Does the More Risk-averse Investor hold a Less Risky Portfolio?*

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Abstract

We study the suitability of using absolute risk aversion as a measure of willingness to take risk in the Arrow-Debreu portfolio framework. A global measure of risk for Arrow-Debreu portfolios is introduced. This measure is termed ‘conservatism’. We show that the concept of ‘more conservative’ is stronger than that of ‘more risk-averse’. A higher absolute risk aversion is only necessary but not sufficient to induce a less risky Arrow-Debreu portfolio. Our results challenge the well-accepted notion that a more risk-averse investor holds a less risky portfolio.

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Abstract

We study the suitability of using absolute risk aversion as a measure of willingness to take risk in the Arrow-Debreu portfolio framework. A global measure of risk for Arrow-Debreu portfolios is introduced. This measure is termed ‘conservatism’. We show that the concept of ‘more conservative’ is stronger than that of ‘more risk-averse’. A higher absolute risk aversion is only necessary but not sufficient to induce a less risky Arrow-Debreu portfolio. Our results challenge the well-accepted notion that a more risk-averse investor holds a less risky portfolio.

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“In the standard portfolio problem, an increase in risk aversion always reduces the optimal risk exposure, where the risk exposure is measured by the amount invested in the risky asset. In the complete markets model, a global measurement of the exposure to risk is less obvious.”

Gollier [2001]

1 Introduction

The literature on portfolio theory has predominantly focused on portfolio choice with two assets, namely, the risk-free asset and the market portfolio. Since investors only hold the risk-free asset and the market portfolio, the relationship between the portfolio payoff and the market return is always linear. We refer to this type of portfolio problem as ‘the linear portfolio problem’ and the portfolios, whose values are perfectly correlated to market return, as ‘linear portfolios’.¹

Consider the one period linear portfolio problem. Let $\alpha^*_i$ be the optimal amount of market portfolio allocation of an investor with utility $u_i$ who is endowed with initial wealth $w_0$. At the end of the period, the payoff of this portfolio $W^*_i(\tilde{r})$ can be written as,

$$W^*_i(\tilde{r}) = (w_0 - \alpha^*_i)r_f + \alpha^*_i\tilde{r},$$

where

- $\tilde{r}$ = the gross rate of return on the market portfolio,
- $r_f$ = the gross risk-free rate.

Equation (1) indicates that the portfolio payoff $W^*_i(\tilde{r})$ is always linear in the market return $\tilde{r}$ when investors are allowed to hold only the risk-free asset and the market portfolio.

In practice, however, investors are not precluded from purchasing assets whose values are not perfectly correlated with the market return. These assets include all state-contingent claims on the market return such as options and other options embedded securities. If the market is perfect and complete, the investor can choose any portfolio payoff whose value is not necessarily linear in the market return. We refer to this broader set of portfolios as ‘complete portfolios’ and the corresponding portfolio problem as ‘the complete portfolio problem’.²

Investors are different in terms of their attitude towards risk. The term ‘more risk-averse’ is used to compare investors’ attitude towards risk. An investor with utility $u_1$ is said to be more risk-averse than an investor with utility $u_2$, if $u_1$
rejects all risks that \( u_2 \) rejects.\(^3\) Pratt [1964] shows that \( u_1 \) is more risk-averse than \( u_2 \), if and only if, the absolute risk aversion of \( u_1 \) is greater than that of \( u_2 \).\(^4\) Thus, unless otherwise stated, when we say ‘\( a \) is more risk-averse than \( b \)’, this implies that \( a \) has a higher absolute risk aversion than \( b \).

The term ‘more risk-averse’ is also used to compare the riskiness of linear portfolios among different investors. Consider the linear portfolio in equation (1). We say that portfolio \( W^{*2}(\tilde{\tau}) \) is more risky than portfolio \( W^{*1}(\tilde{\tau}) \), if the optimal risky allocation of \( W^{*2}(\tilde{\tau}) \) is greater than that of \( W^{*1}(\tilde{\tau}) \), i.e. \( \alpha^*_2 \geq \alpha^*_1 \). By Pratt [1964], \( \alpha^*_2 \geq \alpha^*_1 \), if and only if, \( u_1 \) is more risk-averse than \( u_2 \). Thus, in the linear portfolio problem, the size of the absolute risk aversion (or equivalently the size of the optimal risky allocation \( \alpha^*_i \)) measures the risk of the portfolio.

