Jump-Diffusion International Asset Pricing with Nontraded Consumption Goods

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Abstract

We present a jump-diffusion international asset pricing model with stochastic exchange rates and inflation rates when investors consume both traded and nontraded goods. We argue that in general, the Adler-Duma inflation rate differential may not capture PPP deviation risks, unless all volatilities, drift rates and jumps rates for price levels, exchange rates and asset returns are constant, and there are no PPP deviation jumps. Our model suggests that country-specific demand for risky assets arises from two sources: PPP-deviation risks and nontraded-good-specific inflation-rate-differential risks. Furthermore, equilibrium asset returns can be expressed in a multi-beta linear asset pricing model with a number of benchmark portfolios including hedge portfolios for PPP deviation risks and nontraded-good-specific inflation rate risks. We show that all of these hedge portfolios may be constructed by using portfolios of the reference-country nominal riskfree bond and individual countries' TIPS bonds. We note that in the presence of inflation risks, hedging against exchange rate risks in isolation can actually make the investor's real wealth riskier than no hedging at all. We also show that global investors optimally increase their consumption in both traded nontraded goods as the prices of traded goods of their own countries increase.

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1 Introduction

We present a jump-diffusion international asset pricing model in the presence of stochastic exchange rates and inflation rates when investors consume both traded and nontraded goods. The inflation rates of traded consumption goods of different countries are allowed to fluctuate inconsistently with currency exchange rates across countries, generating PPP (purchasing power parity) deviation risks. New features of our model are threefold: (1) Unlike Adler and Dumas [1983], we explicitly model PPP deviation risks in the presence of inflation rate risks; (2) dynamics of all risky asset returns are driven by jump-diffusion processes where their jumps are resulted from jumps in PPP deviations; and (3) unlike authors in the "home bias" literature, we introduce price risks, as opposed to output risks, of nontraded goods. The economic significance of these features are discussed in turn.

It is well known that exchange rate risks arise because of PPP (purchasing power parity) deviations. Without PPP deviations, exchange rates are nothing more than a translation mechanism from one currency to another, and therefore the international asset pricing model should not be different from the domestic CAPM except that the global market portfolio replaces the domestic market portfolio. In the international asset pricing literature, although the relevance of PPP deviation risks is repeatedly and extensively discussed, existing models fail to isolate the role of PPP deviation risks in the presence of inflation rate risks.[1] In their seminal paper, Adler and Dumas (AD) [1983, footnote 54] state ".... There is no implication that PPP deviations are some kind of separate risk against which any one investor would want to hedge. ...."

We argue that there exists separate demand for PPP deviation hedge portfolios and that in some cases the PPP deviation risks can be hedged by using portfolios of domestic and foreign TIPS (Treasury Inflation Protection Securities) bonds. The separate demand is identified as a consequence of our explicit modelling of PPP deviations. We follow a widely used convention to model the PPP deviation of each country as the logarithm of the real exchange rate, where the real exchange rate is the ratio of the actual to the PPP-theoretical exchange rates. We assume PPP deviations are driven by jump-diffusion processes in the presence of both inflation and exchange rates risks. Jumps in PPP deviations can be expected particularly for countries adopting controlled exchange rate regimes.

Our PPP deviations are in contrast with AD's inflation rate differentials between two countries. Although AD interpret their differentials as PPP deviations, we argue that their interpretation may not be justified in general unless all volatilities and drift rates for price levels and exchange rates are constant and there are no PPP deviation jumps. For instance, even when they do not jump, if PPP deviations exhibit a mean-reverting behavior, AD's differentials may no longer serve as a good measure of PPP deviation risks.

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[1] Solnik [1974]/Sercu [1980] started the international asset pricing literature with exchange rate risks but without inflation risks. In the absence of inflation risks, exchange rate risks are the same as PPP deviation risks.

We also allow asset returns to jump as PPP deviations jump. It is empirically well known that international asset returns exhibit jump behaviors. Bates [1996] uses a jump-diffusion exchange rate model to price currency options contracts. Das and Uppal [2004] introduce systematic and perfectly correlated jumps to capture excess kurtosis and skewness in international asset return data, in order to examine effects of jumps on the effectiveness of international diversification.[2] In Das and Uppal, there are no exchange rate/PPP deviation risks. In spite of the well-known recognition of the importance of jump behaviors in international asset returns, there doesn't seem to exist an international asset pricing model allowing for jumps in asset returns. As a result of our jump-diffusion model, we identify demand for jump-hedge portfolios which can be used as a metric in assessing the effect of jumps on the international diversification/asset allocation in the presence of PPP deviation risks.

In addition to jumps in both PPP deviations and asset returns, our investors are allowed to consume traded and nontraded goods, and consequently our model nests models with a common traded good. This feature is important since it has been recognized that investors' portfolio decisions may be impacted by nontraded goods.[3] In the "home biases" literature, authors argue that in order to hedge against output risks of nontraded goods (produced by domestic assets) in exchange economies, investors may want to hold more domestic assets than they do when there are no nontraded goods. See Stockman and Dellas [1989], Tesar [1993] and Baxter, Jermann and King [1998]. Unlike these authors, we focus on price risks, as opposed to output risks, of nontraded goods and on investors' portfolio decisions in the capital market in order to investigate the role of price levels of nontraded goods in international asset pricing.

As a byproduct of modelling price risks of traded and nontraded goods, we show that holding other things constant, international investors increase their consumption of both traded and nontraded goods as prices of traded goods in their own currencies increase. The intuition of this striking result is that when the country-l price of the traded good increases, the exchange rate decreases, i.e. the value of country-l currency decreases. Since the country-l investor manages his wealth in global financial markets, his real wealth in his own currency increases as country-l currency weakens, and thus he increases his real consumption of both traded and nontraded goods. Note that these increases in consumption would not occur without either of the following two fundamental reasons: the existence of nontraded goods and global financial markets. Without nontraded goods, an increase in the price of the traded good does not affect the traded good consumption, because the increase of his nominal wealth in his country currency can be exactly offset by the increase in the price. Without global financial markets, the country-l investor only invests in domestic assets, and his nominal wealth in his country

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[2]Ang and Bekart [2002] use a regime shift model to support the usefulness of international diversification in spite of high correlations of international equity market returns. Their regime shift model may also be interpreted as a jump-diffusion model in effect.

[3]The importance of nontraded goods in international asset pricing is noted by Stulz [1981]. However, in his article, it is not discussed how exactly the existence of nontraded goods affect asset prices.

currency remains unaffected.

Our jump-diffusion financial markets with traded and nontraded goods are formulated as an incomplete market problem where investors may not be able to fully hedge for all sources of uncertainties affecting asset returns. We use the martingale approach to characterize investors' asset allocation problems. In a simplified case of our general model, we obtain five fund separation: Each investor holds a portfolio of three global funds and two country-specific funds. The three global funds are the global 'log' risky fund, a hedge fund for the reference-country inflation risks, and the reference country's nominal riskfree asset. The country-specific funds consist of two funds: one fund to hedge PPP deviations between the investor's and reference countries, and another to hedge nontraded-good specific inflation rates.

Country-specific demands for risky assets have been intensively studied in the empirical literature on home biases. In this paper, the country-specific demands for assets depend on two sources: (1) the differences between individual countries' PPP deviations and the world weighted average PPP deviation and (2) the differences between individual countries' nontraded good-specific inflation rates and the world weighted average nontraded good-specific inflation rates.[4] It can be shown that under some conditions, the first source is similar to AD's inflation rate differential risks between individual countries and and the world.

In global incomplete market equilibrium, our five fund separation suggests that the excess return on each risky asset can be expressed as a multi-beta linear function of the rate of return on benchmark portfolios which are the global market portfolio, and individual countries' common-good inflation hedge portfolios, PPP deviation-hedge portfolios, and nontraded-good inflation hedge portfolios. Consequently, we have an asset pricing model with $3(L+1)$ betas, where $L+1$ is the total number of countries. If our financial market were to be complete, then there would be an extra set of L benchmark portfolios hedging PPP-deviation jumps, in addition to the $3(L+1)$ portfolios. We further show that under certain conditions, TIPS bonds of all countries can be used to construct the hedge portfolios.

For example, when there are no jumps, a country-l PPP deviation hedge portfolio can simply be formed by a long position in reference-country TIPS bonds indexed on the reference-country common good price, and a short position in country-l TIPS bonds indexed on country-l price of the same good. The intuitive reason for the usefulness of international TIPS bonds in hedging PPP deviation risks is as follows: TIPS bond prices in general fluctuate with inflation rates. However, a position in a foreign TIPS bond brings about not only foreign inflation rate risks, but exchange rate risks. In this case, one can show that the net risks of the foreign TIPS bond consists of the reference-country inflation rate and PPP deviation risks. Thus, the foreign TIPS bond position, combined with a short position in the reference country TIPS bond, can have the reference-country inflation rate risks completely filtered

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[4]Since the they are country-specific, two sources may be related to the "home-bias" literature. However, we leave it for a future empirical investigation.

out, and the net position will be left with PPP deviation risks only. When there are jumps in PPP deviations, the PPP deviation hedge portfolio can take a more complex form, but the basic intuition is similar to the case of no jumps.

The rest of this paper is organized as follows: In the next section, we model PPP deviations. Section 3 presents the formulation of individual investors' portfolio choice problems. We analyze the problems using the martingale approach in section 4. In this section optimal consumption/investment decisions are discussed in terms of five funds consisting of three global funds and two country-specific funds. In section 5, we aggregate individual investors' optimal portfolios to produce our version of the IAPM in a familiar multi-beta form with global and country-specific benchmark portfolios. We discuss how one can use TIPS bonds to construct some country-specific benchmark portfolios, and also discuss forward-rate premium/discount with a note that the premium/discount is equal to market risk premia for exchange rate jump-diffusion risks minus expected exchange rate. Section 6 recovers existing IAPMs as special cases. Finally, a brief summary of the paper is presented in section 7.

2   PPP Deviations

The probability space $(\Omega, \mathcal{F}, P)$ is given. Let $z(t) = (z_1(t), ..., z_N(t))^\top$ be an N-dimensional (column) vector of standard independent Wiener processes, and $\tilde{N}(t) = (\tilde{N}^1(t), ..., \tilde{N}^L(t))^\top$ be an L-dimensional column vector of independent point processes with an L-dimensional intensity $\lambda(t) = (\lambda^1(t), ..., \lambda^L(t))^\top$. Throughout the paper, we use superscript $\top$ to denote the transpose. The "usual" filtration $\{\mathcal{F}_t\}$ is generated by $\{z(t)\}$ and $\{\tilde{N}(t)\}$.

There are $L + 1$ countries. Country $L + 1$ is the reference country. There is a common good globally traded at a price of $P_c^l (> 0, a.s.)$ in country-l currency. The price evolves as follows. For $l = 1, ..., L, L + 1$, and for $\omega \in \Omega$,

$$dP_c^l(t) = P_c^l(t)[\mu_{p_c^l}(t, \omega)dt + \sigma_{p_c^l}^\top(t, \omega)dz(t)], \tag{1}$$

where $\mu_{p_c^l}$ and $\sigma_{p_c^l}$ are $\mathcal{F}_t$-adapted, and they are a scalar and an N-dimensional process, respectively. We set $P_c^{L+1} \equiv P_c^*$.

Country l, $l = 1, ..., L, L + 1$, also has one nontraded good whose price is denoted by $P_n^l (> 0, a.s.)$ and evolves as follows.

$$dP_n^l(t) = P_n^l(t)[\mu_{p_n^l}(t, \omega)dt + \sigma_{p_n^l}^\top(t, \omega)dz(t)], \tag{2}$$

where $\mu_{p_n^l}$ and $\sigma_{p_n^l}$ are $\mathcal{F}_t$-adapted, and they are a scalar and an N-dimensional process, respectively. We also set $P_n^{L+1} \equiv P_n^*$. The arguments (t, ω) will be suppressed for the remainder of the paper unless needed for clarity.

The PPP exchange rate for the common good is

$$S_c^l(t) = \frac{P_c^*(t)}{P_c^l(t)}.$$

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Throughout the paper, we express all exchange rates in terms of common good prices across countries. However, we require no PPP relationships for nontraded goods, because the PPP can only be meaningfully applied to traded goods. Then by Itô's formula,

$$\frac{dS_c^l(t)}{S_c^l(t)} = \left(\mu_{p_c^*} - \mu_{p_c^l} + \sigma_{p_c^*}^\top \sigma_{p_c^*} - \sigma_{p_c^l}^\top \sigma_{p_c^*} \right) dt + \left(\sigma_{p_c^*}^\top - \sigma_{p_c^l}^\top \right) dz(t).$$

Let the actual exchange rate denoted by $S_A^l(t)$, and define the PPP deviation $D^l(t)$ as follows:

$$D^l(t) := \ln \left(\frac{S_A^l(t)}{S_c^l(t)} \right). \tag{3}$$

We assume that for $l = 1, 2, ..., L,$[5]

$$dD^l(t) = \mu_{D^l}(t, \omega)dt + \sigma_{D^l}^\top(t, \omega)dz(t) + \phi_{D^l}(t-, \omega)d\tilde{N}^l(t), \tag{4}$$

where $\mu_{D^l}(t)$, $\sigma_{D^l}(t)$ and $\phi_{D^l}(t)$ are all $\mathcal{F}_t$-adapted, and $\mu_{D^l}(t)$ and $\phi_{D^l}(t)$ are scalers, and $\sigma_{D^l}(t)$ is an N-dimensional column vector. We set $D^{L+1}(t) = D^*(t) = 0$ by convention. The existence of nonzero $D^l(t)$ implies that commodity markets for the common good are imperfect or partially segmented. The case of perfect commodity markets is a special case with $D^l(t) \equiv 0$ for all l. On the other hand, commodity markets for nontraded goods are completely segmented, in the sense that no PPP relationships affect prices of nontraded goods across countries.

Eq.'s (3) and (4) imply

$$\phi_{D^l}(t-) = \ln S_A^l(t) - \ln S_A^l(t-),$$

and by Itô's formula,

$$dS_A^l(t) = S_A^l(t-) \left[\mu_{S_A^l} dt + \left(\sigma_{p_c^*}^\top - \sigma_{p_c^l}^\top + \sigma_{D^l}^\top \right) dz(t) + (e^{\phi_{D^l}} - 1)d\tilde{N}^l(t) \right], \tag{5}$$

where

$$\mu_{S_A^l} := \mu_{p_c^*} - \mu_{p_c^l} + \sigma_{p_c^l}^\top \sigma_{p_c^l} - \sigma_{p_c^l}^\top \sigma_{p_c^*} + \mu_{D^l} + \frac{1}{2} \left(\sigma_{D^l}^\top \sigma_{D^l} + 2\sigma_{p_c^*}^\top \sigma_{D^l} - 2\sigma_{p_c^l}^\top \sigma_{D^l} \right).$$

Note that our exchange rates directly depend on jump risks and common good price risks, but not directly on nontraded goods price risks. Eq.(5) implies that the actual exchange rate risk can be decomposed into two components: PPP exchange rate risk $(\sigma_{p_c^*}^\top - \sigma_{p_c^l}^\top)dz(t)$, and PPP-deviation-related risk $\sigma_D^\top dz(t) + (e^{\phi_{D^l}} - 1)d\tilde{N}^l(t)$.

