A Variance Ratio Related Prediction Tool with Application to the NYSE Index 1825-2002

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Abstract
Cochrane’s variance ratio is a leading tool for detection of deviations from random walks in financial asset prices. This paper develops a variance ratio related regression model that can be used for prediction. We suggest a comprehensive framework for our model, including model identification, model estimation and selection, bias correction, model diagnostic check, and an inference procedure. We use our model to study and model mean reversion in the NYSE index in the period 1825-2002. We demonstrate that in addition to mean reversion, our model can generate other characteristic properties of financial asset prices, such as short-term persistence and volatility clustering of unconditional returns.
Introduction

This paper proposes a simple time-series regression-prediction model compatible with Cochrane’s variance ratio (1988). The regression model uses a price level measure as the explanatory variable. This measure consists of past prices only; it is very easy to use and intuitively appealing. While we use this model primarily to study and model mean reversion in the NYSE index, we also demonstrate that the suggested model can generate other characteristic properties of financial asset prices that have been documented in previous research, such as short-term persistence (e.g., Lo and MacKinlay, 1988, Poterba and Summers, 1988) and volatility clustering of unconditional returns (e.g., Mandelbrot, 1963).

Our model is related to a family of models that assign linear weighting schedules to different order autocorrelations. Three popular models in this group have been used to capture the dynamics of financial prices. These are Cochrane’s variance ratio (1988), the autoregressive model of Fama and French (1988) and the model of Jegadeesh (1991). Cochrane (1988) demonstrates that his variance ratio asymptotically assigns weights that decrease linearly with the order of the autocorrelation. Fama and French use, however, a model that assigns linearly increasing weights to low order autocorrelations, and then linearly decreasing weights to higher orders (see, for example, Richardson and Smith, 1994). Finally, the model of Jegadeesh assigns equal weights to different order autocorrelations (see, for example, Richardson and Smith, 1994).

The evidence of mean reversion in stock markets is mixed. Poterba and Summers (1988) apply the variance ratio methodology of Cochrane (1988) to 17 different stock indices around the world. They find evidence of mean reversion in most of these countries. Fama and French (1988) test several specifications of their autoregressive model and find evidence of mean reversion, particularly at ranges of 3-5 years. Jegadeesh also estimates his autoregressive model for the NYSE index and finds only weak evidence of mean reversion. Several other authors have questioned the statistical significance of some of the evidence of mean reversion in the NYSE index (see, for example, Kim, Nelson and Startz, 1991, and McQueen, 1992).

In this study, we are primarily interested in modelling mean reversion in the NYSE index using our regression model. Our model involves two parameters: a lag parameter for the price level measure, and an aggregation factor for the price level measure. To set possible candidates for these parameters, we employ a simple
identification tool based on autocorrelations. Using this tool, we find that annual returns of the NYSE index exhibit negative autocorrelations between lags 2 and 52. Based on this result, we set the candidate models, estimate them, and select the one with the lowest root mean squared error (RMSE). We follow to correct the bias in the coefficients of the selected model, and then we perform diagnostic checks for the corrected model. Finally, we apply a statistical inference test to the selected model that accounts for data mining. We find that the p-value of the selected model is 14.1%.

This paper is organized as follows: Section 1 documents some preliminary evidence on mean reversion in the NYSE index and in some commodity prices, using Cochrane’s variance ratio. This section also provides variance ratio results for the Consumer Price Index in the United States. Section 2 introduces a simple measure of price level consisting of prices only. In Section 3, we introduce our time-series model. We discuss the properties of our model, including its relation to Cochrane’s variance ratio, and we demonstrate that the model can generate several characteristic properties of financial asset prices, such as short-term persistence, long-term mean reversion, and volatility clustering. In Section 4, we suggest a comprehensive framework for our model, including model identification, model estimation and selection, bias correction, model diagnostic check, and an inference procedure. We apply this framework to the NYSE index for the period 1825-2002, using a new data set for the period 1825-1870 (Goetzmann, Ibboston and Peng, 2000). We conclude with a summary.

1. Applying the variance ratio methodology to study mean reversion in the NYSE index and three commodity prices

In subsequent parts of this paper, we will use a time-series regression model compatible with Cochrane’s variance ratio to model mean reversion in the NYSE index. In this section, we use Cochrane’s variance ratio to study the dynamics of the following series adjusted for inflation: the NYSE index, cotton prices, coffee prices and silver prices over an extended period of almost 200 years. (Refer to appendix D for descriptions of the NYSE index, commodity prices, and inflation.) The inspection of the NYSE index over such a long period of time is possible thanks to a new data set of returns for the NYSE
index provided by the Yale International Centre for Finance in a paper by Goetzmann, Ibboston and Peng (2000). The new dataset of returns, compatible with the CRSP data, begins with 1825. A full chart of the real NYSE index from the end of 1824 to the end of 2002 is given in figure 1.