Since the seminal paper by Pratt [1964], the effect of absolute risk aversion on willingness to take risk becomes obfuscated by generality. A more risk-averse investor is expected to hold a less risky portfolio. The aim of this paper is to examine this well-accepted generality by considering the choice of complete portfolios whose values are not necessarily perfectly correlated with risk. The main research question that we attempt to answer is: Does a more risk-averse investor in the sense of Arrow-Pratt always hold a less risky complete portfolio? We believe such an attempt is important, because without it, one cannot decide whether it is appropriate to apply the concept of risk aversion to the complete portfolio problems.

As suggested by Gollier [2001], the global measure of the risk of the portfolio is not as straightforward in the complete portfolio problem.\(^5\) To date there has been no single answer to this question, although progress has been made through the work by Gollier [2001] and Franke, Stapleton and Subrahmanyam [2004].\(^6\) Thus, an alternative measure of risk for complete portfolios is indispensable to our analysis. We define a global measure of risk for the complete portfolio - ‘conservatism’, which is measured by the sensitivity of portfolio payoff to the market return. ‘Conservatism’ is shown to be inversely proportional to the slope of the payoff function. Comparing two investors, one investor is defined to be ‘more conservative’ than another, if the payoff function of the former is

\[^3\]For simplicity, throughout this paper, we often use the shorthand notations ‘investor \( u_i \)’ or ‘\( u_i \)’ to represent ‘investor with utility \( u_i \)’.

\[^4\]The absolute risk aversion is defined as the negative of the ratio of the second to the first derivative of the utility function.

\[^5\]Unlike the linear case, the optimal allocation does not represent the measure of risk of the optimal portfolio. It is also not obvious how absolute risk aversion can be a measure of risk for a complete portfolio. One obvious choice would be to use variance as a measure of risk. However, unlike the linear case, the computation of variance in the complete setting requires some tedious numerical methods which make the analysis non-tractible.

\[^6\]Gollier [2001] uses the first local derivative of the payoff function around the ‘crossover point’ as a measure of risk for the complete portfolio. The derivative is taken with respect to the pricing density and the ‘crossover point’ is the point at which, one optimal payoff function crosses the other. Franke, Stapleton and Subrahmanyam [2004] examine the impact of a simple increase in background risk on the first derivative of the payoff function, where the derivative is taken with respect to the pricing density. They introduce a concept called ‘generalized risk aversion’. An investor is generalized risk-averse if the first derivative of the payoff function increases globally, given an increase in background risk. They show that the concept of generalized risk aversion is equivalent to standard risk aversion.
everywhere flatter than that of the latter.

With the concept of conservatism, our research question is reduced to: does the concept of more risk-averse imply more conservative? Our results suggest that, in general, a higher absolute risk aversion is only necessary but not sufficient to induce a less risky complete portfolio. Our results not only challenge the generality of using absolute risk aversion as a measure of willingness to take risk, but also suggest a stronger measurement - conservatism - to evaluate the riskiness of the portfolio.

The remainder of the paper is organized as follows. Section 2 develops the portfolio model. Section 3 defines the concept of conservatism and shows that it represents the measure of risk for the complete portfolio. Section 4 examines whether a more risk-averse investor always holds a less risky complete portfolio, or equivalently, whether more risk-averse implies more conservative. Section 5 concludes.

2 The Model

Consider a one period economy where the dates are indexed 0 and $\tau$. Let $\mathcal{R}$ be the set of all possible gross returns on the market portfolio at time $\tau$. $\mathcal{R}$ is time $\tau$ measurable and it is either discrete or continuous on $\mathbb{R}^\tau$. We denote $\mathcal{P}(r \in \mathcal{R})$ as the probability distribution of $\mathcal{R}$.\(^7\) We assume a perfect and complete market for claims on $\mathcal{R}$. This implies that for each state $r \in \mathcal{R}$, there is an associated Arrow-Debreu asset, with a unique (forward) state price $q(r)$, traded in the market. The function $q(.)$ is known as the (forward) state price density. An Arrow-Debreu asset $r$ is an asset that provides one unit of payoff if state $r$ occurs and nothing otherwise. Since $q(r)$ represents the price of a claim which has a non-negative payoff, it must itself be non-negative or else there would be arbitrage. Also, by the law of one price, the sum of the forward state prices across all possible states is equal to 1, which implies $\sum_{r \in \mathcal{R}} q(r) = 1$.\(^8\)

We consider the portfolio choice of a finitely risk averse investor with utility $u$ who is endowed with initial wealth $W_0$.\(^9\) The utility function is assumed to be

\(^7\)Note that we do not impose any restrictions on either the states or the probability distribution of $\mathcal{R}$. This implies that $r \in \mathcal{R}$ and $\mathcal{P}(r \in \mathcal{R})$ can be either discrete or continuous. In this paper, however, we use only the discrete notation. Readers should keep in mind that, with an appropriate set of assumptions, the model works for both discrete and continuous states and probability distributions.