In addition to P_c^l, P_n^l and D^l, the economy may have an m-dimensional column vector of state variables X which evolve according to the following dynamics.

$$dX_t = \mu_X(t, \omega)dt + \sigma_X^\top(t, \omega)dz(t), \tag{6}$$

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[5]See Froot and Rogoff [1995] whose empirical evidence leads them to reject the PPP. However, the mean reverting properties of the PPP appear to be controversial: Dumas [1992] rejects whereas Taylor and Peel [2000] support the properties.

where μ_X is an $\mathcal{F}_t$-adapted K-dimensional column vector and $\sigma_X^\top$ is an $\mathcal{F}_t$-adapted $K \times N$ matrix.

Before the closure of this section, a comment on AD's inflation rate differential is in order. Recall that AD define inflation rates in terms of the reference currency. Let $\varphi_c^l(t) = S_A^l(t)P_c^l(t)$ and $\varphi_n^l(t) = S_A^l(t)P_n^l(t)$. Then $\varphi_c^l(t)$ and $\varphi_n^l(t)$ are price levels of country-l traded and nontraded goods in the reference currency, respectively. Note that by definition we have $\varphi_c^*(t) = P_c^*(t)$ and $\varphi_n^*(t) = P_n^*(t)$ because $S_A^* \equiv 1$. AD interpret their inflation rate differential $d\varphi_c^l/\varphi_c^l - d\varphi_c^*/\varphi_c^*$ as the PPP deviation risk. This interpretation may not be justified in general.

To see this, note that

$$d\varphi_c^l(t) \;\; = \;\; \varphi_c^l(t)\mu_{\varphi_c^l}dt + \varphi_c^l(t)\left(\sigma_{p_c^*}^\top + \sigma_{D^l}^\top\right)dz(t) + \varphi_c^l(t)(e^{\phi_{D^l}} - 1)d\tilde{N}^l(t)$$

where

$$\mu_{\varphi_c^l} \;\; = \;\; \mu_{p_c^*} + \mu_D + \frac{1}{2}\sigma_D^\top\sigma_D + \sigma_{p_c^*}^\top\sigma_{D^l}.$$

Therefore, we have

$$\frac{d\varphi_c^l}{\varphi_c^l} - \frac{d\varphi_c^*}{\varphi_c^*} \;\; = \;\; \left(\mu_D + \frac{1}{2}\sigma_D^\top\sigma_D + \sigma_{p_c^*}^\top\sigma_D\right)dt + \sigma_{D^l}^\top dz(t) + (e^{\phi_{D^l}} - 1)d\tilde{N}^l(t)$$

$$= \left(\frac{1}{2}\sigma_D^\top\sigma_D + \sigma_{p_c^*}^\top\sigma_D\right)dt + dD^l(t) + (e^{\phi_{D^l}} - 1 - \phi^l)d\tilde{N}^l(t) \quad (7)$$

Clearly, AD's inflation rate differential $d\varphi_c^l/\varphi_c^l - d\varphi_c^*/\varphi_c^*$, in general, does not evolve like the PPP deviation $dD^l(t)$. Furthermore, in order to capture the PPP deviation risk, $\sigma_{D^l}^\top dz(t) + \phi_{D^l}(d\tilde{N}^l(t) - \lambda^l dt)$, the AD differential should be, at least, represented in the following form: $(\text{constant})dt + \sigma_{D^l}^\top dz(t) + \phi_{D^l}(d\tilde{N}^l(t) - \lambda^l dt)$. However, this form of representation can be possible only if $\phi_{D^l} = 0$ and the drift $\mu_D + \frac{1}{2}\sigma_D^\top\sigma_D + \sigma_{p_c^*}^\top\sigma_D$ is constant. Thus, roughly speaking, empirical studies on international asset pricing models using inflation rates or inflation differentials may not fully capture the effects of the PPP deviation risk, unless all volatilities and drifts are truly constant and there are no PPP deviation jumps. For instance, if the PPP deviation is mean-reverting, AD differentials may not purely capture PPP deviation risks even without PPP deviation jumps.

3 The Individual Investor's Problem

Assume no taxes, transactions costs and information asymmetry. There are $N + 1$ traded securities which consist of N risky assets and the reference country's nominal riskfree asset. All security prices are measured in the reference currency. Among N risky assets included are individual countries' nominal riskfree assets. The gains process (capital gains plus dividends) for risky security i, is denoted by A_i, is positive and evolves as follows: for $i = 1, 2, ..., N$,

$$dA_i(t) = A_i(t)\left[\mu_i(t,\omega)dt + \sigma_i^\top(t,\omega)dz(t) + \phi_i^\top(t,\omega)d\tilde{N}(t)\right].$$

Let $\mu_A = (\mu_1, ..., \mu_N)^\top$, $\sigma_A = (\sigma_1, \sigma_2, ..., \sigma_N)^\top$ and $\phi_A = (\phi_1, \phi_2, ..., \phi_N)^\top$. Note that μ_A, σ_A and ϕ_A are, respectively, an $N \times 1$ vector, an $N \times N$ matrix, and an $N \times L$ matrix. For security 0, the reference country's nominal riskfree asset,

$$dA_0(t) = r(t, \omega) A_0(t) dt.$$

We assume that $\int_0^T \|\mu_A(t, \omega)\| dt < \infty$ a.s. and $\int_0^T r(t, \omega) dt < \infty$ a.s., and that σ_A has full rank for all $t \in [0, T]$.

Let Q be a probability measure such that $dQ/dP = \xi(T)$, where ξ is an exponential local martingale defined as follows: $\xi(t) = \xi_\nu(t) \xi_\theta(t)$, where

$$\xi_\nu(t) := \exp \left\{ - \int_0^t \nu^\top(s) dz(s) - \frac{1}{2} \int_0^t \nu^\top(s) \nu(s) ds \right\}, \tag{8}$$

$$\xi_\theta(t) := \exp \left\{ \sum_{l=1}^L \int_{(0,t]} \ln \left(\frac{\theta_l(s)}{\lambda_l(s)} \right) d\tilde{N}(s) - \sum_{l=1}^L \int_0^t (\theta_l(s) - \lambda_l(s)) ds \right\}, \tag{9}$$

$$\mu_A(t) - r(t)\mathbf{1} - \sigma_A(t)\nu(t) + \phi_A^\top(t)\theta(t) = \mathbf{0}, \tag{10}$$

where $\mathbf{1}$ and $\mathbf{0}$ are N-dimensional vectors of ones and zeros, respectively; and $\theta(t)$ and $\lambda(t)$ are bounded $L \times 1$ intensity vectors for $\tilde{N}(t)$ under probability measures Q and P, respectively. (For ξ_θ, see Brèmaud [1981, 165f].) Let us assume that $E[e^{\frac{1}{2} \int_0^T \|\nu(t)\|^2 dt}] < \infty$. Then by the Girsanov theorem,

$$z^*(t) = z(t) + \int_0^t \nu(s) ds$$

is a martingale under probability measure Q with $dQ/dP = \xi(T)$, and it is independent of $\tilde{N}$. Also, let $M(t) = \tilde{N}(t) - \int_0^t \lambda(s) ds$ and $M^*(t) = \tilde{N}(t) - \int_0^t \theta(s) ds$. Then $M(t)$ and $M^*(t)$ are P- and Q-martingales, respectively. Condition (10) ensures all assets yield riskfree rates of return r under Q. That is, under this condition, Q becomes a risk-neutral measure to prevent arbitrage opportunities in our global financial markets.

Note that our financial market is incomplete, because there are N risky assets with N diffusion-risk and L jump-risk sources. That is, there are infinitely many pairs of $(\nu(t), \theta(t))$ to satisfy (10), and thus infinitely many risk-neutral measures. Although we note later that in some special cases, the (PPP deviation-induced) jump risk sources can be hedged with TIPS bonds and the global market can be completed (see section 5.1.2), our general model is formulated to allow the incompleteness of the market.

We assume there is one representative investor for each country. Each investor from country $l = 1, 2, ..., L + 1$ maximizes his expected utility of real consumption subject to his budget constraint. The expected utility is given by

$$E \left[\int_0^T U(C_{Rc}^l, C_{Rn}^l, t) dt \right]$$

where $C_{Rc}^l \geq 0$ and $C_{Rn}^l \geq 0$ are real consumption in country-l's common and nontraded goods, respectively.

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Let C_k^l be the nominal consumption of good k, $k = c, n$, in the reference currency. Then, C_k^l/S_A^l is the investor's nominal consumption of good k in his own currency. Thus, his real consumption is given by $C_{Rk}^l = C_k^l/(S_A^l P_k^l)$, $k = c, n$. Equivalently, $C_{Rc}^l = C_c^l/(e^{D^l} P_c^*)$ and $C_{Rn}^l = C_n^l/(e^{D^l}(P_c^*/P_c^l)P_n^l)$. When the absolute PPPs do not hold, one unit of reference currency may give investors of different nationalities different levels of real purchasing power.

The nominal consumption $C_k^l (\geq 0)$ for good k, $k = c, n$, in the reference currency is withdrawn from his nominal wealth W^l which is driven by the following (self-financing) dynamics.

$$W^l(t) = W^l(0) + \int_0^t r(s)W^l(s)(1 - {\alpha_A^l}^\top \mathbf{1})ds + \int_0^t W^l(s){\alpha_A^l}^\top (\mu_A dt + \sigma_A dz(s) + \phi_A d\tilde{N}(t))$$
$$- \int_0^t (C_c^l(s) + C_n^l(s))ds, \tag{11}$$

where $\alpha_A^l := (\alpha_1^l, ..., \alpha_N^l)^\top$ is a set of portfolio weights of risky assets. Note that throughout the paper, the jump integral, $\int_0^t$, should be interpreted as $\int_{(0,t]}$. We assume $\int_0^T \|\alpha(t)\|^2 dt < \infty$ a.s. and $\int_0^T (C_c(t) + C_n(t))dt < \infty$ a.s.

Then, the country-l investor's problem is to choose nominal consumption policies C_c^l and C_n^l, and portfolio policy α_i^l, $i = 1, 2, ..., N$ to maximize

$$E\left[\int_0^T U\left(\frac{C_c^l}{e^{D^l}P_c^*}, \frac{C_n^l P_c^l}{e^{D^l}P_c^* P_n^l}, t\right)dt\right] \tag{12}$$

subject to the following dynamic constraints: C_c^l and C_n^l are self-financed by the wealth process in (11); P_c^* and P_c^l evolve according to (1); and P_n^l and D^l evolve according to (2) and (4), respectively. Note that if $C_n^l \equiv 0$, for all l, then we have the AD case. If $P_c^l \equiv 1$, and $C_n^l \equiv 0$, for all l, then we have the Solnik/Sercu case.[6]

One may approach the above problem by using either the dynamic programming or the martingale method developed by Cox and Huang [1989], Karatzas, Lehoczky and Shreve [1987] and Karatzas [1989] for complete markets; and by He and Pearson [1991] and Karatzas, Lehoczky, Shreve and Xu (KLSX) [1991] for incomplete markets. We rely on the martingale method, which enables us to characterize optimal consumption in our incomplete market without having to solve for the value function.

KLSX [1991] and He and Pearson (1991) characterize incomplete market equilibrium conditions by using the least favorability condition which requires the incomplete-market equilibrium pricing kernel parameters to minimize the representative investor's value function over pricing kernel parameters for fictitiously completed markets. In this sense, one may view the incomplete market equilibrium as a special case of fictitiously completed market equilibria. This implies that necessary conditions for all fictitious market equilibria should be also necessary for our incomplete global financial market equilibrium.

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[6] Later on, one can see that the Solnik/Sercu result also holds with multiple goods if both P_c^l and P_n^l are deterministic.

One may interpret the least favorability condition in the context of our jump-diffusion economy as follows: for all $l = 1, ..., L+1$, let $V^l(\nu(t), \theta(t), t)$ be the value function with a fictitiously completed market and the pricing kernel parameter pair $\{(\nu(t), \theta(t))\}$. Note that the market can be fictitiously completed by introducing L pure jump assets where pure jump asset l, $l = 1, ..., L$, has its price jump when D^l jumps. Then $\{(\nu(t), \theta(t))\}$ supports the incomplete market equilibrium if for all $l = 1, ..., L+1$ and $t \in [0, T]$,

$$(\nu(t), \theta(t)) \in \arg\min_{(\nu', \theta')} V^l(\nu', \theta', t) \quad \text{s.t.} \quad (\nu', \theta') \text{ satisfies Eq.(10).}$$

The above appears to be a very stringent condition. Given the pricing kernel satisfying the least favorability condition, investors find no incentives to complete the market by bringing new assets into the market. However, instead of limiting ourself to particular pairs of $(\nu(t), \theta(t))$ satisfying the above least favorability condition, we try to characterize our incomplete market given any (ν', θ') satisfying Eq.(10) and the self-financing condition. Thus our results throughout the paper can be interpreted as necessary conditions for the incomplete market equilibrium.

We close this section with a remark on exchange rate hedging. The investor's problem stated in (12) implies that the investor is concerned with common- and nontraded-good-price risks arising, respectively, from $e^{D^l} P_c^*$ and $e^{D^l} P_c^* (P_n^l / P_c^l)$, in addition to asset price risks from $A(t)$. For simplicity, let us assume no jumps in exchange rates. Then the risk of $e^{D^l} P_c^*$ is represented by $\sigma_{p_c^*}^\top + \sigma_{D^l}^\top$ which is the same as $\sigma_{p_c^l}^\top + \sigma_{S_A^l}^\top$. Also the risk of $e^{D^l} P_c^* (P_n^l / P_c^l)$ is $\sigma_{p_c^*}^\top + (\sigma_{p_n^l}^\top - \sigma_{p_c^l}^\top) + \sigma_{D^l}^\top \equiv \sigma_{p_n^l}^\top + \sigma_{S_A^l}^\top$. Thus, when he tries to hedge against exchange rate risks, he has to consider interactions between exchange rates S_A^l's and inflation rates, P_c^l and P_n^l. Suppose that the country-l investor expects a foreign currency inflow in the near future, and that the exchange rate, expressed in units of the reference currency per one unit of country-l's domestic currency, happens to be significantly negatively correlated with domestic inflation rates P_c^l and P_n^l. (This negative correlation may happen, because as P^l increases, country-l currency value weakens and thus S_A^l decreases.) Then hedging against the exchange rate alone can make his real wealth riskier than no hedging at all. Thus, in order to avoid this pitfall, he need to hedge against inflation risks, together with the exchange rate risk. This simultaneous hedging can be possible if appropriate TIPS bonds or TIPS bond forward contracts are traded in the foreign country. For example, he may hedge a future cash inflow of one foreign currency unit by taking a short position in a foreign TIPS bond with a face value of one foreign currency unit maturing at the same time as the future cash inflow matures. See Section 5.1.