Figure 1 about here

We begin by presenting the methodology of Cochrane’s variance ratio and then show its empirical results. Cochrane’s variance ratio is defined in the population as

\[ VR(k) = \frac{\text{var}(p_t - p_{t-k})}{k \cdot \text{var}(p_t - p_{t-1})} \]  

(1)

where \( p_t \) is the natural log of the price at time \( t \) and \( k \) is the window of the variance ratio. If the population displays a random walk, the variance ratio equals 1; if it is characterized by mean reversion, the variance ratio declines below 1; if it is characterized by persistence, the variance ratio exceeds 1. (Refer Campbell, Lo and MacKinlay, pages 48-49, for a simple demonstration of this property.) Cochrane (1988) suggests a corrected-for-bias small-sample estimator of the variance ratio that we use here.

Figure 2 reports the results of the variance ratios for the following adjusted for inflation series: the NYSE index, coffee prices, cotton prices, and silver prices.

Figure 2 about here

Figure 2 demonstrates that the variance ratio of the NYSE index declines to values below 0.3 for large values of \( k \). This result suggests that, for very long time horizons, the temporary mean-reverting component is substantial. The extent of mean reversion is considerably greater than reported by Poterba and Summers (1988). They use the variance ratio methodology of Cochrane (1988) to study the dynamics of stock market indices across seventeen countries and find evidence of long-horizon mean reversion in most of them. For the United States, they compute variance ratios for the real NYSE index for the period 1871-1985 in windows of up to 96 months, reporting a minimal variance ratio of 0.769 at a window of 84 months. A close look at figure 2 reveals that the value of the variance ratio at a window of 84 months (7 years) is very close to that reported by Poterba and Summers. But once the interval is increased beyond 8 years, the variance ratio
declines considerably and reaches a minimum at a window of 90 years. Similar dynamics are found in the prices of the three commodities - the variance ratios decline considerably below 1 for lengthy time intervals. The resemblance between the different variance ratios holds for the extent of the mean-reverting component and the lengthy period needed for the mean-reverting component to exhaust itself.

Figure 3 presents results for the variance ratio of the Consumer Price Index in the United States for the years 1820-2002. In contrast to the variance ratios in figure 2, the variance ratio for the Consumer Price Index is characterized by persistence.\(^1\)

2. A price level measure

This section proposes a price level measure comprised only of prices. The suggested measure is simple and intuitive and may be applied to many other econometric settings as an explanatory variable or as a dependent variable. The price level measure is defined as

\[
L_{t,N} = \log \left( \frac{p_t}{\left( \prod_{j=1}^{N} p_{t-j} \right)^{\frac{1}{N}}} \right) = p_t - \frac{1}{N} \sum_{j=1}^{N} p_{t-j}
\]

where \(L_{t,N}\) is the price level measure at time \(t\), \(p_t\) is the price at time \(t\), \(p_i\) is the natural log of the price at time \(t\), and \(N\) is an aggregation factor. Equation (2) demonstrates that the price level measure consists of the ratio (or difference in terms of log price) of the current price to the average of \(N\) prices from preceding periods. Under a regime of a temporary mean-reverting component in prices, the role of the average of past prices is to filter out the temporary component and provide a stable estimate of a long-run benchmark price. The difference between the current price and this average thus reflects a deviation from the long-run price. We regard this deviation as a price level measure. In the model presented below, we use a lagged version of this price level measure as the explanatory variable.
Representation of the price level measure in terms of returns

Equation (2) can be represented as

\[ L_{t,N} = p_t - \frac{1}{N} \sum_{j=1}^{N} p_{t-j} = \frac{1}{N} \sum_{j=1}^{N} (p_{t} - p_{t-j}) \]  

(3)

Let \( r_i \) denote the continuous return from time \( t-1 \) to time \( t \). It follows that

\[ p_t - p_{t-1} = r_t \]
\[ p_t - p_{t-2} = r_t + r_{t-1} \]
\[ p_t - p_{t-3} = r_t + r_{t-1} + r_{t-2} \]
\[ \vdots \]
\[ p_t - p_{t-N} = r_t + r_{t-1} + \ldots + r_{t-N+1} \]

Substituting these expressions in (3), we have

\[ L_{t,N} = \sum_{j=0}^{N-1} \frac{N-j}{N} r_{t-j} \]  

(4)

Equation (4) implies that the measure given in (2) can be represented as a weighted average of past returns, where the weights assigned to these returns are decreasing linearly with the lag. This property of decreasing weights might be appealing in many settings, because it assigns higher weights to more recent information and lower weights to more distant past information.

Figure 4 presents an example of the price level measure for the real NYSE index for the years 1919 to 2002, with aggregation of \( N=90 \).

Figure 4 about here

3. The CAR\(_g(\mathbf{i, N})\) Model

This section introduces the regression model that we use. Our model can be regarded as a return generating model designed to capture the influence of past price level upon the current return. We demonstrate below that our model can be represented as an autoregressive model with coefficients constrained to be linearly decreasing with the lag.
of returns. We denote the model henceforth as \( \text{CAR}_d(i,N) \), where “CAR” stands for “constrained autoregressive”, “d” for “decreasing”, \( i \) for the lag of the price level measure, and \( N \) for the aggregation of the price level measure.