\(^8\)Let, $P_i$, be the price of a claim $i$ with payoff, $\Pi^i(r)$. The law of one price affirms that, $P_i = \sum_{r \in \mathcal{R}} \Pi^i(r)q(r)$.

If the states of nature are continuous, then the law of one price implies

$P_i = \int_{r \in \mathcal{R}} \Pi^i(r)q(r)dr.$

\(^9\)We say an investor is finitely risk-averse if the absolute risk aversion of the investor is less than infinity.
state independent and it is defined over wealth. At time 0, this investor chooses a portfolio, which is represented by function \( W(\cdot) \), where \( W(r) \) is the number of units purchased of each Arrow-Debreu asset \( r \). The portfolio is chosen in order to maximize expected utility defined over final wealth, \( W(\tilde{r}) \). The problem faced by the investor can be written as

\[
\max_{W(\cdot)} E\left[u(W(\tilde{r}))\right],
\]

subject to the budget constraint

\[
\sum_{r \in \mathfrak{R}} W(r)q(r) = W_0, \quad (3)
\]

We refer to this type of portfolio problem and its solution as ‘the complete portfolio problem’ and ‘the complete portfolio’, respectively.

The First Order Condition (F.O.C.) of the optimization problem stated in (2) and (3) is

\[
p(r)u'(W^o(r)) = \lambda q(r) \quad \forall r, \quad (4)
\]

where \( p(.) \) represents the probability density of \( \mathfrak{R} \), \( W^o(.) \) is a function representing the optimal allocation to the Arrow-Debreu assets and \( \lambda \) is the Lagrange multiplier of the budget constraint.\(^{11}\)

Note that the function \( W^o(.) \) has different identities at different times. It represents the optimal allocation to each Arrow-Debreu asset at time 0 and becomes the optimal portfolio payoff at time \( \tau \). At time 0, before nature chooses a state, the investor chooses an optimal allocation \( W^o(r) \) to each Arrow-Debreu asset \( r \). At time \( \tau \), nature chooses a state \( r \) and the payoff of the optimal portfolio is realized. The investor receives a portfolio payoff, which is equivalent to the number of units purchased of Arrow-Debreu asset \( r \), i.e. \( W^o(r) \). Since the function \( W^o(.) \) also represents the payoff of the optimal portfolio, it is also known as ‘the optimal portfolio payoff function’. For simplicity, we hereafter refer to it as ‘the payoff function’.

Let \( \phi(r) \equiv \frac{q(r)}{p(r)} \) denote the probability deflated (forward) pricing density.\(^{12}\) For simplicity, we refer to it as the ‘pricing density’. The pricing density is the

\[
\int_{r \in \mathfrak{R}} W(r)q(r)dr = W_0.
\]

\(^{10}\)Since the state price is the forward price, the initial wealth \( W_0 \) is also a measure in terms of its forward value. In other words, let \( w_0 \) be the initial wealth at time 0, then \( W_0 = w_0r_f \), where \( r_f \) is the gross risk-free interest rate. Also, if the states of nature are continuous, then the budget constraint is rewritten as

\[
\int_{r \in \mathfrak{R}} W(r)q(r)dr = W_0.
\]

\(^{11}\)It is well known that the existence of a solution for F.O.C. (4) requires the utility function to be increasing and concave and to satisfy

\[
\lim_{x \to -\infty} u'(x) = 0.
\]

In the continuous state model, the existence of a solution requires more assumptions.

\(^{12}\)Note that \( \phi(.) \) is also known as the (forward) pricing kernel.
forward state price density divided by the probability density of \( R \). It is a measure of price relative to scarcity in that state. Since the optimal allocation in state \( r \) depends only on the pricing density \( \phi = \phi(r) \) in that state, it can be rewritten as a function of \( \phi \). Let \( W^*(\phi) = W^*(r) \) be the optimal allocation expressed as a function of \( \phi \), then the budget constraint (3) and F.O.C. (4) can be written, respectively, as

\[
E \left[ \tilde{\phi} W^*(\tilde{\phi}) \right] = W_0, \tag{5}
\]

and

\[
u'(W^*(\phi)) = \lambda \phi \quad \forall \phi. \tag{6}
\]

Equation (6) indicates that the marginal utility of consumption in a state is proportional to the relative scarcity. It also implies that the optimal allocation is an increasing function of the market returns. Figure 1 plots a typical optimal allocation as a function of the market returns.