4 Optimal Consumption/Investment Decisions

From now on, we suppress superscript l unless needed for clarity. Rewriting the nominal wealth process in the reference currency, we have

$$W(t) = W(0) + \int_0^t r(s)W(s)ds + \int_0^t W(s)\alpha_A^\top(\sigma_A dz^*(s) + \phi_A dM^*(s)) - \int_0^t (C_c(s) + C_n(s))ds,$$

with the terminal condition being $W(T) = 0$ because of our zero bequest. Define

$$\hat{W}^l(t) = e^{-\int_0^t r(u)du} W^l(t)$$

Then

$$\hat{W}(t) = \hat{W}(0) + \int_0^t \hat{W}(s)\alpha_A^\top(\sigma_A dz^*(s) + \phi_A dM^*(s)) - \int_0^t e^{-\int_0^s r(u)du}(C_c(s) + C_n(s))ds,$$

where $\hat{W}(0) = W(0)$. Furthermore, under Q,

$$E^Q\left[\int_0^T e^{-\int_0^t r(u)du}(C_c(t) + C_n(t))dt\right] = W_0.$$

Thus, the country-l investor's problem is restated as follows:

$$\max_{C_c^l, C_n^l} \quad E\left[\int_0^T U\left(\frac{C_c^l}{e^{D^l}P_c^*}, \frac{C_n^l P_c^l}{e^{D^l}P_c^* P_n^l}, t\right)dt\right] \tag{13}$$

$$\text{s.t.} \quad E^Q\left[\int_0^T e^{-\int_0^t r(u)du}(C_c^l(t) + C_n^l(t))dt\right] = W_0^l.$$

The Lagrangian is

$$\mathcal{L}^l = E\left[\int_0^T U\left(\frac{C_c^l}{e^{D^l}P_c^*}, \frac{C_n^l P_c^l}{e^{D^l}P_c^* P_n^l}, t\right)dt\right] + q^l\left(W_0^l - E^Q\left[\int_0^T e^{-\int_0^t r(u)du}(C_c^l(t) + C_n^l(t))dt\right]\right).$$

Define

$$\Psi^l(t) = q^l\xi(t)e^{-\int_0^t r(u)du},$$

which is the pricing kernel times the Lagrange multiplier q^l. Assuming an interior solution, we have the first order conditions (FOCs) as follows:

$$U_{C_{R_c}}\left(\frac{C_c}{e^D P_c^*}, \frac{C_n P_c}{e^D P_c^* P_n}, t\right) = e^{D(t)}P_c^*(t)\Psi(t) \tag{14}$$

$$U_{C_{R_n}}\left(\frac{C_c}{e^D P_c^*}, \frac{C_n P_c}{e^D P_c^* P_n}, t\right) = e^{D(t)}P_c^*(t)\Psi(t)\frac{P_n(t)}{P_c(t)}. \tag{15}$$

The FOC in Eq.(15) amounts to an implicit assumption that the supply of a nontraded good within a country is determined to maximize the expected utility of the representative investor of that country,

just as in the literature, the supply of the common good is determined to maximize the expected utility of the global central planner, i.e., to maximize a weighted average of expected utilities of all investors by equalizing marginal utilities across countries with respect to the common good. Consequently, via FOCs, country-l nontraded good price is related to country-l common good price which is in turn related to the reference country common good price. In other words, although nontraded goods are not traded, their equilibrium (domestic) prices cannot be completely independent of those of traded goods. This kind of global interaction between prices of traded and nontraded goods occurs because of the presence of the global capital market pricing kernel $e^{-\int_0^t r(u)du}\xi(t)$ in Ψ^l.

Define

$$F^l(t) := e^{\int_0^t r(u)du} E^Q\left[\int_t^T e^{-\int_0^s r(u)du}(C_c^l + C_n^l)ds \,\bigg|\, \mathcal{F}_t\right].$$

$F(t)$ is simply the present value of future total consumption and is equal to $W(t)$. To see this, note that since $\hat{W}(T) = 0$,

$$\hat{W}(t) = -\int_t^T \hat{W}(s)\alpha_A^\top(\sigma_A dz^*(s) + \phi_A dM^*(s)) + \int_t^T e^{-\int_0^s r(u)du}(C_c(s) + C_n(s))ds.$$

Thus

$$\hat{W}(t) = E^Q\left[\int_t^T e^{-\int_0^s r(u)du}(C_c(s) + C_n(s))ds \,\bigg|\, \mathcal{F}_t\right] = e^{-\int_0^t r(u)du}F(t).$$

Therefore, $F(t) = W(t)$. Note that (14) and (15) imply $C_c(t) = C_c(\Psi, D, P_c^*, P_c, P_n, t)$ and $C_n(t) = C_n(\Psi, D, P_c^*, P_c, P_n, t)$. Note that each of $(\Psi, D, P_c^*, P_c, P_n, t)$ can depend on X. When Ψ, D, P_c^*, P_c, P_n and X are jointly Markov, one can write $F(t) = F(\Psi, D, P_c^*, P_c, P_n, X, t)$. With the Markovian and proper differentiability assumptions for F, let us introduce the following notation.

$$\gamma^l(t) := -\frac{\Psi^l(t)F_\Psi^l(t)}{W^l(t)}, \quad \delta_c^*(t) := \frac{F_{P_c^*}(t)P_c^*(t)}{W(t)}, \quad \delta_{D^l}(t) := \frac{F_{D^l}^l(t)}{W^l(t)}.$$

$$\delta_c^l(t) := \frac{F_{P_c}^l(t)P_c^l}{W^l(t)}, \quad \delta_n^l(t) := \frac{F_{P_n}^l(t)P_n^l}{W^l(t)}, \quad \delta_X(t) := \frac{F_X^l(t)}{W^l(t)}.$$

Note that all above γ and δ's are scalers except δ_X, a K-vector. It is well known that γ is a measure of investor's risk tolerance.

Theorem 1 *Assume that $\Psi, D^l, P_c^*, P_c^l, P_n^l$ and X are jointly Markov, and that F is differentiable once in t and twice in all state variables $(\Psi, D^l, P_c^*, P_c^l, P_n^l, X)$. Suppose that there exists an interior solution to the problem stated in (13) and $(U_{C_{R_c}}, U_{C_{R_n}})$ is invertible with respect to (C_{R_c}, C_{R_n}). Then*

i. the optimal portfolio policy $\alpha_A^l(t)$ for $t < T$ is given by

$$\alpha_A^l(t) = \gamma^l(t)(\sigma_A\sigma_A^\top)^{-1}(\mu_A - r\mathbf{1} + \phi_A\theta) + \delta_c^*(t)(\sigma_A^\top)^{-1}\sigma_{p_c^*} + \delta_{D^l}(t)(\sigma_A^\top)^{-1}\sigma_{D^l}$$

$$+\delta_c^l(t)(\sigma_A^\top)^{-1}\sigma_{p_c^l} + \delta_n^l(t)(\sigma_A^\top)^{-1}\sigma_{p_n^l} + (\sigma_A^\top)^{-1}\sigma_X\delta_X(t). \tag{16}$$

ii. Let $Y_c^l := P_c^* \Psi$ and $Y_n^l := P_c^* \Psi(P_n/P_c)$. If (D, Y_c^l, Y_n^l, X) are jointly Markov, then the optimal portfolio policy $\alpha_A^l(t)$ for $t < T$ is given by

$$
\begin{aligned}
\alpha_A^l(t) =\ & \gamma^l(t)(\sigma_A \sigma_A^\top)^{-1}(\mu_A - r\mathbf{1} + \phi_A \theta) + \left(1 - \gamma^l(t)\right)(\sigma_A^\top)^{-1}\sigma_{p_c^*} \\
& + \delta_{D^l}(t)(\sigma_A^\top)^{-1}\sigma_{D^l} + \delta_n^l(t)(\sigma_A^\top)^{-1}\left(\sigma_{p_n^l} - \sigma_{p_c^l}\right) + (\sigma_A^\top)^{-1}\sigma_X \delta_X(t).
\end{aligned}
\tag{17}
$$

iii. If $(\Psi, D^l, P_c^*, P_c^l, P_n^l)$ do not depend on X and have constant drifts, volatilities and jump rates over time, then $\delta_{D^l} = \delta_c^* = 1 - \gamma$, and (16) and (17) are further simplified to

$$
\begin{aligned}
\alpha_A^l(t) =\ & \gamma^l(t)(\sigma_A \sigma_A^\top)^{-1}(\mu_A - r\mathbf{1} + \phi_A \theta) + \left(1 - \gamma^l(t)\right)(\sigma_A^\top)^{-1}\left(\sigma_{p_c^*} + \sigma_{D^l}\right) \\
& + \delta_n^l(t)(\sigma_A^\top)^{-1}\left(\sigma_{p_n^l} - \sigma_{p_c^l}\right).
\end{aligned}
\tag{18}
$$

The joint Markovian assumption implies that each of time-t drifts and volatilities of Ψ, D, P_c^*, P_c, P_n and X can be a function of $\Psi(t), D(t), P_c^*(t), P_c(t), P_n(t), X(t)$ and t. The mean-reverting property of D has been extensively studied in the empirical literature in international finance. In the above theorem, Parts (i) and (ii) allow D to be mean reverting, whereas Part (iii) does not. If D is mean-reverting, say $\mu_D(t) = \bar{\mu}_D - kD(t)$ for some $\bar{\mu}_D \in \mathcal{R}$ and $k > 0$, then we may not have $\delta_D = \delta_c^*$, or $F_D = F_{P_c^*}P_c^*$.[7]

The first term $(\sigma_A \sigma_A^\top)^{-1}(\mu_A - r\mathbf{1} + \phi_A \theta)$ from the right hand sides of Eq.'s (16) to (18) is from the well-known log portfolio, and $(\mu_A - r\mathbf{1} + \phi_A \theta)$ would be an N-dimensional vector of expected excess rates of return on N risky assets if their jumps were completely hedged away. Note that this term can be broken down into two parts as follows:

$$
(\sigma_A \sigma_A^\top)^{-1}(\mu_A - r\mathbf{1} + \phi_A \lambda) - (\sigma_A \sigma_A^\top)^{-1}\phi_A(\lambda - \theta).
\tag{19}
$$

The first part would be the log portfolio if the investor mistook jump-diffusion asset price processes for diffusion asset price processes without jumps, and the second represents an additional (corrective hedging) demand for risky assets because of jump risks. Alternatively put, the second part would be an extra demand for risky assets if pure jump (martingale) risks are added to diffusion asset returns without affecting the expected returns on the assets.

The effect of the second part of (19) on the investor's portfolio allocation decision is numerically examined by Das and Uppal [2004, Tables IV and V] in a simplified setting. Our model captures the jump-induced demand for risky assets in general and in a closed form as above. For simplicity, suppose there is only one risky asset that is expected to make positive jumps if any. Assume the jump risk

---

[7]When D does not exhibit the mean-reverting property, Part (iii) suggests that $F_D = F_{P_c^*}P_c^*$, i.e., both P_c^* and e^D affect the investor's wealth in a similar fashion. However, when D exhibits the mean-reverting property, one may conjecture an increase in e^D may yield smaller impact on the investor's consumption decisions than the same increase in P_c^*, because with a mean reverting property, a current increase in D is likely smoothed out with a decrease in the future. Thus one may conjecture $0 > F_D > F_{P_c^*}P_c^* = 1 - \gamma$. Given the difficulty of obtaining an explicit solution for the investor's value function, we are unable to provide a proof of this conjecture and leave it for future interesting research.

premium $\lambda - \theta$ is positive. Then (19) implies the investor reduces his holding in the risky asset when jump risks are added to the asset return process.

Other terms, $(\sigma_A^\top)^{-1}\sigma_{p_c^*}$, $(\sigma_A^\top)^{-1}\sigma_D$, $(\sigma_A^\top)^{-1}\sigma_{p_c}$, and $(\sigma_A^\top)^{-1}\sigma_{p_n}$ are N-vectors of portfolio weights for risky assets to hedge the diffusion risks of the reference country traded good price, country-l PPP deviation, country-l traded good price, and country-l nontraded good price; and $(\sigma_A^\top)^{-1}\sigma_{p_c^*}$ is a $N \times K$ matrix for hedging risks of K state variables. In Parts ii and iii, $(\sigma_{p_n} - \sigma_{p_c})$ is the diffusion risk of non-traded-good-specific inflation rate. In Part iii, $(\sigma_{c^*} + \sigma_D)$ is the diffusion risk of the country-l exchange rate.

With $1 - \gamma^l \leq 0$, both Parts (ii) and (iii) suggest that the country-l investor would like to take a short position in a portfolio hedging (duplicating) risks of the reference-country inflation rates plus the PPP deviations. However, since the sign of $\delta_n (= F_{P_n}(t)P_n(t)/W(t))$ is not determined yet, it is not completely clear whether the investor would desire to hedge against the country-l nontraded-good-specific price risk. In order to shed light on the sign of F_{P_n}, we assume a Cobb-Douglas utility function.

Proposition 1 *Suppose that all volatilities, drifts and jump rates of $(\Psi, D, P_c^*, P_c, P_n)$ are constant over time, and that country-l investor's utility is given in the following Cobb-Douglas form.*

$$U(C_{R_c}, C_{R_n}, t) = \kappa \left(C_{R_c}\right)^{a_c} \left(C_{R_n}\right)^{a_n},$$

where $a_c, a_n > 0$, and $a_c + a_n = a < 1$. Then,

$$\delta_n = -\frac{a_n}{1-a} < 0, \quad \delta_c = \frac{a_n}{1-a} > 0, \quad \delta_c^* = -\frac{a}{1-a} < 0$$

$$\gamma = \frac{1}{1-a} > 1, \quad and \quad \delta_D = -\frac{a}{1-a} < 0.$$

Furthermore, $(P_c/P_n)^{a_n}(C_{R_c})^a$ is lognormally distributed with jumps. Let (μ_v, η_v) be the pair of the drift and jump rates of $d\left((P_c/P_n)^{a_n}(C_{R_c})^a\right)$. Then the optimal real consumption levels of traded nontraded goods are

$$C_{R_c} = (\kappa a_c)^{\frac{1}{1-a}} \left(\frac{a_n}{a_c}\right)^{\frac{a_n}{1-a}} \left(e^D P_c^* \Psi\right)^{-\frac{1}{1-a}} \left(\frac{P_c}{P_n}\right)^{\frac{a_n}{1-a}} \tag{20}$$

$$C_{R_n} = C_{R_c} \left(\frac{a_n}{a_c}\right) \left(\frac{P_c}{P_n}\right) \tag{21}$$

and the investor's value function is

$$V(t) = \kappa^{\frac{1}{1-a}} a_c^{\frac{a}{1-a}} \left(\frac{a_n}{a_c}\right)^{\frac{a_n}{1-a}} g_v(t) \left(\frac{P_c(t)}{P_n(t)}\right)^{\frac{a_n}{1-a}} \left(e^{D(t)} P_c^*(t)\Psi(t)\right)^{-\frac{a}{1-a}} \tag{22}$$

where

$$g_v(t) = \begin{cases} T - t, & if \ \mu_v + \phi_v^\top \lambda = 0, \\ \frac{e^{(\mu_v + \phi_v^\top \lambda)(T-t)} - 1}{\mu_v + \phi_v^\top \lambda}, & otherwise. \end{cases}$$

Since $F_{P_n} < 0$ by Proposition 1, the country-l investor with the Cobb-Douglas utility function holds a short position in the portfolio hedging the risk of nontraded-good-specific inflation rate. Moreover, with the Cobb-Douglas utility function, both γ and δ are constant over time.

As a byproduct, the value function immediately implies $\partial V/\partial P_c = (V/P_c)(a_n/(1-a)) > 0$. It is striking that holding other things constant, the investor's utility increases as P_c increases. The intuition is that when P_c increases, S_A decreases, i.e. the value of country-l currency decreases. Since the country-l investor manages his wealth in global financial markets, the country-l currency value of his wealth increases. Thus he is better off. In fact, $\partial C_{R_c}/\partial P_c = (C_{R_c}/P_c)(a_n/(1-a)) > 0$ and $\partial C_{R_n}/\partial P_c = (C_{R_c}/P_n)(a_n/a_c)((1-a_c)/(1-a)) > 0$, i.e., real consumption levels of both traded and nontraded goods increase with P_c. Note that these increases in consumption would not occur without either of the following two fundamental reasons: the existence of nontraded goods and global financial markets. Without nontraded goods, an increase in P_c affects neither the value function nor the real consumption. Without global financial markets, the country-l investor only invests in domestic assets, and there will be no exchange rate risks.