### 3.1 The return generating model

The return generating model is

\[
    r_t = \alpha + \beta \cdot L_{t-i,N} + \varepsilon_t 
\]

where \( L_{t-i,N} \) is a price level measure with a lag of \( i \) and aggregation of \( N \) defined as

\[
    L_{t-i,N} = p_{t-i} - \frac{1}{N} \sum_{j=1}^{N} p_{t-i-j} = \sum_{j=0}^{N-1} \frac{N-j}{N} r_{t-i-j} 
\]

and where \( \alpha \) is the regression intercept, \( \beta \) is the regression slope, and \( \varepsilon_t \) is the residual at time \( t \).

Our model can be represented in a wider context as \( r_t = f(X, L_{t-i,N}) + \varepsilon_t \), where \( X \) is a vector of other explaining variables. In particular, \( X \) can include autoregressive components, moving average components, and/or other exogenous variables.

It is easy to demonstrate that our model can be represented as an autoregressive model that constrains the size of the coefficients of lagged returns. To show this, we substitute (6) into (5) and have

\[
    r_t = \alpha + \beta \sum_{j=0}^{N-1} \frac{N-j}{N} r_{t-i-j} + \varepsilon_t 
\]

The regression model in (7), however, can also be represented as

\[
    r_t = \alpha + \sum_{j=0}^{N-1} \left( \frac{\beta}{N} (N-j) \right) r_{t-i-j} + \varepsilon_t 
\]

which is clearly an autoregressive model that constrains the coefficients of past returns to be linearly decreasing with the lag.

### 3.2 The relation between the \( \text{CAR}_d \) model and Cochrane’s variance ratio

Cochrane (1988) demonstrates that as the number of observations goes to infinity his variance ratio can be represented in terms of autocorrelations as

\[
    VR(k) = 1 + 2 \sum_{j=1}^{k} \frac{k-j}{k} \rho_j 
\]
We demonstrate in appendix A that a variance ratio with a window of $k=N+1$ has the following asymptotic representation:

$$VR(N+1) = 1 + 2 \frac{N}{N+1} \sum_{j=1}^{N} N - j + 1 \rho_j$$  \hspace{1cm} (9)

We also demonstrate in appendix A that the regression coefficient of a CAR$_d$(1,N) model, $\beta_{1,N}$, can be represented as

$$\beta_{1,N} = \frac{\sigma^2(L_{1-N})}{\sigma^2(t_i)} \sum_{j=1}^{N} \left( \frac{N - j + 1}{N} \right) \rho_j$$  \hspace{1cm} (10)

Because both (9) and (10) use the expression $\sum_{j=1}^{N} \left( \frac{N - j + 1}{N} \right) \rho_j$, it follows that

$$\beta_{1,N} \frac{\sigma^2(L_{1-N})}{\sigma^2(t_i)} = \left( \frac{VR(N+1)-1}{2} \right) \left( \frac{N+1}{N} \right)$$  \hspace{1cm} (11)

Equation (11) demonstrates the asymptotic relation between Cochrane’s variance ratio with a window of $N+1$ and the slope coefficient of a CAR$_d$ model with a lag of 1 and aggregation of $N$. This relation results because the two tools, the asymptotic variance ratio and the CAR$_d$(1,N) model, assign to autocorrelations weights that decrease linearly with the order of the autocorrelation. Equation (11) implies that when the variance ratio is smaller than 1 (mean reversion), the slope coefficient of the compatible CAR$_d$(1,N) model is negative. Conversely, when the variance ratio is larger than 1 (persistence), the slope coefficient of the compatible CAR$_d$(1,N) model is positive. Finally, when the variance ratio is equal to 1 (random walk), the slope coefficient of the compatible CAR$_d$(1,N) model is 0.

3.3 Properties generated by the CAR$_d$ model: short-term persistence, long-term mean reversion, and volatility clustering

Several interesting properties have been documented for financial asset prices. These properties include short-term persistence (e.g., Lo and MacKinlay, 1988, Poterba and Summers, 1988), long-term mean reversion (e.g., Poterba and Summers, 1988), and volatility clustering (e.g., Mandelbrot, 1963). In this subsection, we demonstrate that the CAR$_d$ model can generate short-term persistence and long-term mean reversion, and then show that it can also generate volatility clustering.
To demonstrate that a CAR$_d$ model can generate short-term persistence and long-term mean reversion, we ran several simulations with the following models, all with a negative slope: CAR$_d$(1,6), CAR$_d$(2,6), CAR$_d$(4,6), CAR$_d$(6,6), and CAR$_d$(8,6). Figure 5 presents the autocorrelation functions for these models, with the autocorrelations corrected for bias using Fuller’s (1976, 1996) bias correction formula, given in equation (15) below.

Figure 5 about here

Figure 5 demonstrates that all the CAR$_d$ models create a series of positive autocorrelations at low orders and then a series of negative autocorrelations at higher orders. The only exception is the CAR$_d$(1,6) model that creates a series of negative autocorrelations starting from order 1. The simulations demonstrate that when $i$ is relatively small, the negative autocorrelations start at order $i$ (which is the lag of the price level measure); however, when $i$ is relatively large, the negative autocorrelations can start before $i$ (refer to the results for the CAR$_d$(6,6) and the CAR$_d$(8,6) models).