In a perfect and complete market, the investor can choose any desired payoff function which is not necessarily linear in the market return. In fact, the payoff function will be linear in the market return, if and only if,

\[
\frac{dW^*(\phi)}{dr} = k \quad \forall r, \tag{7}
\]

where \( k \) is any state-independent non-negative constant. Thus, it is never optimal for an investor to hold a linear portfolio unless equation (7) holds. Indeed, the popularity of the derivative contracts has suggested that it may not always be optimal for an investor to hold a linear portfolio. In Appendix, we provide the necessary and sufficient condition under which an investor will hold a linear portfolio.

3 The Riskiness of the Complete Portfolios

Intuitively, one portfolio is said to be less risky than the other if the change in portfolio value of the former is less sensitive to the change in the market return

\[ \text{Notice that since both } p(r) \text{ and } q(r) \text{ are non-negative, the pricing density must also be non-negative, i.e. } \phi(r) \geq 0. \text{ Moreover, } \sum_{r \in R} q(r) = 1 \text{ implies that } \]

\[ E \left[ \phi(\tilde{r}) \right] = \sum_{r \in R} p(r) \frac{q(r)}{p(r)} = 1. \]

It is a measure of price relative to scarcity because a lower probability in state \( r \) (i.e. the state is relatively scarce) induces a higher pricing density in that state.

\[ \text{Note that, throughout this paper, we will use } \phi \text{ and } \phi(r) \text{ interchangeably.} \]

\[ \text{Assuming that the representative investor exists, Rubinstein [1974] shows that there exists a representative utility function } v(r), \text{ such that } \]

\[ v'(r) = \phi. \tag{F1} \]

Equation (6) suggests that \( W^* \) increases as \( \phi \) decreases. Since equation (F1) indicates that \( \phi \) decreases as \( r \) increases, it implies that \( W^* \) is increasing in \( r \).
than that of the latter. In an extreme case, where the investor is infinitely risk-averse, she will always choose a portfolio whose payoff is certain regardless of the change in the market return (i.e. the risk-free portfolio). Using this logic, we define a global measure of risk for the complete portfolio - ‘conservatism’, which is measured by the sensitivity of portfolio payoff to the change in the market return.

Let $W^{*i}(\phi)$ be the optimal complete portfolio for an investor with utility $u_i$ who is endowed with initial wealth $W_0$. The conservatism of the complete portfolio $W^{*i}(\phi)$ is measured inversely by $\frac{dW^{*i}(\phi)}{dr}$ (i.e. the lower the value of $\frac{dW^{*i}(\phi)}{dr}$, the more the conservatism of the investor). Using the chain rule, the conservatism can be written as

$$\frac{dW^{*i}(\phi)}{dr} = \frac{dW^{*i}(\phi)}{d\phi} \frac{d\phi}{dr} \forall r.$$  

Differentiating equation (6) with respect to $\phi$ and eliminating $\lambda$, one can show that

$$\frac{dW^{*i}(\phi)}{dr} \propto \frac{1}{A_i(W^{*i}(\phi))},$$  

where $A_i = -\frac{u_i''}{u_i'}$ represents the absolute risk aversion of $u_i$.

Equation (9) indicates that $\frac{dW^{*i}(\phi)}{dr}$ is greater than (equal to) zero for all finitely (infinitely) risk-averse investors. Figure 2 plots the optimal allocations for an infinitely risk-averse investor with utility $u_1$ and a finitely risk-averse investor with utility $u_2$. The infinitely risk-averse investor holds a risk-free portfolio whose value is insensitive to the change in the market return. In contrast, the finitely risk-averse investor holds a portfolio whose value is increasing in the

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market return.\(^{17}\)

Let \(\mathcal{P}\) be the probability distribution of \(\mathfrak{R}\). We define the concept of more conservative as follows.