4.1 Global Mutual Funds

In order to obtain an insight into the structure of the investor's optimal portfolio, let us assume no state variables X and rearrange Eq.(17) as follows:

$$
\begin{aligned}
\left[\begin{array}{c} \alpha_A \\ 1 - \mathbf{1}^\top \alpha_A \end{array}\right] &= \gamma \left[\begin{array}{c} (\sigma_A \sigma_A^\top)^{-1}(\mu_A - r\mathbf{1} + \phi_A \theta) \\ 1 - \mathbf{1}^\top (\sigma_A \sigma_A^\top)^{-1}(\mu_A - r\mathbf{1} + \phi_A \theta) \end{array}\right] \\
&\quad + (1-\gamma)\left[\begin{array}{c} (\sigma_A^\top)^{-1}\sigma_{p_c^*} \\ 1 - \mathbf{1}^\top (\sigma_A^\top)^{-1}\sigma_{p_c^*} \end{array}\right] - (\delta_D + \delta_n)\left[\begin{array}{c} \mathbf{0} \\ 1 \end{array}\right] \\
&\quad + \delta_D \left[\begin{array}{c} (\sigma_A^\top)^{-1}\sigma_D \\ 1 - \mathbf{1}^\top (\sigma_A^\top)^{-1}\sigma_D \end{array}\right] + \delta_n \left[\begin{array}{c} (\sigma_A^\top)^{-1}(\sigma_{p_n} - \sigma_{p_c}) \\ 1 - \mathbf{1}^\top (\sigma_A^\top)^{-1}(\sigma_{p_n} - \sigma_{p_c}) \end{array}\right] \\
&= \gamma \alpha_{\log} + (1-\gamma)\alpha_{infl^*} - (\delta_D + \delta_n)\alpha_{rf^*} + \delta_D \alpha_D + \delta_n \alpha_{H_n}. \quad (23)
\end{aligned}
$$

The above implies that the country-l investor's portfolio consists of five funds: three global mutual funds and two country-specific funds. Since it is based on Eq.(17), the above five fund result should still hold even when the PPP deviation is mean-reverting.

The five funds are as follows: (1) α_{log}, the log portfolio, the optimal risk portfolio for investors with log utility.[8]; (2) α_{infl^*}, a hedge portfolio against the inflation risks of the reference currency; (3) α_b, the reference country's nominal riskfree asset r; (4) α_D, a hedge portfolio against PPP deviations between the investor's and reference countries; and (5) α_{H_n}, a hedge portfolio against nontraded-good-specific inflation rates. Note that there is a separate demand for a PPP deviation hedge portfolio. Moreover, if there is a home bias, it should depend on the two country-specific funds, α_D and α_{H_n}.

Special cases of our five fund result is summarized in the following remarks.

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[8]For a detailed interpretation of this portfolio, see Adler and Prasad [1992].

Remark 1: No inflation risks and nontraded goods under fixed exchange rate systems imply that $\underline{\alpha}_{infl*} = \underline{\alpha}_D = \underline{\alpha}_{H_n} = \mathbf{0}$, and thus each investor, regardless of her nationality, holds a combination of two global portfolios which are the global market portfolio and the reference currency riskfree asset, i.e., $\gamma \underline{\alpha}_{log} + (1-\gamma) \underline{\alpha}_{rf*}$. Also assume no jumps. Then the IAPM becomes identical to the domestic CAPM.

Remark 2: No inflation risks and nontraded goods under floating exchange rate systems imply that $\underline{\alpha}_{infl*} = \underline{\alpha}_{H_n} = \mathbf{0}$. Then we have three fund separation with the reference-country riskfree asset and two risky funds $\underline{\alpha}_{log}$ and $\underline{\alpha}_D$, i.e., $\gamma \underline{\alpha}_{log} - \delta_D \underline{\alpha}_{rf*} + \delta_D \underline{\alpha}_D$. Further, if $(\Psi, D, P_c^*, P_c, P_n)$ do not depend on X and have constant drifts, and volatilities without jumps, and thus D is not mean-reverting, then $\delta_D = 1 - \gamma$, and thus the investor's portfolio expressed in the investor's own currency collapses to the two funds as seen in Sercu's [1980] equation (2). Since without inflation risks, PPP risks are the same as exchange rates risks, one may write $\underline{\alpha}_{S_A}$ for $\underline{\alpha}_D$.

Remark 3: If there are no nontraded goods, then $\delta_n = 0$ and by Eq.(23), the investor's portfolio is $\gamma \underline{\alpha}_{log} + (1-\gamma) \underline{\alpha}_{infl*} + \delta_D (\underline{\alpha}_D - \underline{\alpha}_{rf*})$. Further, if $(\Psi, D, P_c^*, P_c, P_n)$ do not depend on X and have constant drifts and volatilities without jumps, then the investor's portfolio becomes $\gamma \underline{\alpha}_{log} + (1-\gamma)(\underline{\alpha}_{infl*} + \underline{\alpha}_D - \underline{\alpha}_{rf*})$, which is the same as AD's two fund separation as seen in AD's equation (9), because $\sigma_{p_c^*} + \sigma_D$ is the diffusion rate of AD's country-l inflation φ_c^l.

4.2   Country-specific demand for risky assets

In the home bias literature, country-specific demands for risky assets resulting from investors' optimal portfolio decisions are of particular interest. In an attempt to rationalize "home biases," (empirical) researchers have tried, without much success, to relate home biases to country-specific demands for risky assets implied by international asset pricing models. The country-specific demands for risky assets under intensive empirical investigation have been country-specific hedging demands against Solnik/Sercu's exchange rate risks and AD's inflation rate differential risks.

Cooper and Kaplanis [1994] find that domestic assets do not hedge AD's inflation rate differential risks. Rather, since domestic inflation rates are typically positively correlated with rates of returns on domestic assets, inflation hedging motives can deepen the home bias puzzles. (See Lewis [1999] and Karolyi and Stulz [2001] for reviews on the home bias literature.)

Nonetheless, in this paper, when investors are allowed to consume both traded and nontraded goods, country-specific hedging demands can arise from two sources: (1) PPP deviation rate differential diffusion risks and (2) and nontraded-good-specific inflation-rate-differential risks. Let W^l be the wealth of country l. Also let

$$w^l = \frac{W^l}{\sum_{k=1}^{L+1} W^k}.$$

Following Cooper and Kaplanis [1994], let us assume that γ^l's are the same across all countries. We also assume δ^l's are the same as well. Then Eq.(17) implies that the portfolio weights for risky assets in the global market portfolio are given by

$$\alpha_A^m = \gamma(\sigma_A\sigma_A^\top)^{-1}(\mu_A - r\mathbf{1} + \phi_A\theta) + (1-\gamma)(\sigma_A^\top)^{-1}\sigma_{p_c^*} + \delta_D(\sigma_A^\top)^{-1}\sum_{l=1}^{L} w^l\sigma_{D^l}$$

$$+\delta_n(\sigma_A^\top)^{-1}\sum_{l=1}^{L+1} w^l\left(\sigma_{p_n^l} - \sigma_{p_c^l}\right) + (\sigma_A^\top)^{-1}\sigma_{p_c^*}\delta_X.$$

Recall that $\sigma_{D^{L+1}} = \sigma_{D^*} = 0$. Therefore, country-specific portfolio weights (or country-specific demand) for risky assets can be expressed by

$$\alpha_A^l - \alpha_A^m = \delta_D(\sigma_A^\top)^{-1}\left(\sigma_{D^l} - \sum_{k=1}^{L} w^k\sigma_{D^k}\right)$$

$$+\delta_n(\sigma_A^\top)^{-1}\left\{\left(\sigma_{p_n^l} - \sigma_{p_c^l}\right) - \sum_{k=1}^{L+1} w^l\left(\sigma_{p_n^k} - \sigma_{p_c^k}\right)\right\}.$$

The above indicates that there are two sources of risks for country-specific demand: diffusion risks of PPP-deviation-rate differentials, and nontraded-good-specific inflation-rate-differential risks. To compute the diffusion risks, let us look at PPP deviation rate differential as follows.

$$dD^l - \sum_{k=1}^{L} w^k dD^k$$

$$= (\mu_{D^l} - \sum_{k=1}^{L} w^k\mu_{D^k})dt + (\phi_{D^l}d\tilde{N}^l(t) - \sum_{k=1}^{L}\phi_{D^k}d\tilde{N}^k(t)) + (\sigma_{D^l}^\top - \sum_{k=1}^{L} w^k\sigma_{D^k}^\top)dz(t), \quad (24)$$

which implies the diffusion risks can be captured only after the PPP deviation differential process is completely de-meaned with all jump effects taken into account.[9] Complete de-meaning is important particularly when the PPP deviation exhibits mean-reverting behavior. When the PPP deviations do not have jumps and their drifts are all constant, then the above PPP deviation differential and AD differential agree with each other except by some constant, which is also implied by Eq.(7).[10]

On the other hand, nontraded-good-specific inflation-rate-differential risks can be computed from the following nontraded-good specific inflation-rate-differential process,

$$\left(\frac{dP_n^l}{P_n^l} - \frac{dP_c^l}{P_c^l}\right) - \sum_{k=1}^{L+1} w^l\left(\frac{dP_n^k}{P_n^k} - \frac{dP_c^k}{P_c^k}\right). \quad (25)$$

---

[9]Note that $\sum_{k=1}^{L} w^k dD^k$ is the world average PPP deviation, which is similar to the well-known *real effective exchange rate*, a measure of global competitiveness of the reference country. The real effective exchange rate is a weighted average of real exchange rates, with weights being proportions of trades done with each of the reference country's trading partners. Our world average PPP deviations is a weighted average of logarithms of real exchange rates with weights being sizes of foreign economies in fractions of the world total wealth.

[10]Another issue in the empirical estimation of PPP deviations risks can arise in the presence of nontraded assets. PPP deviations should be computed using common (traded) good price levels rather than a generic CPI (Consumer Price Index). Officer [1976] points out that the use of the CPI is inadequate in computing PPP deviations, because the index is from a mixture of traded and nontraded goods and services and the mixture may not be comparable across countries.

provided that it is properly de-meaned. Note that the country l-specific demand for a portfolio hedging nontraded-good-specific inflation-rate-differential risks is measured in terms of price risks of nontraded goods $\{P_n^l; l = 1, ...L+1\}$ as seen in Eq.(25), whereas in the literature, the hedging demand is typically measured in terms of (production) output risks of nontraded goods. See Stockman and Dellas [1989], Tesar [1993] and Baxter, Jermann and King [1998].

5 International Asset Pricing Model

Note that in our incomplete market economy, one may construct hedge portfolios for various diffusion risk sources by using traded securities. Let $H_{p_c^*}$, H_{D^l}, H_n^l and H_X be hedge portfolios for the common-good-price inflation rate risks, country-l PPP deviations risks over the common good, country-l nontraded good-price inflation rate risks, and other state variables, respectively. Then $H_{p_c^*} = ((\sigma_A^\top)^{-1} \sigma_{p_c^*}, 1 - \mathbf{1}^\top (\sigma_A^\top)^{-1} \sigma_{p_c^*})$, i.e., fractions of the investor's wealth, $(\sigma_A^\top)^{-1} \sigma_{p_c^*}$, invested in N risky assets and the rest $1 - \mathbf{1}^\top (\sigma_A^\top)^{-1} \sigma_{p_c^*}$ in the reference country's riskfree asset. Similarly, $H_{D^l} = (\sigma_{D^l}^\top (\sigma_A)^{-1}, 1 - \mathbf{1}^\top \sigma_{D^l}^\top (\sigma_A)^{-1})$, $H_n^l = (\sigma_{p_n^l}^\top (\sigma_A)^{-1}, 1 - \mathbf{1}^\top \sigma_{p_n^l}^\top (\sigma_A)^{-1})$, and $H_{X_k} = (\sigma_{X_k}^\top (\sigma_A)^{-1}, 1 - \mathbf{1}^\top \sigma_{X_k}^\top (\sigma_A)^{-1})$, where $\sigma_{X_k}^\top$ is the k-th row of $\sigma_X^\top$, and H_{X_k} is a hedge portfolio for the k-th state variable, $k = 1, ..., m$.

Further, let A_m be the global market portfolio index, and also Γ be the variance-covariance matrix for diffusion risks among variables $(A_m, P_c^*, D_1, ..., D_L, P_c^1, ..., P_c^{L+1}, P_n^1, ..., P_n^{L+1}, X_1, ..., X_K)$, which are the market index, the reference-country traded good price, other countries' traded- and nontraded-good prices, and K state variables, respectively. Then in global market equilibrium, all investors agree on the rate of return on each asset and we have the following multi-beta linear IAPM.

Proposition 2 *Suppose that country-l investor's optimal portfolios are given by Eq.(16) for all $l = 1, 2, ..., L+1$. Assume Γ is nonsingular. Then in global market equilibrium, the expected instantaneous rate of return on asset i in the reference currency satisfies the following multi-beta linear relationship: For $i = 1, 2, ..., N$,*

$$
\begin{aligned}
\mu_i + \phi_i^\top \lambda &= r + \phi_i^\top (\lambda - \theta) + \beta_{im}(\mu_m - r + \phi_m^\top \theta) + \beta_{iH_{p_c^*}}(\mu_{H_{p_c^*}} - r + \phi_{H_{p_c^*}}^\top \theta) \\
&\quad + \sum_{l=1}^{L} \beta_{iH_{D^l}}(\mu_{H_{D^l}} - r + \phi_{H_D^l}^\top \theta) + \sum_{l=1}^{L} \beta_{iH_c^l}(\mu_{H_c^l} - r + \phi_{H_c^l}^\top \theta) \\
&\quad + \sum_{l=1}^{L+1} \beta_{iH_n^l}(\mu_{H_n^l} - r + \phi_{H_n^l}^\top \theta) + \beta_{iH_X}(\mu_{H_X} - r + \phi_{H_X}^\top \theta), \quad (26)
\end{aligned}
$$

where $\mu_{H_{p_c^*}} - r + \phi_{H_{p_c^*}}^\top \theta = \sigma_{p_c^*}^\top (\sigma_A)^{-1}(\mu_A - r\mathbf{1} + \phi_A \theta)$, $\mu_{H_{D^l}} - r + \phi_{H_{p_c^*}}^\top \theta = \sigma_{D^l}^\top (\sigma_A)^{-1}(\mu_A - r\mathbf{1} + \phi_A \theta)$, $\mu_{H_c^l} - r + \phi_{H_c^l}^\top \theta = \sigma_{p_c^l}^\top (\sigma_A)^{-1}(\mu_A - r\mathbf{1} + \phi_A \theta)$, and $\mu_{H_n^l} - r + \phi_{H_n^l}^\top \theta = \sigma_{p_n^l}^\top (\sigma_A)^{-1}(\mu_A - r\mathbf{1} + \phi_A \theta)$. That is, $\mu_{H_{p_c^*}}$, $\mu_{H_{D^l}}$, $\mu_{H_n^l}$, and μ_{H_X} are the continuous drifts of hedging portfolios, respectively, for the

reference-country inflation risk, country-l PPP deviation risk, country-l nontraded-good inflation risk, and other state variable risks.