While mean reversion in our simulations stems directly from the negative value of the regression slope, the short-term persistence that CAR$_d$ models create is not so easily explained. It seems that short-term persistence is generated by the fact that successive price level measures overlap by definition and are therefore characterized by positive autocorrelations at low orders. If the regression coefficient, $\beta$, is different from zero, then these positive autocorrelations shift to the simulated returns through the return generating model.

We now demonstrate that a CAR$_d$ model also creates volatility clustering of unconditional returns; there are periods where returns have high volatility, while there are other periods with relatively low volatility. To demonstrate this property, recall that equation (5) defines the CAR$_d$ model as

$$ r_t = \alpha + \beta \cdot L_{t-i,N} + \varepsilon_t $$

(12)

Taking variances from both sides of (12), we have

$$ \text{var}(r_t) = \text{var}(\alpha + \beta \cdot L_{t-i,N} + \varepsilon_t) $$

Assuming that the predictor and the residual are uncorrelated, it follows that

$$ \text{var}(r_t) = \beta^2 \text{var}(L_{t-i,N}) + \text{var}(\varepsilon_t) $$

10
Substituting the definition of the variance of the price level measure, we get
\[
\text{var}(r_t) = \beta^2 E(L_{t-i,N} - \bar{L})^2 + \text{var}(\epsilon_t)
\] 
(13)

Because successive values of \( L_{t+i,N} \) are highly positively correlated, it follows that successive values of \( (L_{t+i,N} - \bar{L})^2 \) tend to be similar. This volatility clustering of \( L \) shifts to \( r_t \) because of the positive relation between the variance of \( r_t \) and the variance of \( L \), as implied by (13).

4. Model identification, model estimation and selection, bias correction, model diagnostic check and statistical inference

This section provides a comprehensive CAR\(_d\) model framework, with the following steps:

1. Identification of lags and aggregations for the price level measure (subsection 4.1).
2. Estimation of candidate models and model selection (subsection 4.2).
3. Small-sample bias correction (subsection 4.3).
4. Diagnostic checks (subsection 4.4).
5. Statistical inference on the significance of the selected model (subsection 4.5).

We follow to apply this setting to the real NYSE index for the period 1825-2002.

4.1 Identification of lags and aggregations for the price level measure

In this section we use a simple identification tool to identify possible candidates for the two parameters of our model, a lag parameter for the price level measure and an aggregation factor for the price level measure. Why do we need to consider lags different from 1 for the price level measure? Our simulation results in subsection 3.3 demonstrate that a CAR\(_d\) model with a negative slope coefficient typically creates positive autocorrelations at low orders and a series of negative autocorrelations at higher orders. The exception is a CAR\(_d\) model with a lag of one for the price level measure, which tends to create negative autocorrelations starting at order one. (Refer to the simulation results of the CAR\(_d\)(1,6) model.) If we set the lag of the price level measure at 1, then, consistent with our simulations, the estimated CAR\(_d\) model should produce negative autocorrelations starting at order 1, which might be inconsistent with short-term persistence. Because there
is some evidence that stock returns are characterized by short-term persistence (e.g. Lo and MacKinlay, 1988, Poterba and Summers, 1988), it may be useful to also consider lags larger than one.

The tool that we use for identification of candidate values for the lag of the price level measure and the aggregation of the price level measure is a simple sum of autocorrelations defined in the population as

$$WACF_e(k) = \sum_{j=1}^{k} \rho_j$$

(14)

where “WACF” is an acronym for “weighted autocorrelation function”, and “e” short for equal weights.\(^2\)

The main advantage of the WACF\(_e\) tool is its capacity to identify sequences of autocorrelations of the same sign. If, for example, a time series is characterized by positive autocorrelation at orders 1, 2 and 3 followed by negative autocorrelations at higher orders, then the WACF\(_e\) will reveal a maximum at order 3, indicating that negative autocorrelations start at order 4.

The problem of small-sample bias of autocorrelations is well documented (e.g. Marriot and Pope, 1954, Kendall 1954) and is present in the sample estimator of (14). There are two possible approaches to adjust for this problem. The first applies correction under the null of zero autocorrelation in the population, and the second corrects the bias under the observed size of the sample autocorrelation. The second approach is more appropriate here because the examination in this subsection is not associated with an inference procedure. Fuller (1996, 1976) suggests the following approximation for a bias-corrected autocorrelation: \(^3\)

$$\hat{\rho}_{ub,j} = \hat{\rho}_j + \frac{T - j}{(T - 1)^2} \left(1 - \hat{\rho}_j^2\right)$$

(15)

where \(\hat{\rho}_{ub,j}\) is the sample corrected for bias estimator of the j-th order autocorrelation and \(\hat{\rho}_j\) is the sample biased estimator of the j-th order autocorrelation defined as
The unbiased estimator of (14) is given thus by

\[ WACF_e(k) = \sum_{j=1}^{k} \hat{\rho}_{ub,j} \]  

(16)

Although useful as a preliminary identification tool, the WACF_e tool, being heuristic in nature, cannot be used to set the value of the lag in our regression model definitively. We suggest the WACF_e be used only to suggest possible values of the lag. If the WACF_e tool identifies that negative autocorrelations start at order k*, then a simple possible strategy is to examine several values of the lag for the CAR_d model in the vicinity of k*.