**Definition 1** A complete portfolio, \(W^{*1}(\phi)\), is more conservative than a complete portfolio, \(W^{*2}(\phi)\), if for all probability distributions \(\mathcal{P}\) of \(\mathfrak{R}\) and all initial wealth \(W_0\), the slope of the complete portfolio \(W^{*1}(\phi)\) is everywhere flatter than that of \(W^{*2}(\phi)\). That is,

\[
\forall \mathcal{P}, W_0, r : \frac{dW^{*1}(\phi)}{dr} \leq \frac{dW^{*2}(\phi)}{dr},
\]

or equivalently,

\[
\forall \mathcal{P}, W_0, r : A_1(W^{*1}(\phi)) \geq A_2(W^{*2}(\phi)).
\]

Note that the concept of more conservative is defined in an analogous way to that of more risk-averse. \(u_1\) is said to be more risk-averse than \(u_2\), if and only if,

\[
\forall \mathcal{P}(x), \forall W_0 : \alpha_1^{*} \leq \alpha_2^{*},
\]

where \(\alpha_i^{*}\) represents the optimal amount of market portfolio allocation of an investor with utility \(u_i\) who is endowed with initial wealth \(W_0\). In words, a more risk-averse investor will always hold a less risky linear portfolios regardless of the economy in which she operates. Indeed, if we restrict investors to hold only the linear portfolios, condition (10) becomes identical to condition (12) -

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\(^{17}\)This property is referred to as the notion of second order risk aversion. The risk-averse investor is willing to bear risk in order to take advantage of consuming an Arrow-Debreu asset with a higher return.
The concepts of more conservative and more risk-averse converge. The different between the two concepts arises when investors are not precluded from holding ‘non-linear’ portfolios.

Figure 3 plots the optimal portfolios of two investors, where investor $u_1$ is more conservative than investor $u_2$. One can see that regardless of the size of change in the market return, the change in $W^{*1}$ is always smaller than that of $W^{*2}$, given a change in the market return. This is a direct consequence of definition 1.

After defining the concept of conservatism, one critical issue is to determine whether more risk-averse implies more conservative; or equivalently, does the more risk-averse investor hold a less risky complete portfolio? This is important because if this is true, the role of conservatism will be made redundant by the degree of absolute risk aversion. We can simply use absolute risk aversion as a measure of riskiness for complete portfolios. Conversely, if this is not true, then we cannot treat the two concepts as equivalent.

We provide a simple numerical example to illustrate that more risk-averse does not always imply more conservative. Consider the following constant relative risk averse (CRRA) utility

$$u_i(w) = \frac{1 - \gamma_i}{\gamma_i} \left( \frac{w}{1 - \gamma_i} \right)^{\gamma_i},$$

(13)
Using simple differentiation, we derive the following equations

\[ u_i'(w) = \left( \frac{w}{1 - \gamma_i} \right)^{\gamma_i - 1}, \quad (14) \]

\[ u_i''(w) = -\left( \frac{w}{1 - \gamma_i} \right)^{\gamma_i - 2}, \quad (15) \]

\[ A_i(w) = 1 - \gamma_i w. \]

We assume that there are only ten states of nature in the market and we let the set of all possible gross returns on the market portfolio be \( R = \{ \frac{2}{10}, \frac{2}{9}, \ldots, 2 \} \). The occurrence of each state is assumed to be equally likely, so that the probability density of \( R \) is written as

\[ p(r) = \frac{1}{10}, \quad \forall r = \frac{2}{10}, \frac{2}{9}, \ldots, 2. \]

The pricing density associated with each state \( r \) is assumed to have the following functional form

\[ \phi(r) = \frac{20}{55r}, \quad \forall r = \frac{2}{10}, \frac{2}{9}, \ldots, 2. \]

Notice that equation (16) satisfies the conditions of being a probability measure because \( 0 \leq p(r) \leq 1 \) for all \( r \) and \( \sum_r p(r) = 1 \). Similarly, equation (17) satisfies the conditions of being a pricing density because \( \phi(r) \geq 0, \phi'(r) \leq 0 \) for all \( r \) and \( E[\phi(\bar{r})] = 1 \).

We consider the complete portfolio choice of investor with utility \( u_i \), where \( i = 1, 2 \). We assume that \( \gamma_1 = 0.2 \) and \( \gamma_2 = 0.5 \). Both investors are endowed with initial wealth \( W_0 = 1 \). Using the budget constraint (5), F.O.C. (6), probability distribution (16) and pricing density (17), the optimal allocation \( W^*_i(\phi) \) can be solved by the following system of equations for \( i = 1, 2 \)

\[ \left( \frac{W^*_i(\phi)}{1 - \gamma_i} \right)^{\gamma_i - 1} = \frac{20\lambda}{55r}, \quad \text{s.t.} \quad \sum_r \frac{W^*_i(\phi)}{r} = \frac{55}{4}, \quad \forall r = \frac{2}{10}, \frac{2}{9}, \ldots, 2. \]