Corollary 1 *If we assume $(\Psi, D^l, P_c^*, P_c^l, P_n^l)$ do not depend on X and have constant drifts, volatilities and jump rates over time, the IAPM in (26) can be greatly simplified as follows: For $i = 1, 2, ..., N$,*

$$
\begin{aligned}
\mu_i + \phi_i^\top \lambda &= r + \phi_i^\top (\lambda - \theta) + \beta_{im}(\mu_m - r + \phi_m^\top \theta) + \beta_{iH_{P_c^*}}(\mu_{H_{P_c^*}} - r + \phi_{P_c^*}^\top \theta) \\
&\quad + \sum_{l=1}^{L} \beta_{iH_{D^l}}(\mu_{H_{D^l}} - r + \phi_{H_D^l}^\top \theta) + \sum_{l=1}^{L+1} \beta_{i\bar{H}_n^l}(\mu_{\bar{H}_n^l} - r + \phi_{\bar{H}_n^l}^\top \theta), \quad (27)
\end{aligned}
$$

where $(\mu_{\bar{H}_n^l} - r + \phi_{\bar{H}_n^l}^\top \theta) = (\sigma_{p_n^l}^\top - \sigma_{p_c^l}^\top)(\sigma_A)^{-1}(\mu_A - r\mathbf{1} + \phi_A \theta)$, and $\bar{H}_n^l$ is a portfolio hedging country-l nontraded-good specific inflation rate risks.

Note that Eq.(26) clearly shows how PPP deviation risks affect asset pricing. The left hand side of the equation is the expected instantaneous rate of return on asset i. The right hand side implies that the risk premium on a jump-diffusion asset consists of two components: a set of jump-risk premia, $\phi_i^\top (\lambda - \theta)$ and another set of diffusion risk premia, i.e., the beta terms. For example, $\phi_{il}^\top (\lambda^l - \theta^l)$ and $\beta_{iH_{D^l}}(\mu_{H_{D^l}} - r + \phi_{H_D^l}^\top \theta)$ are, respectively, jump risk and diffusion risk premia on asset i due to the jump and diffusion risks of the country-l PPP deviation.

With all constant parameters, Eq.(27) implies that asset returns can be approximately/empirically expressed by the following regression model:

$$
\frac{\Delta A_i}{A_i} = a + \sum_{l=1}^{L} \phi_{il} \Delta \tilde{N}^l + \beta_{im} \frac{\Delta A_m}{A_m} + \beta_{iH_{P_c^*}} \frac{\Delta P_c^*}{P_c^*} + \sum_{l}^{L} \beta_{iH_{D^l}} \frac{\Delta D^l}{D^l} + \sum_{l}^{L+1} \beta_{iH_n^l} \left(\frac{\Delta P_n^l}{P_n^l} - \frac{\Delta P_c^l}{P_c^l} \right) + \varepsilon, \quad (28)
$$

where ΔN^l is country-l PPP jump, and ΔA_m, ΔP_c^*, ΔD^l, ΔP_c^l, and ΔP_n^l are changes in continuous parts of A_m, P_c^*, D^l, P_c^l and P_n^l, respectively. Unlike the literature, the above equation allows asset returns to be regressed directly on PPP deviation risks.

Currently, empirical literature seems to confirm the significance of inflation and exchange rates risks in asset prices. See Dumas and Solnik [1995] for exchange rate risks and Vassalou [2000] for inflation rate risks. Although there are numerous discussions on PPP deviations in the asset pricing literature, they only focus on either exchange rate or inflation rate risks separately, and surprisingly it seems there are no empirical studies relating PPP deviations directly. Furthermore, it would be also interesting to see empirical roles of nontraded-good-specific price risks and PPP jumps in international asset pricing.

5.1 Hedge portfolios

Given the importance of PPP-deviation and nontraded-good-specific-inflation hedge portfolios in international asset pricing, we discuss how one can construct these hedge portfolio using familiar TIPS instruments. Let us assume that TIPS bonds are globally traded and that there are two types of TIPS

bonds in each country, one indexed on the common good and the other indexed on the nontraded good. Thus there are $2 \times (L+1)$ TIPS bonds globally traded. We first derive equilibrium prices and dynamics of TIPS bonds, and then show how one can use TIPS bonds to construct the hedge portfolios.

For simplicity, assume all μ's, σ's and ϕ's are constant over time.

5.1.1 Price dynamics of reference country TIPS bonds

Consider a TIPS bond strip (or a zero-coupon TIPS bond) with an initial face value of one reference currency unit maturing at time T. Without loss of generality, assume $\bar{P}_k^*(0) = 1$, $k = c, n$. Then its terminal value (the accrued principal at time T) in the reference currency is $P_k^*(T)$. We know that

$$P_k^*(T) = P_k^*(t)e^{(\mu_{p_k^*} - \frac{1}{2}\sigma_{p_k^*}^\top \sigma_{p_k^*})(T-t) + \sigma_{p_k^*}^\top (z(T) - z(t))}.$$

Let the value of the TIPS bond at time t be $V_{TP}^*(t)$. Then

$$
\begin{aligned}
V_{TP}^*(t) &= E_t\left[P_k^*(T) \frac{\xi(T)}{\xi(t)} \right] \\
&= P_k^*(t)e^{\left(\mu_{p_k^*} - r - \sigma_{p_k^*}^\top \nu\right)(T-t)}.
\end{aligned}
$$

Thus, the dynamics of the TIPS bond price are given by

$$\frac{dV_{TP}^*(t)}{V_{TP}^*(t)} = (r + \sigma_{p_k^*}^\top \nu)dt + \sigma_{p_k^*}^\top dz(t).$$

That is, the reference-country TIPS bond indeed perfectly hedges P_k^*, the reference-country good-k inflation risks measured in the reference currency. This property of TIPS bonds implies that TIPS-bond price changes can be used as a proxy for the inflation risks. It can be a useful property in empirical studies requiring continuous inflation rates while actual inflation rates are not being published continuously.

5.1.2 Price dynamics of country l TIPS bonds

Consider a TIPS bond strip (or a zero-coupon TIPS bond) with an initial face value of one country-l currency unit maturing at time T. The accrued principal at time T is $P_k^l(T)$, $k = c, n$. Without loss of generality, assume $P_k^l(0) = 1$, $k = c, n$. Then its terminal value in the reference currency is $S_A^l(T)P_k^l(T)$. Let $\mu_{SP_k^l}$ and $\sigma_{SP_k^l}$ be the drift and volatility of $S_A^l(T)P_k^l(T)$, respectively. Then

$$\mu_{SP_k} = \mu_{P_k} - \mu_{S_A} + \sigma_{P_k}^\top \sigma_{S_A},$$

and

$$\sigma_{SP_k} = \begin{cases} \sigma_{P_c^*} + \sigma_{D^l}, & \text{if } k = c, \\ \sigma_{P_c^*} + \left(\sigma_{P_n^l} - \sigma_{P_c^l}\right) + \sigma_{D^l}, & \text{if } k = n. \end{cases}$$

We know that with superscript l suppressed,

$$S_A(T)P_k(T) = S_A(t)P_k(t)e^{(\mu_{SP} - \frac{1}{2}\sigma_{SP}^\top \sigma_{SP})(T-t) + \sigma_{SP}(z(T) - z(t)) + \phi_{Dl}(\tilde{N}^l(T) - \tilde{N}(t))}.$$

Let $V_{TP}^l(t)$ be the market price at time t for the the above cashflow in the reference currency. Then we have

$$
\begin{aligned}
V_{TP_k}^l(t) &= E_t\left[S_A(T)P_k(T)\frac{\xi(T)}{\xi(t)}\right] \\
&= S_A(t)P_k(t)e^{\left(\mu_{SP_k} - r - \sigma_{SP_k}^\top \nu + \theta_l(e^{\phi_{Dl}} - 1)\right)(T-t)}
\end{aligned}
$$

Thus,

$$
\frac{dV_{TP_k}^l(t)}{V_{TP_k}^l(t-)} = \left(r + \sigma_{SP_k}^\top \nu - \theta_l(e^{\phi_{Dl}} - 1)\right)dt + \sigma_{SP_k}^\top dz + (e^{\phi_{Dl}} - 1)d\tilde{N}^l. \tag{29}
$$

The above TIPS-bond price dynamics imply if there are TIPS bonds available for all countries, then the market can be completed as far as PPP deviation risks are concerned, even when those PPP deviations are driven by jump-diffusion processes.

For the rest of this subsection, we only consider TIPS bonds indexed on the common good prices, just for simplicity. Consider a portfolio consisting of L TIPS bonds from L countries, asset i and the risk free asset of the reference country, as follows: for $i = 1, 2, ..., N$,

$$
\left(-\frac{\phi_{i1}}{e^{\phi_{D1}} - 1}, ..., -\frac{\phi_{iL}}{e^{\phi_{DL}} - 1}, 1, \sum_{l=1}^{L}\frac{\phi_{il}}{e^{\phi_{Dl}} - 1}\right).
$$

Then, this portfolio completely hedges asset i against its jump risks. Let H_i be the value of the portfolio. Then we have

$$
\frac{dH_i(t)}{H_i(t-)} = \mu_{H_i}dt + \sigma_{H_i}^\top dz(t), \tag{30}
$$

where,

$$
\mu_{H_i} = \mu_i - \sum_{l=1}^{L}\frac{\phi_{il}}{e^{\phi_{Dl}} - 1}(\mu_{TP^l} - r)
$$

$$
\sigma_{H_i}^\top = \sigma_i^\top - \sum_{l=1}^{L}\frac{\phi_{il}}{e^{\phi_{Dl}} - 1}(\sigma_{P_c^*}^\top + \sigma_{Dl}^\top).
$$

Similarly one can create N diffusion-risk assets without jumps, $H1, ..., HN$. Let $\sigma_{H_A} := (\sigma_{H1}, ..., \sigma_{HN})^\top$ and assume σ_H is nonsingular. Also let $\mu_{H_A} = (\mu_{H1}, ..., \mu_{HN})^\top$. Note that since $H_A(t)e^{-rt}$ has to be an N-dimensional vector of Q-martingales, and since $z(t) + \int_0^t \nu dt$ is an N dimensional vector of independent standard Wiener processes under Q measure, we must have $\nu = \sigma_{H_A}^{-1}(\mu_{H_A} - r\mathbf{1})$.

Given the above set of diffusion assets, one can combine this set of portfolios with each TIPS bonds to create pure jump assets. Consequently, the whole market becomes complete. For example, a pure jump asset for country-l PPP deviation jumps can be constructed as a portfolio

$$\left(-(\sigma_{P_c^*}^\top + \sigma_{D^l}^\top)\sigma_{H_A}^{-1}, 1, (\sigma_{P_c^*}^\top + \sigma_{D^l}^\top)\sigma_{H_A}^{-1}\mathbf{1}\right)$$

of the N diffusion-risk assets, the country-l TIPS bond, and the reference-country risk free asset, respectively. In words, this portfolio can be achieved with \$100 initial investment as follows: Short \$100 $\times (\sigma_{P_c^*}^\top + \sigma_{D^l}^\top)\sigma_{H_A}^{-1}$ worth of the N diffusion-risk assets, long \$100 worth of the country-l TIPS bond, and long \$100$(\sigma_{P_c^*}^\top + \sigma_{D^l}^\top)\sigma_{H_A}^{-1}\mathbf{1}$ worth of the reference-country risk free asset. The market price of this portfolio jumps only when country-l PPP deviation jumps, without being affected by diffusion risks at all. Furthermore, the price of this country-l pure jump-risk asset, $H_J^l(t)$ evolves as follows.

$$\frac{dH_J^l(t)}{H_J^l(t)} = (r - \theta_l(e^{\phi_{D^l}} - 1))dt + (e^{\phi_{D^l}} - 1)dN^l(t). \tag{31}$$

Once the above asset is constructed, one can not only use it to complete the market, but to estimate θ from its drift.

5.1.3 PPP deviation hedge portfolios

PPP deviation hedge portfolios can also be constructed as follows. Let

$$\left(1, -1, -1 + \frac{\phi_{D^l}}{e^{\phi_{D^l}} - 1}, 2 - \frac{\phi_{D^l}}{e^{\phi_{D^l}} - 1}\right)$$

be a portfolio of assets in (29), (30), and (31) and the reference-country riskfree asset, respectively. Then the market price of the portfolio, H_D^l evolves as follows:

$$\frac{dH_D^l}{H_D^l} = \mu_{H_D^l}dt + \sigma_{D^l}^\top dz(t) + \phi^l d\tilde{N}^l(t).$$

Clearly the above dynamics duplicate those of country-l PPP deviation risks.

If there were no jumps in PPP deviations, then a country-l PPP deviation hedge portfolio can be easily constructed by using a portfolio of the following three positions: long reference currency bonds for an amount of \$$\bar{H}_{D^l}(t)$, long country-$l$ TIPS bonds $\bar{H}_{X_c^l}$ for the same amount, and short the reference currency TIPS bonds $\bar{H}_{p_c^*}$ also for the same amount. Then the current value of this portfolio is \$$\bar{H}_{D^l}(t)$ and its rate-of-return dynamics are given by

$$\frac{d\bar{H}_{D^l}}{\bar{H}_{D^l}} = (r + \sigma_{D^l}^\top \nu)dt + \sigma_{D^l}^\top dz(t).$$

This portfolio perfectly hedges the country-l PPP deviation risk in the world of no PPP deviation jumps.

5.1.4   Nontraded-good-specific inflation hedge portfolios

Recall that $\sigma_{SP_c} = \sigma_{p_c^*} + \sigma_{D^l}$ and $\sigma_{SP_n} = \sigma_{p_c^*} + \left(\sigma_{p_n^l} - \sigma_{p_c^l}\right) + \sigma_{D^l}$, and that jump-rates of both V_{TP_c} and V_{TP_n} are the same. Based on this observation, consider a portfolio of the following three positions: long reference currency bonds for an amount of $\$H_n^l(t)$, long country-$l$ TIPS bonds indexed on nontraded good price for $\$H_n^l(t)$, and short country-$l$ TIPS bonds indexed on common good price for $\$H_n^l(t)$. Then the current value of this portfolio is $\$H_n^l(t)$ and its rate of return is given by

$$\frac{dH_n^l(t)}{H_n^l(t)} = \left(r + (\sigma_{p_n^l} - \sigma_{p_c^l})^\top \nu\right) dt + \left(\sigma_{p_n^l} - \sigma_{p_c^l}\right)^\top dz(t).$$

Then this portfolio perfectly hedges the country-l nontraded-good-specific inflation risk.

5.2   Forward exchange rate premium/discount

Let $F^l(t, T)$ be the country-l forward exchange rate for time T determined at time t. Then the interest rate parity implies that $F(t, T) = S(t)e^{(r - R^l)(T - t)}$. The expected (instantaneous) interest rate differential $r - R^l$ is frequently referred to as the forward exchange rate premium.

The following proposition provides no-arbitrage condition for the interest rate differential relationship between the forward exchange rate and the expected future exchange rate.

Proposition 3 *No arbitrage implies*

$$R^l - r = \sigma_{S_A}^\top \nu + (\lambda_l - \theta_l)(e^{\phi_{D^l}} - 1) - (\mu_{S_A} + \lambda_l(e^{\phi_{D^l}} - 1)). \tag{32}$$

and

$$F^l(t, T) = e^{\left\{-\sigma_{S_A}^\top \nu - (\lambda_l - \theta_l)(e^{\phi_{D^l}} - 1)\right\}(T - t)} E_t[S^l(T)]. \tag{33}$$

Proof: Consider the country-l nominal riskfree bond with a face value of one country-l currency maturing at time T. Then the time-t reference-currency price of this bond is $f(t, T) = S_A(t)e^{-R^l(T - t)}$, where R^l is the country-l nominal riskfree interest rate. Note that

$$\frac{df^l(t, T)}{f^l(t, T)} = (R^l + \mu_{S_A})dt + (\sigma_{p_c^*}^\top - \sigma_{p_c^l}^\top + \sigma_{D^l}^\top)dz(t) + (e^{\phi_{D^l}} - 1)d\tilde{N}^l(t).$$

The above asset can also be duplicated with portfolio H_f,

$$\left(1, -\sigma_{p_c^l}^\top \sigma_{H_A}^{-1}, \sigma_{p_c^l}^\top \sigma_{H_A}^{-1} \mathbf{1}\right),$$

of country-l TIPS bond, a portfolio of N diffusion-risk assets to hedge country-l common-good price (P_c^l) risk, and the reference-country nominal risk free asset, respectively. Then, we have

$$\frac{dH_f}{H_f} = \left(r + \sigma_{SP}^\top \nu - \theta_l(e^{\phi_{D^l}} - 1) - \sigma_{p_c^l}^\top \sigma_{H_A}^{-1}(\mu_{H_A} - r\mathbf{1})\right) dt + (\sigma_{p_c^*}^\top - \sigma_{p_c^l}^\top + \sigma_{D^l}^\top)dz + (e^{\phi_{D^l}} - 1)d\tilde{N}^l.$$

However, no arbitrage implies that

$$R^l - r = \sigma_{SP}^\top \nu - \mu_{S_A} - \theta_l(e^{\phi_{D^l}} - 1) - \sigma_{p_c^l}^\top \sigma_{H_A}^{-1}(\mu_{H_A} - r\mathbf{1}).$$

Moreover, since $\nu = \sigma_{H_A}^{-1}(\mu_{H_A} - r\mathbf{1})$ as noted before, we have Eq.(32).

