investment. The higher is the value of future R&D investments the lower is the marginal return on R&D investment (assuming that \( r \) and \( g(0) \) are held constant). Better R&D growth opportunities increase the size of the cake, and the increased value leads to a lower required marginal return on R&D investments.

8. An example

Consider a firm with the following growth function

\[ g(\theta_{RD}) = g(0) + b\theta_{RD}^{\lambda-\lambda} \]  

(41)
where $0 < \lambda < 1$ in order to satisfy the requirement of concavity of the growth function and $b > 0$ is a constant.

The derivative of the growth function with respect to $\theta_{RD}$ is

$$g'(\theta_{RD}) = b(1-\lambda)\theta_{RD}^{-\lambda}$$  \hspace{1cm} (42)

Applying the optimality condition in (21) and using (40), we have

$$b(1-\lambda)\theta_{RD}^{-\lambda} = \frac{r - g(\theta_{RD})}{1-\theta_{RD}}$$  \hspace{1cm} (43)

Suppose that $b = 0.08$, $\lambda = 0.7$, $r = 0.1$, $g(0) = 0.01$. Under these values it is found that the optimal $\theta_{RD}$ is $\theta_{RD}^* = 0.365$, the growth rate is $g^*(\theta_{RD}^*) = 0.069$, the average return on R&D is $r_{\theta}^*(\theta_{RD}^*) = 0.162$, the marginal return on R&D investment is $g'(\theta_{RD}^*) = 0.049$, and assuming $GCF_0^*$ is equal to $1$ dollar the value of the firm is $V_0^*(\theta_{RD}^*) = $20.57.

Note that substituting $\theta_{RD}^* = 0.365$, $r_{\theta}^*(\theta_{RD}^*) = 0.049$, $r_{\theta}^*(\theta_{RD}^*) = 0.162$ and $g(0) = 0.01$ into (31), we get the cost of capital $r = 0.1$. Also note that the results satisfy the condition in (37).

Figures 1 and 2 illustrate the characteristic features of this problem. Figure 1 plots the average return on distributions, the average return on R&D, and the marginal return on R&D. The optimal allocation to R&D is determined from the intersection of the marginal return on R&D and the average return on distributions. Figure 2 plots the value of the firm as a function of $\theta_{RD}$. The maximum value, $20.57$, is obtained with $\theta_{RD}^* = 0.365$.

Figures 1 and 2 about here
Conclusions

This paper studies the topic of optimal investment in R&D and physical capital in the context of a valuation model, which is a particular version of the Gordon growth model. The topic of optimal investment in the context of the Gordon model has been studied in the 1960s, but the studies on the topic did not distinguish between R&D investments and investments in physical capital. This drawback appears to be critical, because the results in this paper show that the two types of investments have distinct optimality criteria. Optimal investment in physical capital conforms to the standard profit maximization criterion, while for R&D investments, it is shown that the relevant hurdle rate is equal to ratio of the firm’s cash flows (before R&D expenditures) to the value of the firm.

Another topic that this paper considers is the possibility of negative or positive growth rates even if the firm does not make any investment at all in R&D. This property of the growth function is feasible due to externalities, such as the research spillover effect and the business stealing effect. It is found that externalities have implications in two respects. First, they have an impact on a bound from above on the hurdle rate for investments in R&D, and second, they affect the way in which returns on R&D investments are reconciled with returns on financial assets.

One of the features of endogenous growth models, the property of constant or increasing returns to capital, has been under attack recently as this property has been rejected by the empirical evidence. This paper shows that an endogenous growth model can be reconciled with diminishing returns to capital. The property of diminishing returns to capital is obtained by assuming that the growth function, relating R&D investment to the growth of the firm, is concave.
Appendix A

Derivation of $V_0$ with respect to $K_0$

\[
\frac{\partial V_0}{\partial K_0} = \frac{\partial}{\partial K_0} \left[ (1 - \theta_{R&D}) \left( p_0 K_0^\alpha L_0^\beta - K_0 z_0 - w_0 L_0 \right) \right] = \alpha p_0 K_0^\alpha L_0^\beta - z_0 = 0
\]

And it follows that

\[
K_0 = \left( \frac{z_0}{\alpha p_0 L_0^\beta} \right)^{\frac{1}{a-1}}
\]

Derivation of $V_0$ with respect to $K_{\Delta t}$

\[
\frac{\partial V_0}{\partial K_{\Delta t}} = \frac{\partial}{\partial K_{\Delta t}} \left[ p_0 e^{(1-a)g(\theta_{RD})\Delta t} K_{\Delta t}^\alpha L_0^\beta - K_{\Delta t} z_0 - e^{g(\theta_{RD})\Delta t} w_0 L_0 \right] = \alpha p_0 e^{(1-a)g(\theta_{RD})\Delta t} K_{\Delta t}^{\alpha-1} L_0^\beta - z_0 = 0
\]

And it follows that

\[
K_{\Delta t} = e^{g(\theta_{RD})\Delta t} \left( \frac{z_0}{\alpha p_0 L_0^\beta} \right)^{\frac{1}{a-1}}
\]
Appendix B

Differentiating equation (19) with respect to $\theta_{RD}$ and equating to zero

\[
\frac{\partial V_\theta(\theta_{RD})}{\partial \theta_{RD}} = \left[ \frac{(-1)}{r - g(\theta_{RD})} + (-1) \frac{(1 - \theta_{RD})}{(r - g(\theta_{RD}))^2} (-g'(\theta_{RD})) \right] = 0
\]

Rearranging

\[
\left[ \frac{-1}{r - g(\theta_{RD})} + \frac{(1 - \theta_{RD})g'(\theta_{RD})}{(r - g(\theta_{RD}))^2} \right] = 0 \quad (44)
\]

Multiplying by $r - g(\theta_{RD})$

\[
\left[ \frac{(1 - \theta_{RD})g'(\theta_{RD})}{r - g(\theta_{RD})} \right] = 1
\]

Rearranging, we get the optimality condition

\[
g'(\theta_{RD}) = \frac{r - g(\theta_{RD})}{1 - \theta_{RD}}
\]
References


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Figure 1: The average return on R&D investment, the marginal return on R&D investment, and the average return on the distributions

Figure 1 plots the average return on R&D investment, the marginal return on R&D investment, and the average return on distributions for the case \( r = 0.1 \) and a growth function given by \( g(\theta_{RD}) = 0.01 + 0.08\theta_{RD}^3 \). The average return on R&D investment is \( r_{\varepsilon}(\theta_{RD}) \). The marginal return on R&D investment is \( g'(\theta_{RD}) \). The average return on distributions is \( r_{\psi}(\theta_{RD}) \). Optimal investment in R&D is determined by the condition \( g'(\theta_{RD}) = r_{\psi}(\theta_{RD}) \). In this example the optimal allocation to R&D is \( \theta_{RD} = 0.365 \).
Figure 2: Firm’s value as a function of the proportion of gross cash flows allocated to R&D

Figure 2 presents the value of the firm as a function of the allocation to R&D investment. The results are generated for the case $r = 0.1$, a growth function given by $g(\theta_{RD}) = 0.01 + 0.08 \theta_{RD}^{0.3}$, and a gross cash flow of $GCF_0 = $1. The optimal allocation to R&D is 0.365, resulting in a firm value of $20.57$. 