Figure 4 plots the optimal allocations \( W^*_1(\phi) \) and \( W^*_2(\phi) \) on the vertical axis with \( r \) on the horizontal axis. One can see from equation (15) that \( u_1 \) is more risk-averse than \( u_2 \). We want to check whether \( u_1 \) is more conservative than \( u_2 \). Figure 5 plots the conservatism, \( \frac{dW^*_1(\phi)}{dr} \) and \( \frac{dW^*_2(\phi)}{dr} \), on the vertical axis with \( r \) on the horizontal axis.\(^\text{18}\) It shows that, for \( r \leq \frac{2}{10} (r > \frac{2}{9}) \), the conservatism of \( u_1 \) is less (greater) than that of \( u_2 \). Thus, more risk-averse does not always imply more conservative. This result challenges the well-accepted notion that a more risk-averse investor holds a less risky portfolio.

\(^{18}\)Recall that the conservativeness is measured inversely by \( \frac{dW^*_i(\phi)}{dr} \). Therefore, the conservativeness can be measured directly by \( -\frac{dW^*_i(\phi)}{dr} \).
Figure 4: The optimal allocations of investors with $u_1$ and $u_2$. The optimal allocation is plotted on the vertical axis with the gross market returns on the horizontal axis. The solid (dash) line represents the optimal allocation for investor with utility $u_1$ ($u_2$). The optimal allocations are drawn by connecting the points which solve the system of equation (18).
Figure 5: The conservatism of investors with utility $u_1$ and $u_2$. The conservatism is plotted on the vertical axis with the gross market returns on the horizontal axis. The conservatism is computed discretely by $W^*(\phi_{x+1}) - W^*(\phi_x)$, where $x$ represents the state of nature. The solid (dash) line represents the conservatism of investor with utility $u_1$ ($u_2$).
4 Risk Aversion and conservatism

In the complete portfolio choice problem, investors can hold any portfolios where the linear portfolio is only a special case. Intuitively, we expect the concept of more conservative to be stronger than that of more risk-averse. However, as shown from the previous numerical example that little intuition can be dangerous. We, therefore, grace the intuition by formal analysis.

Consider two investors with utility $u_1$ and $u_2$, respectively. Using conditions (10) and (12), the concept of more conservative is stronger than that of more risk-averse if

$$
\frac{dW^{*1}(\phi)}{dr} \leq \frac{dW^{*2}(\phi)}{dr} \quad \forall P, W_0, r \implies \alpha^*_1 \leq \alpha^*_2 \quad \forall P, W_0
$$

or equivalently

$$
A_1(W^{*1}(\phi)) \geq A_2(W^{*2}(\phi)) \quad \forall P, W_0, r \implies A_1(z) \geq A_2(z) \quad \forall z.
$$

where $A_i(\cdot) = -\frac{u''(\cdot)}{u'(\cdot)}$ represents the absolute risk aversion of utility $u_i$.

The left hand side of condition (19),

$$
A_1(W^{*1}(\phi)) \geq A_2(W^{*2}(\phi)) \quad \forall P, W_0, r,
$$

represents the condition that investor $u_1$ is more conservative than investor $u_2$.

Let $P^\phi$ be the probability distribution of the degenerated random variable which takes value of $\phi$ with probability 1. Since $E[\tilde{\phi}] = 1$, the probability deflated state price $\phi$ must be equal to 1 under $P^\phi$. Using the budget constraint (5), we obtain the optimal allocation

$$
E[\tilde{\phi}W^{*i}(\tilde{\phi})] = W^{*i}(1) = W_0.
$$

Hence, under $P^\phi$, condition (20) can be written as

$$
\forall W_0: A_1(W_0) \geq A_2(W_0).
$$

Condition (21) is precisely the condition that $u_1$ is more risk-averse than $u_2$. Under the definition of conservatism, there is no restriction imposed on the probability distribution of $\mathcal{R}$ and $P^\phi$ is only a special case. Hence the concept of more risk-averse is necessary, but not sufficient for more conservative. This result together with the numerical example provided in the previous section leads to the following proposition.

Proposition 1 The concept of more conservative is stronger than the concept of more risk-averse.

Proposition 1 indicates that while a more conservative investor always holds a less risky linear portfolio, a more risk-averse investor does not necessarily hold...
a less risky complete portfolio. Our results not only challenge the generality of using absolute risk aversion as a measure of willingness to take risk, but also suggest a stronger measurement - conservatism - to evaluate the riskiness of the portfolio.