For the second part, by Eq.(32),

$$F^l(t,T) = S^l(t)e^{\left\{-\sigma_{S_A}^\top \nu + (\mu_{S_A} + \lambda_l(e^{\phi_{D^l}} - 1)) - (\lambda_l - \theta_l)(e^{\phi_{D^l}} - 1)\right\}(T-t)}.$$

However, since $E_t[S^l(T)] = S^l(t)e^{\left\{\mu_{S_A} + \lambda_l(e^{\phi_{D^l}} - 1)\right\}(T-t)}$, we have (33). $\qquad\square$

Eq.(32) implies that the interest rate differential $R^l - r$ is the exchange rate diffusion-risk premium, $\sigma_{S_A}^\top \nu$ plus the exchange rate jump-risk premium, $(\lambda_l - \theta_l)(e^{\phi_{D^l}} - 1)$ minus the expected rate of change in the exchange rate, $(\mu_{S_A} + \lambda_l(e^{\phi_{D^l}} - 1))$. The third term of the RHS of Eq.(32), $-(\mu_{S_A} + \lambda_l(e^{\phi_{D^l}} - 1))$, is simply a correction term for the exchange rate effect: The forward exchange rate premium $r - R^l$ increases as the foreign currency is in an increasing trend against the domestic currency. For example, if the exchange rate risk is globally unsystematic with no jumps, then the forward premium is completely determined by μ_{S_A}, the expected rate of change in the exchange rate.

This second result of Proposition (3) is related to the unbiased expectations hypothesis. Although we know that the forward rate is determined simply based on the interest rate parity, the unbiased expectation hypothesis states that the forward exchange rate is the expectation of future exchange rate. In this paper, Eq.(33) suggests that the forward and expected future exchange rates are related just by no-arbitrage condition, not by the expectation. In particular, it implies that assuming that the exchange rate risk demands a positive market premium, the forward exchange rate should be smaller than the expected future exchange rate.

6   Special Cases

Now we recover existing international and domestic asset pricing models for perfect capital markets as special cases of our model. We assume no jumps in PPP deviations/exchange rates, and asset prices.

6.1   Adler and Dumas (AD)

There are no nontraded goods. Thus $\beta_{iH_n^l} = 0$ for all l, and Eq.(27) implies

$$\mu_i = r + \beta_{im}(\mu_m - r) + \beta_{iH_{p_c^*}}(\mu_{H_{p_c^*}} - r) + \sum_{l=1}^{L} \beta_{iH_{D^l}}(\mu_{H_{D^l}} - r) \tag{34}$$

Eq.(34) is an alternative expression of AD [1983, equation (16)].

6.2 Solnik/Sercu

All inflation rates are zero, $\sigma_{p_c^l} = \sigma_{p_n^l} = 0$ for all l. Then $\sigma_{S_A^l} \equiv \sigma_{D^l}$, and Eq.(27) implies that for $i = 1, 2, ..., N$,

$$\mu_i = r + \beta_{im}(\mu_m - r) + \sum_{l=1}^{L} \beta_{iH_{S_A^l}}\left(\mu_{H_{S_A^l}} - r\right) \tag{35}$$

where $\mu_{H_{S_A^l}} - r = \sigma_{S_A^l}^{\top}(\sigma_A)^{-1}(\mu_A - r\mathbf{1})$, which is the excess return in the reference currency on the portfolio hedging the exchange rate risk between the reference country and country-l. Note that without inflation rates, country-l nominal riskfree bond is an asset hedging the exchange rate risk and that Itô's formula implies the rate of return is $R^l + \mu_{S_A^l} - r$, where R^l is the rate of return in country-l currency on the country-l nominal riskfree bond. Thus, we have $\mu_{H_{S_A^l}} - r = R^l + \mu_{S_A^l} - r$. Therefore, Eq.(35) is identical to Sercu [1980, equation (11b)].

Note that Eq.(35) holds even in the presence of nontraded goods as long as all inflation rates are zeros. One may claim that Solnik/Sercu is also a special case of AD if there is a single consumption good in the world. However this claim may not be valid if there are multiple consumption goods, because with multiple consumption goods, Eq.(35) can still hold whereas Eq.(34) cannot.

6.3 A single country CAPM under inflationary Risks

Assume the PPP holds, i.e., $\sigma_{D^l} \equiv 0$ for all l. Also assume no nontraded goods. Then by Eq.(27), we have, for $i = 1, 2, ..., N$,

$$\mu_i = r + \beta_{im}(\mu_m - r) + \beta_{iH_{p_c^*}}(\mu_{H_{p_c^*}} - r) \tag{36}$$

Similar results with discrete-time models can be found in Grauer, Litzenberger and Stehle [1976], and Friend, Landskroner and Losq [1976]. The CAPMs in their discrete time models are presented as single beta models with the beta computed against the market portfolio properly adjusted for inflation. In our continuous time model, Eq.(36) indicates that the CAPM is a straightforward linear model with two betas, one for the market portfolio and the other for the inflation rate hedge portfolio.

6.4 The CAPM

Assume the PPP holds, i.e., $\sigma_{D^l} \equiv 0$ for all l. Also all inflation rates are deterministic. Then, by Eq.(27), we have, for $i = 1, 2, ..., N$,

$$\mu_i = r + \beta_{im}(\mu_m - r). \tag{37}$$

Note this relationship holds even in the presence of multiple consumption goods. Of course, this is the famous Sharpe-Linter-Mossin's CAPM.

7 Conclusion

In this paper, we have isolated PPP deviation risks in international asset pricing. We have shown that when investors consume both traded and nontraded goods, country-l investors hold three global funds and two country-l specific funds. The three global funds are the global market portfolio, reference-country inflation-risk hedge portfolio, and reference-country bond portfolio. The two country-dependent portfolios are a hedge portfolio against PPP deviation risks and another hedge portfolio against nontraded-good-specific inflation-rate risks. Consequently, country-specific demand for risk assets arises from two sources: PPP deviation-rate differential risks, and nontraded-good-specific inflation-rate differential risks. The first source is well yet indirectly recognized in the literature sometimes as exchange rate risks and sometimes as inflation risks. We believe the second source is new.

Allowing investors to consume nontraded goods not only brings about the above extra country-specific demand but real income effects with changes in traded good prices. We have shown a striking result that as the traded good price of country-l increases while the nontraded good price is held constant, the real income of the globally investing country-l investor increases, thereby increasing consumption of both traded and nontraded goods. In our framework, if nontraded goods do not exist, changes in traded good prices do not cause the real income.

We have argued that PPP deviation hedge portfolios can be created by using domestic and foreign TIPS bonds indexed on common good prices; also that nontraded-good-specific inflation hedge portfolio can be constructed by using domestic TIPS bonds indexed on common good and nontraded good prices. Another application of TIPS bonds suggests that unlike what the unbiased expectations hypothesis suggests, the forward exchange rate is related to the expected exchange rate by no arbitrage, not by the expectation.

Our model leads to the multi-beta IAPM with which existing IAPMs and domestic CAPMs are its special cases. Our multi-beta IAPM require $2(L+1)+1$ benchmark portfolios plus the reference country nominal riskfree asset in order to price all assets. They are the global market portfolio, one portfolio hedging against the reference-country inflation rates, L portfolios hedging against L PPP deviation risks, and $L+1$ portfolios hedging against $L+1$ nontraded-good-specific inflation rate risks.

We have also noted that in the presence of inflation risks, hedging against exchange rate risks in isolation can actually make the investor's real wealth riskier than no hedging at all. In order to avoid this pitfall and to protect her real wealth, the investor may need to simultaneously hedge both inflation rate and exchange rate risks, for example, by using foreign TIPS bonds.

Appendix

A   Proof of Theorem 1

Recall $\varphi_c = S_A P_c = e^D P_c^*$ and $\varphi_n = S_A P_n = e^D P_c^*(P_n/P_c)$. Let (G_c, G_n) be the inverse of $(U_{C_{R_c}}, U_{C_{R_n}})$ is invertible with respect to (C_{R_c}, C_{R_n}). Then the solution to (14) and (15) is given by

$$C_c(t) = \varphi_c(t) G_c(\varphi_c(t)\Psi(t), \varphi_n(t)\Psi(t), t), \tag{A.1}$$

$$C_n(t) = \varphi_n(t) G_n(\varphi_c(t)\Psi(t), \varphi_n(t)\Psi(t), t). \tag{A.2}$$

And the optimal consumption budget is

$$E^Q\left[\int_0^T e^{-\int_0^t r(u)du}(C_c + C_n)dt\right]$$

$$= \frac{1}{q}E\left[\int_0^T q e^{-\int_0^t r(u)du}\xi(t)(C_c + C_n)dt\right]$$

$$= \frac{1}{q}E\left[\int_0^T \Psi(t)\left\{\varphi_c(t)G_c + \varphi_n(t)G_n\right\}dt\right]$$

Thus, by the Bayes rule

$$E^Q\left[\int_t^T e^{-\int_0^s r(u)du}(C_c + C_n)ds \;\middle|\; \mathcal{F}_t\right]$$

$$= \frac{1}{q\xi(t)}E\left[\int_t^T q^l e^{-\int_0^s r(u)du}\xi(s)(C_c + C_n)ds \;\middle|\; \mathcal{F}_t\right]$$

$$= e^{-\int_0^t r(u)du}\frac{1}{\Psi(t)}E\left[\int_t^T \Psi(s)\left\{\varphi_c(s)G_c + \varphi_n(s)G_n\right\}ds \;\middle|\; \mathcal{F}_t\right]$$

$$= e^{-\int_0^t r(u)du}\frac{1}{\Psi(t)}E\left[\int_t^T \Psi(s)e^{D(s)}P_c^*(s)\left\{G_c + \frac{P_n(s)}{P_c(s)}G_n\right\}ds \;\middle|\; \mathcal{F}_t\right]$$

$$= e^{-\int_0^t r(u)du}F(\Psi(t), D(t), P_c^*(t), P_c(t), P_n(t), X(t), t) = e^{-\int_0^t r(u)du}F(t).$$

For the last equality/definition, we have utilized the joint Markovian assumption. Then $F(0) = W(0)$ and $F(T) = 0$. Now we use the standard procedure slightly modified for jumps, in order to derive an expression of optimal portfolio policy as in Eq.(A.9).

By Itô's formula,

$$e^{-\int_0^t r(u)du}F(.,t) = W(0) - \int_0^t r(s)e^{-\int_0^s r(s)ds}F(.,s)ds + \int_0^t e^{-\int_0^s r(u)du}dF$$

However, since

$$d\Psi(t) = -rdt - \nu^\top dz(t) + \sum_{l=1}^{L}\left(\frac{\theta^l(t)}{\lambda^l(t)} - 1\right)(dN^l(t) - \lambda^l(t)dt),$$

by substitution, we have

$$dF = \mu_F dt + \sigma_F dz + \sum_{l=1}^{L} \Delta F^l(t) dN^l(t),$$

where

$$\mu_F = F_t + F_\Psi \Psi \mu_\Psi + F_D \mu_D + F_{P_c^*} P_c^* \mu_{p_c^*} + F_{P_c} P_c \mu_{p_c} + F_{P_n} P_n \mu_{p_n} + F_X \mu_X + \frac{tr}{2} \mathcal{H}(F) \mathcal{M}(F),$$

$$\sigma_F = -F_\Psi \Psi \nu^\top + F_D \sigma_D^\top + F_{P_c^*} P_c^*(t) \sigma_{p_c^*}^\top + F_{P_c} P_c(t) \sigma_{p_c}^\top + F_{P_n} P_n(t) \sigma_{p_n}^\top + F_X \sigma_X^\top,$$

$$\Delta F^l(t) = F\left(\frac{\theta_l(t)}{\lambda_l(t)} \Psi(t-), D^l(t-) + \phi_l(t-), ..., t \right) - F(Z(t-), D^l(t-), ..., t-)$$

and $\mathcal{H}(F)$ and $\mathcal{M}(F)$ are the Hessian and mutual variation matrix processes with respect to continuous parts of $(\Psi, D, P_c^*, P_c, P_n, X)$ at time t. Thus

$$e^{-\int_0^t r(u)du} F(.,t) + \int_0^t e^{-\int_0^s r(u)du} (C_c + C_n) ds$$

$$= W(0) - \int_0^t e^{-\int_0^s r(u)du} \left(r(s) F - \mu_F + \sigma_F \nu - C_c - C_n - \sum_{l=1}^{L} \Delta F^l(s-) \theta_l(s) \right) ds \quad (A.3)$$

$$+ \int_0^t e^{-\int_0^s r(u)du} \sigma_F dz^* + \sum_{l=1}^{L} \int_{(0,t]} e^{-\int_0^s r(u)du} \Delta F^l(s) dM^{l*}(s). \quad (A.4)$$

We know that $e^{-\int_0^t r(u)du} F(.,t) + \int_0^t e^{-\int_0^s r(s)ds} (C_c + C_n) ds$ is a martingale under Q (because it is the conditional expectation of $\int_0^T e^{-\int_0^t r(u)du} (C_c + C_n) dt$ under Q). In fact

$$e^{-\int_0^t r(u)du} F(.,t) + \int_0^t e^{-\int_0^s r(u)du} (C_c + C_n) ds = E^Q\left[\int_0^T e^{-\int_0^s r(u)du} (C_c + C_n) ds \ \middle| \ \mathcal{F}_t \right].$$

Therefore,

$$\int_0^t e^{-\int_0^s r(u)du} \left(r(s) F - \mu_F + \sigma_F \nu - C_c - C_n - \sum_{l=1}^{L} \Delta F^l(s-) \theta_l(s) \right) ds$$

also has to be a martingale, which implies that for all $t \in [0,T]$ a.s.,

$$r(t) F - \mu_F + \sigma_F \nu - C_c - C_n - \sum_{l=1}^{L} \Delta F^l(t-) \theta_l(t) = 0. \quad (A.5)$$

Thus by Eq.'s (A.4) and (A.5), we have

$$e^{-\int_0^t r(u)du} F(.,t)$$

$$= W(0) - \int_0^t e^{-\int_0^s r(u)du} (C_c + C_n) ds$$

$$+ \int_0^t e^{-\int_0^s r(u)du} \left[-F_\Psi \Psi \nu^\top + F_D \sigma_D^\top + F_{P_c^*} P_c^*(t) \sigma_{p_c^*}^\top + F_{P_c} P_c(t) \sigma_{p_c}^\top + F_{P_n} P_n(t) \sigma_{p_n}^\top + F_X \sigma_X^\top \right] dz^*(s)$$

$$+ \sum_{l=1}^{L} \int_0^t e^{-\int_0^s r(u)du} \Delta F^l(s) dM^{l*}(s). \quad (A.6)$$

On the other hand, the self-financing budget constraint yields

$$\hat{W}(t) = W(0) + \int_0^t \hat{W}(s)\alpha_A^\top \left(\sigma_A dz^*(s) + \phi_A dM^*(s)\right) - \int_0^t e^{-\int_0^s r(u)du}(C_c(s) + C_n(s))ds. \qquad (A.7)$$

Recall that $\hat{W}(0) = W_0$. Since $\hat{W}(t) = e^{-\int_0^t r(u)du}F(.,t)$, by matching Eq.'s (A.7) and (A.6), we have

$$W(t)\alpha_A^\top \phi_A = (\Delta F^1(t), ..., \Delta F^L(t))^\top \qquad (A.8)$$

$$W(t)\alpha_A^\top \sigma_A = -F_\Psi \Psi \nu^\top + F_D \sigma_D^\top + F_{P_c^*} P_c^*(t)\sigma_{p_c^*}^\top$$

$$+ F_{P_c} P_c(t)\sigma_{p_c}^\top + F_{P_n} P_n(t)\sigma_{p_n}^\top + F_X \sigma_X^\top. \qquad (A.9)$$

Since $F > 0$ by the assumption of an interior solution, Eq.(A.9) can be rewritten as in Eq.(16).