Figure 6 presents results of the WACF_e tool for the NYSE index for the period 1825-2002.

Figure 6 about here

Figure 6 demonstrates that the sample WACF_e peaks at k=1, implying that negative autocorrelations start at order 2. Consistent with the above discussion, we set the candidate values of the lag so that a lag of 2 is included in the set of candidate lags. The following lags are considered: i=1, 2 and 3.

For the setting of candidate aggregations for the price level measure, we use again the WACF_e tool for preliminary identification. The WACF_e tool presented in figure 6 reveals a maximum at lag 1 and a minimum at lag 52. It follows that annual returns of the NYSE index exhibit negative autocorrelations between lags 2 and 52. We do not use these results to set a deterministic value for the aggregation because of the heuristic nature of the WACF_e tool. However, the fact that negative autocorrelations range between lags 2
and 52 is an indication that the aggregation in our model may be very large. We set the candidate values of the aggregation to the following: N = 10, 20, 30, 40, 50, 60, 70, 80 and 90 years.

4.2 Estimation of candidate models and model selection

We use ordinary least squares to estimate the 27 CAR$_d$ models (all intersections of 3 lags and 9 aggregations). All the models are estimated with the same set of data for the dependent variable. The estimation results are given in table 1.

Table 1 about here

The mean squared error (MSE) or its root (RMSE) is widely used for model selection (e.g. McQuurrie and Tsay, 1998). Applying the minimal RMSE criterion, we select CAR$_d$(2,90) as the leading model among the candidates. The sections below implement several procedures regarding this model, including a small-sample bias correction, a diagnostic check, and an inference procedure.

4.3 A small-sample bias correction

We follow the slope correction procedure of MacKinnon and Smith (1998) but extend their procedure to add a simple correction for the intercept of the regression equation. MacKinnon and Smith use a bias function that describes the sample bias of the slope as a function of the unbiased slope. The approach can be applied when the bias is a linear function of the unbiased slope, or when the bias is a non-linear function of the unbiased slope (the more general case). The two applications are based on the same principle, but differ in the number of required iterations; the linear approach needs only one, whereas the non-linear case might need many. Because we do not know the form of the bias function, we follow the more general non-linear approach. We first describe the method, then implement it for the CAR$_d$(2,90) model.
The biased estimator of a regression slope, \( \hat{\beta} \), can be represented as a sum of an unbiased estimator of the slope, \( \hat{\beta}_{ub} \), and an estimator of the bias, \( \hat{b} \), in the following manner:

\[
\hat{\beta} = \hat{\beta}_{ub} + \hat{b} \tag{17}
\]

The non-linear approach, discussed above, assumes that the bias, \( \hat{b} \), is locally linear in the unbiased estimator, \( \hat{\beta}_{ub} \). Therefore, the bias can be estimated, locally, by

\[
\hat{b} = \gamma + \lambda \cdot \hat{\beta}_{ub} \tag{18}
\]

The procedure starts by arbitrarily picking two values for the slope of the CAR\(d\) model, denoted as \( \hat{\beta}_{ub,0} \) and \( \hat{\beta}_{ub,1} \). After setting the two initial slopes, the corresponding intercepts must be computed. Let \( \hat{\alpha}_{ub} \) denote the intercept of the regression equation. Appendix C demonstrates that the intercept given by

\[
\hat{\alpha}_{ub} = T - \hat{\beta}_{ub} \cdot L \tag{19}
\]

satisfies the conditions of zero average residual and minimal sum of squared errors (where \( T \) is the average value of the dependent variable, and \( L \) is the average value of the explanatory variable). Using (19) we find two intercepts \( \hat{\alpha}_{ub,0} \) and \( \hat{\alpha}_{ub,1} \), which, together with \( \hat{\beta}_{ub,0} \) and \( \hat{\beta}_{ub,1} \), form two artificial CAR\(d\) models. Each model is then used to derive a biased average slope computed from repetitive simulation experiments. From these two biased slopes, two biases, \( \hat{b}_0(\hat{\beta}_{ub,0}) \) and \( \hat{b}_1(\hat{\beta}_{ub,1}) \), are calculated, using equation (17). The two points on the bias function, \( (\hat{\beta}_{ub,0}, \hat{b}_0(\hat{\beta}_{ub,0})) \) and \( (\hat{\beta}_{ub,1}, \hat{b}_1(\hat{\beta}_{ub,1})) \), can be used to find first estimates, \( \hat{\gamma}_1 \) and \( \hat{\lambda}_1 \), for \( \gamma \) and \( \lambda \) in equation (18) with

\[
\hat{\gamma}_1 = \frac{\hat{b}_1(\hat{\beta}_{ub,1}) - \hat{b}_0(\hat{\beta}_{ub,0})}{\hat{\beta}_{ub,1} - \hat{\beta}_{ub,0}}
\]

and

\[
\hat{\lambda}_1 = \hat{b}_0(\hat{\beta}_{ub,0}) - \hat{\gamma}_1 \hat{\beta}_{ub,0}
\]