Recall from the definition that a more conservative (more risk-averse) investor holds a less risky complete (linear) portfolio regardless of the economy in which she operates. Under these definitions, the comparative risk taking behaviour can never be altered by different economies. However, it is noteworthy that if an investor is more risk-averse than another investor in a particular economy, she will always behave in a more risk-averse way in all other economies. In contrast, it is unclear whether a more conservative investor in a particular economy will always behave in a more conservative way in all other economies. Thus, definition 1 of more conservative could be far more demanding than one would expect. To this end, we weaken our definition of more conservative.

Let $\mathcal{P}$ be the set of all possible probability distributions of the economy. We define the concept of ‘weakly’ more conservative as follows.

**Definition 2** $W^{*1}(\phi)$ is weakly more conservative than $W^{*2}(\phi)$ in $\mathcal{P} \in \mathcal{P}$, if under $\mathcal{P}$

$$\frac{dW^{*1}(\phi)}{dr} \leq \frac{dW^{*2}(\phi)}{dr} \quad \forall W_0, r,$$

or equivalently,

$$A_1(W^{*1}(\phi)) \geq A_2(W^{*2}(\phi)) \quad \forall W_0, r.$$  \hspace{1cm} (23)

Clearly, definition 2 is weaker than definition 1. A weakly more conservative investor in one economy may not necessarily behave in a more conservative way in the other economies.

We examine whether the concept of weakly more conservative is stronger than that of more risk-averse. Let $\mathcal{C}$ be the set of all possible probability distributions of the economy under which, $W^{*1}(\phi)$ is weakly more conservative than $W^{*2}(\phi)$. The concept of weakly more conservative is stronger than that of more risk-averse if

$$\forall \mathcal{P} \in \mathcal{C} : A_1(W^{*1}(\phi)) \geq A_2(W^{*2}(\phi)) \quad \forall W_0, r \implies A_1(z) \geq A_2(z) \quad \forall z.$$  \hspace{1cm} (24)

While the intuition is strong for the concept of weakly more conservative to imply more risk-averse, surprisingly, it is rather difficult to show that condition (24) holds for all utility classes. However, we show that condition (24) holds at least for the most widely used utility class - the harmonic absolute risk aversion ($\textit{HARA}$). The $\textit{HARA}$ class of utility has become known as being particularly useful to derive analytical results. A function exhibits $\textit{HARA}$ if the inverse of its absolute risk aversion is linear in wealth. The $\textit{HARA}$ utility for investor $i$ can be written as follows

$$u_i(w) = \frac{1 - \gamma_i}{\gamma_i} \left( a_i + \frac{w}{1 - \gamma_i} \right)^{\gamma_i},$$  \hspace{1cm} (25)

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for some constants $a_i$ and $\gamma_i$ which subject to the condition, $a_i + \frac{w}{1 - \gamma_i} > 0$.\(^{19}\)

Using simple differentiation, we derive the following equations

$$u_i'(w) = \left(a_i + \frac{w}{1 - \gamma_i}\right) \gamma_i^{-1}, \quad (26)$$

$$u_i''(w) = -\left(a_i + \frac{w}{1 - \gamma_i}\right) \gamma_i^{-2}, \quad (27)$$

$$A_i(w) = \left(a_i + \frac{w}{1 - \gamma_i}\right)^{-1}, \quad (28)$$

We say that $W^{*1}(\phi)$ is weakly more conservative than $W^{*2}(\phi)$ in $\mathcal{P} \subset \mathcal{P}$ if

$$A_1(W^{*1}(\phi)) \geq A_2(W^{*2}(\phi)) \quad \forall W_0, r, \quad (29)$$

under $\mathcal{P}$. Using equation (28), condition (29) can be rewritten as

$$a_1 + \frac{W^{*1}(\phi)}{1 - \gamma_1} \leq a_2 + \frac{W^{*2}(\phi)}{1 - \gamma_2} \quad \forall W_0, r.$$ 

Multiplying both sides by $\phi$, taking expectation and using the budget constraint (5) yields

$$\left(a_1 + \frac{W_0}{1 - \gamma_1}\right)^{-1} \geq \left(a_2 + \frac{W_0}{1 - \gamma_2}\right)^{-1} \quad \forall W_0. \quad (30)$$

Condition (30) is precisely the condition that $u_1$ is more risk-averse than $u_2$. Thus, the concept of weakly more conservative implies more risk-averse under \textit{HARA}. This result together with the numerical example provided in the previous section leads to the following proposition.