For Part (ii), note that G_c and G_n are functions of (D, Y_c^l, Y_n^l). Since (D, Y_c^l, Y_n^l, X) are jointly Markov, one can alternatively write F as follows:

$$e^{-\int_0^t r(u)du}F(.,t) = e^{-\int_0^t r(u)du}\frac{1}{\Psi(t)}E\left[\int_t^T \Psi(s)e^{D(s)}P_c^*(s)\left\{G_c + \frac{P_n(s)}{P_c(s)}G_n\right\}ds \,\middle|\, \mathcal{F}_t\right]$$

$$= e^{-\int_0^t r(u)du}\frac{1}{\Psi(t)}E\left[\int_t^T \left\{e^{D(s)}Y_c(s)G_c + e^{D(s)}Y_n(s)G_n\right\}ds \,\middle|\, D(t), Y_c(t), Y_n(t), X(t)\right]$$

$$= e^{-\int_0^t r(u)du}\frac{1}{\Psi(t)}H(D(t), Y_c(t), Y_n(t), X(t), t).$$

Thus we must have

$$F(\Psi(t), D(t), P_c^*(t), P_c(t), P_n(t), X(t), t) \equiv \frac{1}{\Psi}H(D(t), Y_c(t), Y_n(t), X(t), t).$$

Therefore,

$$F_\Psi = -\frac{1}{\Psi^2}H + \frac{1}{\Psi}H_{Y_c}P_c^* + \frac{1}{\Psi}H_{Y_n}\frac{P_c^* P_n}{P_c}$$

$$F_{P_c^*} = H_{Y_c} + H_{Y_n}\frac{P_n}{P_c}$$

$$F_{P_c} = -H_{Y_n}\frac{P_c^* P_n}{P_c^2}$$

$$F_{P_n} = H_{Y_n}\frac{P_c^*}{P_c}.$$

Thus,

$$F_\Psi \Psi = -F + F_{P_c^*}P_c^*$$

and

$$P_c F_{P_c} + P_n F_{P_n} = 0.$$

Therefore, (A.9) becomes

$$W(t)\alpha_A^\top \sigma_A = -F_\Psi \Psi \nu^\top + (F + F_\Psi \Psi)\sigma_{p_c^*}^\top + F_D \sigma_D^\top + F_{P_n} P_n(t)(\sigma_{p_n}^\top - \sigma_{p_c}^\top) + F_X \sigma_X^\top. \qquad (A.10)$$

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Thus we have Eq.(17).

To prove Part (iii), let $\bar{Y}_c^l(t) := e^{D(t)}Y_c^l(t)$ and $\bar{Y}_n^l(t) := e^{D(t)}Y_n^l(t)$. When volatilities, drifts and jump rates of traded and nontraded prices, exchange rates and asset prices are all constant over time, $(\bar{Y}_c^l, \bar{Y}_n^l)$ are Markov. Thus since G_c and G_n are functions of $(\bar{Y}_c^l, \bar{Y}_n^l)$, F can be rewritten as follows:

$$e^{-rt}F(\Psi(t), D(t), P_c^*(t), P_c(t), P_n(t), t)$$
$$= e^{-rt}\frac{1}{\Psi(t)}E\left[\int_t^T \left\{\bar{Y}_c(s)G_c + \bar{Y}_n(s)G_n\right\} ds \,\middle|\, \bar{Y}_c(t), \bar{Y}_n(t)\right]$$

Define

$$\bar{H}(\bar{Y}_c^l(t), \bar{Y}_n^l(t), t) := E\left[\int_t^T \left\{\bar{Y}_c^l(s)G_c + \bar{Y}_n^l(s)G_n\right\} ds \,\middle|\, \bar{Y}_c^l(t), \bar{Y}_n^l(t)\right].$$

Then we must have

$$F(\Psi(t), D(t), P_c(t), P_n(t), t) \equiv \frac{1}{\Psi(t)}\bar{H}(\bar{Y}_c(t), \bar{Y}_n(t), t).$$

Therefore,

$$F_\Psi = -\frac{1}{\Psi^2}\bar{H} + \frac{1}{\Psi}\bar{H}_{\bar{Y}_c}e^D P_c^* + \frac{1}{\Psi}\bar{H}_{\bar{Y}_n}\frac{e^D P_c^* P_n}{P_c}$$

$$F_{P_c^*} = \bar{H}_{\bar{Y}_c}e^D + \bar{H}_{Y_n}\frac{e^D P_n}{P_c}$$

$$F_D = \bar{H}_{\bar{Y}_c}e^D P_c^* + \bar{H}_{Y_n}\frac{e^D P_c^* P_n}{P_c} = P_c^* F_{P_c^*}$$

$$F_{P_c} = -\bar{H}_{Y_n}\frac{e^D P_c^* P_n}{P_c^2}$$

$$F_{P_n} = \bar{H}_{Y_n}\frac{e^D P_c^*}{P_c}.$$

Thus,

$$F_\Psi\Psi = -F + F_{P_c^*}P_c^*$$

$$0 = P_c F_{P_c} + P_n F_{P_n}$$

$$F_D = F_{P_c^*}P_c^*,$$

and (A.9) becomes

$$W(t)\alpha_A^\top\sigma_A = -F_\Psi\Psi\nu^\top + (F + F_\Psi\Psi)(\sigma_{p_c^*}^\top + \sigma_D^\top) + F_{P_n}P_n(t)(\sigma_{p_n}^\top - \sigma_{p_c}^\top).$$

Therefore, we have Eq.(18). $\square$

B   Proof of Proposition 1

Recall

$$C_{R_c}(t) = \frac{C_c(t)}{e^{D(t)}P_c^*(t)}, \quad \text{and} \quad C_{R_n}(t) = \frac{C_n(t)}{e^{D(t)}(P_c^*(t)/P_c(t))P_n(t)}.$$

Let
$$\bar{Y}_c(t) := \frac{e^{D(t)} P_c^*(t)}{Z(t)}, \quad \text{and} \quad \bar{Y}_n(t) := \frac{e^{D(t)} (P_c^*(t)/P_c(t)) P_n(t)}{Z(t)}.$$

Then the FOCs are

$$\kappa a_c \left(C_{R_c}\right)^{a_c-1} \left(C_{R_n}\right)^{a_n} = \bar{Y}_c,$$

$$\kappa a_n \left(C_{R_c}\right)^{a_c} \left(C_{R_n}\right)^{a_n-1} = \bar{Y}_n.$$

Thus the above two imply that

$$C_{R_c} = \left(\frac{a_c}{a_n}\right) C_{R_n} \frac{\bar{Y}_n}{\bar{Y}_c}.$$

By substituting the above back into the FOCs, we have

$$C_{R_c} = (\kappa a_c)^{\frac{1}{1-a}} \left(\frac{a_n}{a_c}\right)^{\frac{a_n}{1-a}} \bar{Y}_c^{\frac{a_n-1}{1-a}} \bar{Y}_n^{\frac{-a_n}{1-a}},$$

$$C_{R_n} = (\kappa a_n)^{\frac{1}{1-a}} \left(\frac{a_c}{a_n}\right)^{\frac{a_c}{1-a}} \bar{Y}_c^{\frac{-a_c}{1-a}} \bar{Y}_n^{\frac{a_c-1}{1-a}}.$$

Note that both C_{R_c} and C_{R_n} are lognormally distributed with jumps because P_c^*, P_c^l, and P_n^l are lognormally distributed, and Z and e^D are lognormally distributed with jumps.[11] Define $\eta_c(t) = \bar{Y}_c(t)C_{R_c}(t)$ and $\eta_n(t) = \bar{Y}_n(t)C_{R_n}(t)$. Then

$$\eta_c(t) = (\kappa a_c)^{\frac{1}{1-a}} \left(\frac{a_n}{a_c}\right)^{\frac{a_n}{1-a}} \left(\bar{Y}_c(t)\right)^{\frac{-a_c}{1-a}} \left(\bar{Y}_n(t)\right)^{\frac{-a_n}{1-a}},$$

$$\eta_n(t) = (\kappa a_n)^{\frac{1}{1-a}} \left(\frac{a_c}{a_n}\right)^{\frac{a_c}{1-a}} \left(\bar{Y}_c(t)\right)^{\frac{-a_c}{1-a}} \left(\bar{Y}_n(t)\right)^{\frac{-a_n}{1-a}}.$$

Since $\bar{Y}_c$ and $\bar{Y}_n$ are lognormally distributed with jumps, so are η_c and η_n. Thus we can express the dynamics of $\eta_c(t)$ and $\eta_n(t)$ as in the following forms:

$$\frac{d\eta_c(t)}{\eta_c(t-)} = \mu_{\eta_c} dt + \sigma_{\eta_c}^\top dz(t) + \phi_{\eta_c}^\top dN(t)$$

$$\frac{d\eta_n(t)}{\eta_n(t-)} = \mu_{\eta_n} dt + \sigma_{\eta_n}^\top dz(t) + \phi_{\eta_n}^\top dN(t).$$

---

[11] Notes on some basics: Suppose

$$\frac{dA(t)}{A(t-)} = \mu dt + \sigma^\top dz(t) + \sum_{l=1}^{L} \phi_l dN_l(t),$$

where μ, σ, and ϕ are all constant. Then, the solution is

$$A(t) = A(0) e^{\left(\mu + \frac{1}{2}\sigma^\top \sigma\right) t + \sigma^\top Z(t) + \sum_{l=1}^{L} \ln(1+\phi_l) N_l(t)}$$

and

$$E[A(T) \mid A(t)] = A(t) e^{\left(\mu + \sum_{l=1}^{L} \lambda_l \phi_l\right)(T-t)}.$$

where μ's, σ's and ϕ's for η_c and η_n are all constant. In fact, since $\eta_c(t) = (a_c/a_n)\eta_n(t)$, the dynamics of both η_c and η_n are identical to each other except their initial values. That is, $\mu_{\eta_c} = \mu_{\eta_n}$, $\sigma_{\eta_c} = \sigma_{\eta_n}$ and $\phi_{\eta_c} = \phi_{\eta_n}$. Therefore,

$$
\begin{aligned}
K_c &:= E_t\left[\int_t^T \eta_c(s)ds\right] = E\left[\int_t^T \eta_c(s)ds \,\middle|\, \bar{Y}_c(t), \bar{Y}_n(t)\right] \\
&= \eta_c(t)E_t\left[\int_t^T \frac{\eta_c(s)}{\eta_c(t)}ds\right] \\
&= \frac{1}{\mu_{\eta_c} + \phi_{\eta_c}^\top \lambda}\eta_c(t)\left(e^{(\mu_{\eta_c}+\phi_{\eta_c}^\top\lambda)(T-t)} - 1\right), \\
K_n &:= E_t\left[\int_t^T \eta_n(s)ds\right] = \frac{a_n}{a_c}K_c.
\end{aligned}
$$

If μ_{η_c} is zero, then K_c and K_n become the limits of the above quantities as μ_{η_c} approaches zero.

Recall $F(t, .) = Z(t)(K_c + K_n)$, i.e., the current level of wealth is the same as the present value of future consumption. Since K_c and K_n are positive almost surely for $t < T$, we have $F > 0$ almost surely. However,

$$(K_c)_{Y_c} = \frac{a_c}{a-1}\frac{1}{\mu_{\eta_c}+\phi_{\eta_c}^\top\lambda}(\kappa a_c)^{\frac{1}{1-a}}\left(\frac{a_c}{a_n}\right)^{\frac{a_n}{a-1}}(\bar{Y}_c)^{\frac{1-a_n}{a-1}}(\bar{Y}_n)^{\frac{a_n}{a-1}}(e^{(\mu_{\eta_c}+\phi_{\eta_c}^\top\lambda)(T-t)} - 1) < 0.$$

Since

$$(K_c)_{\bar{Y}_c} = \frac{a_c}{a-1}\frac{K_c}{\bar{Y}_c}, \qquad (K_n)_{\bar{Y}_c} = \frac{a_n}{a_c}(K_c)_{\bar{Y}_c},$$

$$(K_c)_{\bar{Y}_n} = \frac{a_n}{a-1}\frac{K_c}{\bar{Y}_n}, \quad \text{and} \quad (K_n)_{\bar{Y}_n} = \frac{a_n}{a_c}(K_c)_{\bar{Y}_n},$$

we have $(K_n)_{\bar{Y}_c}$, $(K_c)_{\bar{Y}_n}$, $(K_n)_{\bar{Y}_n} < 0$. Also since $F = Z(K_c + K_n) = ZK_c\frac{a}{a_c}$,

$$
\begin{aligned}
\frac{F_{P_n}P_n}{F} &= \frac{P_n}{F}Z\frac{a}{a_c}(K_c)_{\bar{Y}_n}\frac{\partial\bar{Y}_n}{\partial P_n} = -\frac{a_n}{1-a} < 0 \\
\frac{F_{P_c}P_c}{F} &= \frac{P_c}{F}Z\frac{a}{a_c}(K_c)_{\bar{Y}_n}\frac{\partial\bar{Y}_n}{\partial P_c} = \frac{a_n}{1-a} > 0 \\
\frac{F_{P_c^*}P_c^*}{F} &= \frac{P_c^*}{F}Z\frac{a}{a_c}\left((K_c)_{\bar{Y}_c}\frac{\partial\bar{Y}_c}{\partial P_c^*} + (K_c)_{\bar{Y}_n}\frac{\partial\bar{Y}_n}{\partial P_c^*}\right) = -\frac{a}{1-a} < 0 \\
\gamma = \frac{ZF_Z}{F} &= \frac{Z}{F}\left\{\frac{a}{a_c}K_c + \frac{a}{a_c}Z\left((K_c)_{\bar{Y}_c}\frac{\partial\bar{Y}_c}{\partial Z} + (K_c)_{\bar{Y}_n}\frac{\partial\bar{Y}_n}{\partial Z}\right)\right\} = \frac{1}{1-a} > 1 \\
\frac{F_D}{F} &= \frac{Z}{F}\frac{a}{a_c}\left((K_c)_{\bar{Y}_c}\frac{\partial\bar{Y}_c}{\partial D} + (K_c)_{\bar{Y}_n}\frac{\partial\bar{Y}_n}{\partial D}\right) = -\frac{a}{1-a} < 0.
\end{aligned}
$$