Let \( \hat{\beta}_{ub,2} \) denote the first corrected estimate of the slope we want to correct, \( \hat{\beta} \). From (17), it follows that

\[
\hat{\beta} = \hat{\beta}_{ub,2} + \hat{b} \tag{20}
\]
Recall also that with the calculated values, $\hat{\gamma}_1$ and $\hat{\lambda}_1$, equation (18) can be represented as

$$\hat{b} = \hat{\gamma}_1 + \hat{\lambda}_1 \cdot \hat{\beta}_{ub,2}$$

Equations (20) and (21) can be solved to find $\hat{\beta}_{ub,2}$. The solution for $\hat{\beta}_{ub,2}$ is given by (equation (7) in MacKinnon and Smith)

$$\hat{\beta}_{ub,2} = \frac{\hat{\beta} - \hat{\gamma}_1}{1 + \hat{\lambda}_1}$$

To verify the accuracy of $\hat{\beta}_{ub,2}$, we first use (19) to generate the intercept that corresponds to $\hat{\beta}_{ub,2}$, then generate a new average bias, $\hat{b}_2(\hat{\beta}_{ub,2})$, from the corrected regression model in repetitive simulations. Then the following accuracy condition is examined:

$$\left| \frac{\hat{\beta} - (\hat{\beta}_{ub,2} + \hat{b}_2(\hat{\beta}_{ub,2}))}{\hat{\beta}} \right| < \varphi$$

where $\varphi$ is an arbitrary percentage number set by the researcher. If (23) is satisfied, then the procedure ends. If, however, the condition in (23) is not satisfied, then an additional iteration is required, starting with the points, $(\hat{\beta}_{ub,1}, \hat{b}_1(\hat{\beta}_{ub,1}))$ and $(\hat{\beta}_{ub,2}, \hat{b}_2(\hat{\beta}_{ub,2}))$. The procedure continues until the criterion

$$\left| \frac{\hat{\beta} - (\hat{\beta}_{ub,z} + \hat{b}_z(\hat{\beta}_{ub,z}))}{\hat{\beta}} \right| < \varphi$$

is met. As MacKinnon and Smith indicate, however, this procedure does not necessarily converge. Two measures are possible in this case. One is to take a slower, more gradual approach to parameter adjustment, where the new value of the corrected slope is calculated as a weighted average of the current estimated corrected slope and its predecessor. (See equation (16) page 213 in MacKinnon and Smith.) Another possibility is to increase the accuracy of the derived average bias by increasing the number of replications in each iteration.

We now apply the non-linear approach to the CAR$_d$(2,90) model. The estimated CAR$_d$(2,90) regression equation for the NYSE index (see table 1) is

$$\hat{r}_t = 0.3903 - 0.1229 \cdot L_{t-2,90}$$
We start by arbitrarily picking two values of the slope. These values are \( \hat{\beta}_{\text{ub},0} = 0 \) and \( \hat{\beta}_{\text{ub},1} = -0.15 \). Using (19) the corresponding intercepts are \( \hat{\alpha}_{\text{ub},0} = 0.0681 \) and \( \hat{\alpha}_{\text{ub},1} = 0.4613 \). Implementing 20,000 randomization experiments for each of the two artificial models, we find the corresponding average biased slopes are \(-0.0630\) and \(-0.1823\). These results imply that the corresponding biases are \( \hat{\beta}_0(\hat{\beta}_{\text{ub},0}) = -0.0630 \) and \( \hat{\beta}_1(\hat{\beta}_{\text{ub},1}) = -0.0323 \), respectively. We now use the following two points, \((0.0, -0.0630)\) and \((-0.15, -0.0323)\), to find a first estimate for (18). This estimate is

\[
\hat{b} = -0.0630 - 0.2046 \cdot \hat{\beta}_{\text{ub},2}
\]

Using the identity

\[-0.1229 = \hat{\beta}_{\text{ub},2} + \hat{b}\]

we have two equations with two unknowns resulting with \( \hat{\beta}_{\text{ub},2} = -0.0754 \). Using (19), the corresponding intercept is \( \hat{\alpha}_{\text{ub},2} = 0.2657 \). The corrected CAR\(_d\) regression equation is given thus by

\[
\hat{t}_r = 0.2657 - 0.0754 \cdot L_{t-2,90}
\]

(26)

Assume now we set \( \phi \) in (24) to 1%, implying that the bias of \( \hat{\beta}_{\text{ub},2} \) must be less than 1%. To check whether (24) is satisfied, we conduct 20,000 randomization experiments with the model in (26). We find that the average value of the biased slope is \(-0.1156\). It follows that the accuracy of the unbiased slope, \(-0.0754\), is

\[
\left| \frac{-0.1229 - (-0.1156)}{-0.1229} \right| = 5.95\%.
\]

This value does not satisfy the 1% accuracy requirement, so we continue with an additional iteration.