**Proposition 2** \textit{The concept of weakly more conservative is stronger than the concept of more risk-averse under \textit{HARA}.}

Proposition 2 indicates that more risk-averse is only necessary but not sufficient for an investor to hold a less risky complete portfolio. Our results is strong even when the definition of more conservative has been significantly weaken. Our results provide strong evidence, which challenge the well-accepted generality that a more risk-averse investor holds a less risky portfolio. Our results also suggest a stronger measurement - conservatism - to evaluate the riskiness of the portfolio.

\(^{19}\)It is noteworthy that most commonly used utility functions can be obtained as special cases of the \textit{HARA} family. This can be done by choosing particular values of $a_i$ and $\gamma_i$. For example, the risk neutral utility ($\gamma_i = 1$), the quadratic utility ($\gamma_i = 2$), the constant absolute risk aversion utility ($\gamma_i = -\infty$), and the constant relative risk aversion utility ($a_i = 0$).
5 Conclusion

The literature on portfolio theory has predominantly focused on the choice of bond-stock portfolio. Under this framework, the relationship between portfolio payoff and stock return is always linear and the riskiness of portfolio is measured by absolute risk aversion. More risk-averse investors are expected to hold less risky bond-stock portfolios. In practice, however, investors are not precluded from purchasing assets whose values are not perfectly correlated with the stock return. In this case, the measurement of riskiness of portfolio becomes less obvious.

This paper considers the choice of Arrow-Debreu (complete) portfolios whose values are not necessarily perfectly correlated to the stock return. The suitability of using absolute risk aversion to measure the riskiness of Arrow-Debreu portfolio is examined. A global measure of risk for Arrow-Debreu portfolios is introduced. This measure is termed ‘conservatism’, which is measured by the sensitivity of portfolio payoff to the market return. ‘Conservatism’ is shown to be inversely proportional to the slope of the payoff function. Comparing two investors, one is said to be ‘more conservative’ than the other if the slope of the former payoff function is everywhere flatter than that of the latter. Our results show that the concept of more conservative is stronger than that of more risk-averse. A higher absolute risk aversion is only necessary but not sufficient to induce a more conservative portfolio. Our results not only challenge the well-accepted notion that a more risk-averse investor holds a less risky portfolio, but also suggest a stronger measurement - conservatism - to evaluate the riskiness of the portfolio.
Appendix

Our objective is to derive the necessary and sufficient condition under which an investor will hold a linear portfolio. Recall that, in a complete market, it is optimal for an investor to hold a linear portfolio, if and only if,

\[ \frac{dW^*(\phi)}{dr} = k \quad \forall r, \]  

where \( k \) is any state-independent non-negative constant.

Using the chain rule, equation (31) can be rewritten as,

\[ \frac{dW^*(\phi)}{dr} = \frac{dW^*(\phi)}{d\phi} \frac{d\phi}{dr} = k. \]  

Recall that

\[ \nu'(r) = \phi, \]  

where \( \nu'(r) \) is the marginal utility of the representative investor.

Differentiating both sides of equation (33) with respect to \( r \) yields

\[ \nu''(r) = \frac{d\phi}{dr}. \]  

Differentiating equation (6) with respect to \( \phi \) and eliminating \( \lambda \) yields,

\[ \frac{dW^{*i}(\phi)}{d\phi} = -\frac{1}{\phi A_i(W^{*i}(\phi))}. \]  

Using equations (35), (33) and (34), equation (32) can be rewritten as

\[ \frac{A_\nu(r)}{A(W^*(\phi))} = k, \]  

where \( A_\nu \) represents the absolute risk aversion of the representative investor.

Differentiating both sides of equation (36) with respect to \( r \) and rearranging yields

\[ -\frac{A'(W^*(\phi))}{A(W^*(\phi))^2} = -\frac{A'_\nu(r)}{(A_\nu(r))^2}. \]  

Let \( T \equiv \frac{1}{A} \) be the absolute risk tolerance. Using the fact that \( T' = -\frac{A'}{A^2} \), equation (37) can be rewritten as

\[ T'(W^*(\phi)) = T'_\nu(r). \]  

Thus, it is optimal for an investor to hold a linear portfolio, if and only if, the derivative of the investor’s absolute risk tolerance is equal to the derivative of the representative investor’s absolute risk tolerance (i.e. condition (38) holds).

In reality, the representative investor may not exist and even if she exists we may not be able to obtain her utility functional form. Thus, it is difficult to check whether condition (38) holds and investors may not be optimal to hold linear portfolios.
References