Next, to find the value function, note that

$$U(., t) = \kappa\left(\frac{a_n}{a_c}\right)^{a_n}\left(\frac{P_c(t)}{P_n(t)}\right)^{a_n}C_{R_c}^a(t).$$

Thus, optimal U is lognormally distributed with jumps. Let μ_v and ϕ_v be the drift rate and the L-dimensional jump rate vector for the dynamics of the optimal $U(.,t)$. Then, since μ_v and ϕ_v are constant over time, the value function V is

$$V(t) = E\left[\int_t^T U(.,s)ds \,\middle|\, U(.,t)\right] = \frac{1}{\mu_v + \phi_v^\top \lambda} U(.,t)(e^{(\mu_v + \phi_v^\top \lambda)(T-t)} - 1).$$

Therefore, by substitutions, the value function is computed to be (22). The above value function is well defined for $\mu_v + \phi_v^\top \lambda \neq 0$. If $\mu_v + \phi_v^\top \lambda = 0$, then the value function can be computed as the limit of the above quantity as $\mu_v + \phi_v^\top \lambda$ approaches zero. This completes the proof. $\qquad \square$

C Proof of Proposition 2

Rewriting Eq.(16), we have

$$\gamma^l(t)(\mu_A - r\mathbf{1} + \phi_A\lambda) = (\sigma_A\sigma_A^\top)\alpha_A^l(t) + \gamma^l(t)\phi_A(\lambda - \theta) - \delta_{p_c^*}\sigma_A\sigma_{p_c^*}$$
$$-\delta_{D^l}(t)\sigma_A\sigma_{D^l} - \delta_{p_c^l}(t)\sigma_A\sigma_{p_c^l} - \delta_{p_n^l}(t)\sigma_A\sigma_{p_n^l} - \sigma_A\sigma_X\delta_X^l.$$

Multiply both sides by W^l, sum over l, and divide both sides by $\sum_l^{L+1} W^l$.

$$\gamma^m(t)(\mu_A - r\mathbf{1} + \phi_A\lambda)$$
$$= (\sigma_A\sigma_A^\top)\sum_l^{L+1} \frac{\alpha_A^l W^l}{\sum_l^{L+1} W^l} + \gamma^m(t)\phi_A(\lambda - \theta) - \delta_{p_c^*}^m\sigma_A\sigma_{p_c^*} - \sigma_A\sigma_X\delta_X^m$$
$$-\sigma_A\sum_l^{L+1}\frac{W^l}{\sum_l^{L+1} W^l}\delta_{D^l}\sigma_{D^l} - \sigma_A\sum_l^{L+1}\frac{W^l}{\sum_l^{L+1} W^l}\delta_{p_c^l}\sigma_{p_c^l} - \sigma_A\sum_l^{L+1}\frac{W^l}{\sum_l^{L+1} W^l}\delta_{p_n^l}\sigma_{p_n^l},$$

where $\delta_X^m = (\delta_{X_1}^m, ..., \delta_{X_K}^m)^\top$, and for $k = 1, ..., K$,

$$\delta_{X_k}^m = \sum_l^{L+1}\frac{W^l\delta_{X_k}^l}{\sum_l^{L+1} W^l}.$$

Note that
$$\sum_l^{L+1}\frac{W^l\alpha_A^l}{\sum_l^{L+1} W^l}$$
is the global market portfolio of risky assets. Thus

$$\mu_A - r\mathbf{1} + \phi_A\lambda = \frac{1}{\gamma^m(t)}\sigma_{Am} + \phi_A(\lambda - \theta) - \frac{\delta_{p_c^*}^m}{\gamma^m(t)}\sigma_A\sigma_{p_c^*} - \frac{1}{\gamma^m(t)}\sigma_A\sigma_X\delta_X^m$$
$$-\sum_l^L\frac{\hat{\delta}_{D^l}}{\gamma^m(t)}\sigma_A\sigma_{D^l} - \sum_l^{L+1}\frac{\hat{\delta}_c^l}{\gamma^m(t)}\sigma_A\sigma_{p_c^l} - \sum_l^{L+1}\frac{\hat{\delta}_n^l}{\gamma^m(t)}\sigma_A\sigma_{p_n^l}. \qquad (\text{A.11})$$

where

$$\hat{\delta}_{D^l} = \frac{W^l\delta_{D^l}}{\sum_l^{L+1} W^l}, \quad \hat{\delta}_c^l = \frac{W^l\delta_{p_c^l}}{\sum_l^{L+1} W^l}, \quad \hat{\delta}_n^l = \frac{W^l\delta_{p_n^l}}{\sum_l^{L+1} W^l}.$$

The above equation suggests that the expected instantaneous excess rate of return on the market portfolio is

$$\mu_m - r + \phi_m^\top \lambda = \frac{1}{\gamma^m} \sigma_{mm} + \phi_m^\top (\lambda - \theta) - \frac{\delta_{p_c^*}^m}{\gamma^m(t)} \sigma_{mp_c^*} - \frac{1}{\gamma^m(t)} \sigma_{mX} \delta_X^m$$

$$- \sum_j^L \frac{\hat{\delta}_{D^j}}{\gamma^m} \sigma_{mD^j} - \sum_j^{L+1} \frac{\hat{\delta}_c^j}{\gamma^m} \sigma_{mp_c^j} - \sum_j^{L+1} \frac{\hat{\delta}_n^j}{\gamma^m} \sigma_{mp_n^j}$$

where for the market portfolio X_m,

$$\phi_m^\top = X_m^\top \phi_A.$$

On the other hand, note that the expected excess instantaneous rate of return on the reference-country common-good-price hedge portfolio $H_{p_c^*}$ is $\mu_{H_{p_c^*}} - r + \phi_{H_{p_c^*}}^\top \lambda = \sigma_{p_c^*}^\top (\sigma_A)^{-1} (\mu_A - r\mathbf{1} + \phi_A \lambda)$. However, by Eq.(A.11), we also have

$$\sigma_{p_c^*}^\top (\sigma_A)^{-1} (\mu_A - r\mathbf{1} + \phi_A \lambda) = \frac{1}{\gamma^m} \sigma_{p_c^* m} + \phi_{H_{p_c^*}}^\top (\lambda - \theta) - \frac{\delta_{p_c^*}^m}{\gamma^m(t)} \sigma_{p_c^* p_c^*} - \frac{1}{\gamma^m(t)} \sigma_{p_c^* X} \delta_X^m$$

$$- \sum_l^L \frac{\hat{\delta}_{D^l}}{\gamma^m} \sigma_{p_c^* D^l} - \sum_l^{L+1} \frac{\hat{\delta}_c^l}{\gamma^m} \sigma_{p_c^* p_c^l} - \sum_l^{L+1} \frac{\hat{\delta}_n^l}{\gamma^m} \sigma_{p_c^* p_n^l}.$$

Similarly, $\mu_{H_{D^l}} - r + \phi_{H_{D^l}}^\top \lambda = \sigma_{D^l}^\top (\sigma_A)^{-1} (\mu_A - r\mathbf{1} + \phi_A \lambda)$, and by Eq.(A.11), for $l = 1, 2, ..., L+1$,

$$\sigma_{D^l}^\top (\sigma_A)^{-1} (\mu_A - r\mathbf{1} + \phi_A \lambda) = \frac{1}{\gamma^m} \sigma_{D^l m} + \phi_{H_{D^l}}^\top (\lambda - \theta) - \frac{\delta_{p_c^*}^m}{\gamma^m(t)} \sigma_{D^l p_c^*} - \frac{1}{\gamma^m(t)} \sigma_{D^l X} \delta_X^m$$

$$- \sum_l^L \frac{\hat{\delta}_{D^l}}{\gamma^m} \sigma_{D^l D^l} - \sum_l^{L+1} \frac{\hat{\delta}_c^l}{\gamma^m} \sigma_{D^l p_c^l} - \sum_l^{L+1} \frac{\hat{\delta}_n^l}{\gamma^m} \sigma_{D^l p_n^l}.$$

Also the expected instantaneous excess rates of return on hedge portfolios for country-l traded and nontraded-good proces and all other state variables are, respectively, $\mu_{H_c^l} - r + \phi_{H_c^l} \lambda = \sigma_{p_c^l}^\top (\sigma_A)^{-1} (\mu_A - r\mathbf{1} + \phi_A \lambda)$, $\mu_{H_n^l} - r + \phi_{H_n^l} \lambda = \sigma_{p_n^l}^\top (\sigma_A)^{-1} (\mu_A - r\mathbf{1} + \phi_A \lambda)$, and $\mu_{H_X} - r + \phi_{H_X} \lambda = \sigma_X^\top (\sigma_A)^{-1} (\mu_A - r\mathbf{1} + \phi_A \lambda)$. Consequently, by Eq.(A.11), for $l = 1, 2, ..., L+1$,

$$\sigma_{p_c^l}^\top (\sigma_A)^{-1} (\mu_A - r\mathbf{1} + \phi_A \lambda) = \frac{1}{\gamma^m} \sigma_{p_c^l m} + \phi_{H_{p_c^l}}^\top (\lambda - \theta) - \frac{\delta_{p_c^*}^m}{\gamma^m(t)} \sigma_{p_c^l p_c^*} - \frac{1}{\gamma^m(t)} \sigma_{p_c^l X} \delta_X^m$$

$$- \sum_l^L \frac{\hat{\delta}_{D^l}}{\gamma^m} \sigma_{p_c^l D^l} - \sum_l^{L+1} \frac{\hat{\delta}_c^l}{\gamma^m} \sigma_{p_c^l p_c^l} - \sum_l^{L+1} \frac{\hat{\delta}_n^l}{\gamma^m} \sigma_{p_c^l p_n^l}$$

$$\sigma_{p_n^l}^\top (\sigma_A)^{-1} (\mu_A - r\mathbf{1} + \phi_A \lambda) = \frac{1}{\gamma^m} \sigma_{p_n^l m} + \phi_{H_{p_n^l}}^\top (\lambda - \theta) - \frac{\delta_{p_c^*}^m}{\gamma^m(t)} \sigma_{p_n^l p_c^*} - \frac{1}{\gamma^m(t)} \sigma_{p_n^l X} \delta_X^m$$

$$- \sum_l^L \frac{\hat{\delta}_{D^l}}{\gamma^m} \sigma_{p_n^l D^l} - \sum_l^{L+1} \frac{\hat{\delta}_c^l}{\gamma^m} \sigma_{p_n^l p_c^l} - \sum_l^{L+1} \frac{\hat{\delta}_n^l}{\gamma^m} \sigma_{p_n^l p_n^l}$$

$$\sigma_X^\top (\sigma_A)^{-1} (\mu_A - r\mathbf{1} + \phi_A \lambda) = \frac{1}{\gamma^m} \sigma_{Xm} + \phi_{H_X}^\top (\lambda - \theta) - \frac{\delta_{p_c^*}^m}{\gamma^m(t)} \sigma_{Xp_c^*} - \frac{1}{\gamma^m(t)} \sigma_{XX} \delta_X^m$$

$$- \sum_l^L \frac{\hat{\delta}_{D^l}}{\gamma^m} \sigma_{XD^l} - \sum_l^{L+1} \frac{\hat{\delta}_c^l}{\gamma^m} \sigma_{Xp_c^l} - \sum_l^{L+1} \frac{\hat{\delta}_n^l}{\gamma^m} \sigma_{Xp_n^l}.$$

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Therefore we have

$$
\begin{bmatrix}
\mu_m - r + \phi_m^\top \theta \\
\mu_{H_{p_c^*}} - r + \phi_{p_c^*}^\top \theta \\
\mu_{H_{D^l}} - r + \phi_{H_D^l}^\top \theta \\
.... \\
\mu_{H_n^{L+1}} - r + \phi_{H_n^{L+1}}^\top \theta \\
\mu_{H_X} - r + \phi_{H_X}^\top \theta
\end{bmatrix}
= \Gamma \times
\begin{bmatrix}
1/\gamma^m \\
-\delta_{p_n^*}^m/\gamma^m \\
-\hat{\delta}_{D^1}^m/\gamma^m \\
.... \\
-\hat{\delta}_{p_n^{L+1}}/\gamma^m \\
-(1/\gamma^m)\delta_X^m
\end{bmatrix}.
\tag{A.12}
$$

But

$$
\begin{aligned}
\mu_A - r\mathbf{1} + \phi_A\theta &= \frac{1}{\gamma^m(t)}\sigma_{Am} - \frac{\delta_{p_c^*}^m}{\gamma^m(t)}\sigma_A\sigma_{p_c^*} - \sum_l^L \frac{\hat{\delta}_{D^l}}{\gamma^m(t)}\sigma_A\sigma_{D^l} \\
&\quad - \sum_l^{L+1} \frac{\hat{\delta}_c^l}{\gamma^m(t)}\sigma_A\sigma_{p_c^l} - \sum_l^{L+1} \frac{\hat{\delta}_n^l}{\gamma^m(t)}\sigma_A\sigma_{p_n^l} - \frac{1}{\gamma^m(t)}\sigma_A\sigma_X\delta_X^m \\
&=
\begin{bmatrix}
\sigma_{Am}^\top \\
\sigma_{p_c^*}^\top \sigma_A^\top \\
\sigma_{D^l}^\top \sigma_A^\top \\
.... \\
\sigma_{p_n^{L+1}}^\top \sigma_A^\top \\
\sigma_X^\top \sigma_A^\top
\end{bmatrix}^\top
\times
\begin{bmatrix}
1/\gamma^m \\
-\delta_{p_n^*}^m/\gamma^m \\
-\hat{\delta}_{D^1}^m/\gamma^m \\
.... \\
-\hat{\delta}_{p_n^{L+1}}/\gamma^m \\
-(1/\gamma^m)\delta_X^m
\end{bmatrix} \\
&=
\begin{bmatrix}
\sigma_{Am}^\top \\
\sigma_{p_c^*}^\top \sigma_A^\top \\
\sigma_{D^l}^\top \sigma_A^\top \\
.... \\
\sigma_{p_n^{L+1}}^\top \sigma_A^\top \\
\sigma_X^\top \sigma_A^\top
\end{bmatrix}^\top
\times \Gamma^{-1} \times \Gamma \times
\begin{bmatrix}
1/\gamma^m \\
-\delta_{p_n^*}^m/\gamma^m \\
-\hat{\delta}_{D^1}^m/\gamma^m \\
.... \\
-\hat{\delta}_{p_n^{L+1}}/\gamma^m \\
-(1/\gamma^m)\delta_X^m
\end{bmatrix} \\
&=
\begin{bmatrix} \beta_{Am}, \beta_{AP_c^*}, \beta_{AD}, \beta_{AH_c}, \beta_{AH_n}, \beta_{AH_X} \end{bmatrix}
\begin{bmatrix}
\mu_m - r + \phi_m^\top \theta \\
\mu_{H_{p_c^*}} - r + \phi_{p_c^*}^\top \theta \\
\mu_{H_{D^l}} - r + \phi_{H_D^l}^\top \theta \\
.... \\
\mu_{H_n^{L+1}} - r + \phi_{H_n^{L+1}}^\top \theta \\
\mu_{H_X} - r + \phi_{H_X}^\top \theta
\end{bmatrix}
\end{aligned}
$$

where β_{Am} and $\beta_{AP_c^*}$ are N-dimensional vectors, and β_{AH_D}, β_{AH_c}, β_{AH_n} and β_{AX} are, respectively, $N \times L$, $N \times L$, $N \times (L+1)$ and $N \times K$ matrices. Therefore, we have the multi-beta linear relationship as stated in Eq.(26). □

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