We generate a new point on the bias function \((\hat{\beta}_{\text{ub},2}, \hat{\beta}_2(\hat{\beta}_{\text{ub},2}))\), with \( \hat{\beta}_{\text{ub},2} = -0.0754 \) and \( \hat{\beta}_2(\hat{\beta}_{\text{ub},2}) = -0.1156 - 0.0754 = -0.0402 \). Using the points \((-0.15, -0.0323)\) and \((-0.0754, -0.0402)\), we estimate a new locally linear bias function given by

\[
\hat{b} = -0.0483 - 0.1065 \cdot \hat{\beta}_{\text{ub},3}
\]

Using the identity

\[-0.1229 = \hat{\beta}_{\text{ub},3} + \hat{b}\]
we obtain two equations with two unknowns from which we derive the next corrected slope, \( \hat{\beta}_{ub,3} = -0.0836 \), intercept, \( \hat{\alpha}_{ub,3} = 0.2871 \), and the following corrected equation

\[
\hat{t}_i = 0.2871 - 0.0836 \cdot L_{t-2,90}
\]

(27)

We find the average corresponding (biased) slope across 20,000 randomization experiments is \(-0.1228\). Applying (24),

\[
\left| \frac{-0.1229 - (-0.1228)}{-0.1229} \right| = 0.1\%.
\]

which satisfies our accuracy requirement of 1%. The resulting corrected equation for the \( \text{CAR}_d(2,90) \) model is thus given by equation (27).

4.4 Diagnostic checks

In this subsection we suggest two simple diagnostic checks based on weighted sums of autocorrelations. The first diagnostic check compares the sample WACF\(_e\) curve to that generated by the candidate model. According to this criterion, a good candidate model would produce WACF\(_e\) similar to that of the data. We compare the WACF\(_e\) produced by the \( \text{CAR}_d(2,90) \) model of equation (27) to the sample WACF\(_e\) of the NYSE index. The results are seen in figure 7. The bold line is the WACF\(_e\) produced by the candidate model, and the thin line is that of the NYSE index. Note that both the sample WACF\(_e\) and that generated by the model are bias corrected using Fuller’s formula, given in equation (15).

Figure 7 about here

Another possible diagnostic check is to use a tool that assigns linearly decreasing weights to autocorrelations. We denote this tool by WACF\(_d\), where "d" stands for decreasing weights. The WACF\(_d\) tool is defined in the population as

\[
\text{WACF}_d(k) = \sum_{j=1}^{k} \frac{k-j+1}{k} \rho_j
\]

(28)

An asymptotic relation between the WACF\(_d\) tool and Cochrane’s variance ratio is given in appendix B.

The sample estimator of (28) is given by
\[
\bar{WACF}_d(k) = \sum_{j=1}^{k} \frac{k-j+1}{k} \hat{\rho}_{ab,j}
\]  

(29)

Figure 8 compares the sample WACF of the NYSE index to that generated by the CAR\(_d\)(2,90) model.

Figure 8 about here

### 4.5 Statistical inference on the significance of the CAR\(_d\)(2,90) model

Four computer-intensive methods of statistical hypothesis-testing are available in the literature. These methods are (a) sampling from the population observations; (b) sampling virtual observations from a known distribution; (c) bootstrapping; and (d) randomization (permutation). We now describe all four methods and then choose one to conduct the statistical test. Method (a) is consistent with the definition of a Monte Carlo simulation as given in Good (2000) and Noreen (1989). They define a Monte Carlo simulation as a method based on drawing samples, most commonly, without replacement, from the population’s observations. It seems, however, that the term "Monte Carlo simulation" is used more loosely by researchers for methods (a) and (b) where, in both, the sampling can be with or without replacement. Whereas in Monte Carlo simulations we sample from a known distribution or a known population of observations, in randomization (Fisher, 1935) and bootstrapping (Efron, 1979) we sample from the sample itself. There is one difference between bootstrapping and randomization – the drawing method. In bootstrapping the drawing is with replacement, in randomization without.

Because we have only a sample, method (a) is inappropriate for our study. We choose not to use method (b) because the accurate distribution of stock returns is still controversial. Between the remaining two methods, bootstrapping and randomization, we choose randomization, following Noreen (1989, page 10) who writes that randomization is adequate for stochastic statistical inference. We are unaware of any advantage that bootstrapping may have over randomization in the context of regression models. (For another study using randomization, see Kim, Nelson and Startz (1991), who use it to estimate the significance of mean reversion in the NYSE index over the period 1926-1986.)
We now briefly describe the randomization procedure that we use and then provide the resulting p-value. In section 4.2 we use the minimum root mean squared error (RMSE) for model selection. The minimum RMSE distribution is generated as follows. Each replication starts with shuffling the order of returns, after which the price level measures are composed and 27 CAR$_d$ models estimated (recall that the 27 models consist of all the possible combinations of $i=1,2,3$ with $N=10,20,30,40,50,60,70,80,90$). The minimal RMSE is recorded in each replication. This procedure is repeated 20,000 times, and the 20,000 values form the distribution of the minimal RMSE. Using this distribution, we find the one sided p-value for the RMSE of the CAR$_d(2,90)$ model is 14.1%.
Summary

Cochrane’s variance ratio is a leading tool for detection of deviations from random walks in financial asset prices. This paper proposes a simple variance ratio related regression model that can be used for prediction. Our regression model uses a price level measure as the explanatory variable. This measure is comprised only of prices; it is very easy to use and intuitively appealing. The two tools, Cochrane’s variance ratio and our regression model, are asymptotically related because they both assign to autocorrelations weights that decrease linearly with the order of the autocorrelation.

We use the proposed model to primarily study and model mean reversion in the NYSE index. We demonstrate that in addition to mean reversion, our model can generate other characteristic properties of financial asset prices that have been documented in previous research, including short-term persistence and volatility clustering of unconditional returns.

We suggest a comprehensive framework for our model consisting of the following steps:
1. Identification of lags and aggregations for the price level measure.
2. Estimation of candidate models and model selection.
3. Small-sample bias correction.
4. Diagnostic checks.
5. Statistical inference on the significance of the selected model.

We apply this framework to the real NYSE index for the period 1825-2002. In the identification stage, we find that annual returns of the NYSE index exhibit negative autocorrelations between lags 2 and 52. Using this preliminary result, we set candidates for the two parameters of our model (a lag parameter for the price level measure and an aggregation parameter for the price level measure). We then estimate the candidate models and find that the best model (in terms of RMSE) has a lag of 2 years and an aggregation of 90 years. We next correct the bias in the coefficients of this model, and then perform diagnostic checks for the corrected model. Finally, we apply a statistical inference test to the selected model that accounts for data mining. We find that the p-value of the selected model is 14.1%.
Appendix A: The asymptotic relation between the CAR$_d$ model and Cochrane’s variance ratio

**Proposition:**
The slope coefficient of a CAR$_d$(1,N) model is asymptotically related to Cochrane’s variance ratio with a window of N+1.

**Proof:**
Cochrane (1988) demonstrates that his variance ratio can be represented asymptotically as
\[
VR(k) = 1 + 2 \sum_{j=1}^{k} \frac{k-j}{k} \rho_j
\]
Increasing the window from $k$ to $k+1$
\[
VR(k+1) = 1 + 2 \sum_{j=1}^{k} \frac{k-j+1}{k+1} \rho_j
\]
which is equivalent to
\[
VR(k+1) = 1 + 2 \frac{k}{k+1} \sum_{j=1}^{k} \frac{k-j+1}{k} \rho_j
\]
Replacing $k$ with $N$
\[
VR(N+1) = 1 + 2 \frac{N}{N+1} \sum_{j=1}^{N} \frac{N-j+1}{N} \rho_j
\]
Let $\beta_{1,N}$ denote the regression coefficient of a CAR$_d$ model with a lag of 1 and aggregation of N. The ordinary least squares estimator of $\beta_{1,N}$ is given by
\[
\beta_{i,N} = \frac{\text{cov}(r_i, L_{t-1,N})}{\sigma^2(L_{t-1,N})}
\]
Using the definition of the price level measure given in (6) with $i=1$
\[
L_{t-1,N} = \sum_{j=0}^{N-1} \frac{N-j}{N} r_{t-1-j}
\]
Substituting $L_{t-1,N}$ in the numerator of (32)
Because (31) and (33) use the expression \( \sum_{j=1}^{N} \left( \frac{N - j + 1}{N} \right) \rho_j \), it follows that

\[
\beta_{1,N} \frac{\sigma^2(L_{1-1,N})}{\sigma^2(r_1)} = \left( \frac{\text{VR}(N+1) - 1}{2} \right) \left( \frac{N+1}{N} \right)
\]  

Equation (34) demonstrates the asymptotic relation between Cochrane’s variance ratio with a window of N+1 and the slope coefficient of a \( \text{CAR}_{d}(1,N) \) model.

Q.E.D.
Appendix B: The asymptotic relation between Cochrane’s variance ratio and the WACF_d tool

Proposition:
Cochrane’s variance ratio with a window of N+1 is asymptotically related to the WACF_d(N) tool.

Proof:
From (30)
\[ VR(k+1) = 1 + 2 \frac{k}{k+1} \sum_{j=1}^{k} \frac{k-j+1}{k} \rho_j \]  \hspace{1cm} (35)

Recall that the population’s version of the WACF_d tool is defined (equation (28)) as
\[ WACF_d(k) = \sum_{j=1}^{k} \frac{k-j+1}{k} \rho_j \]  \hspace{1cm} (36)

From (35) and (36) it follows that
\[ VR(k+1) = 1 + 2 \frac{k}{k+1} WACF_d(k) \]  \hspace{1cm} (37)

Equation (37) demonstrates an asymptotic relation between Cochrane’s variance ratio and the WACF_d tool.
Q.E.D.
Appendix C: Bias correction for the regression intercept

Proposition:
Assume that a procedure sets the estimate of the regression slope to \( \tilde{\beta} \). Given this value of the slope, the following estimator of the intercept
\[
\tilde{\alpha} = \bar{r} - \tilde{\beta}\bar{L}
\]
has unbiased predictions and minimal sum of squared errors (where \( \bar{r} \) is the average of the dependent variable and \( \bar{L} \) is the average of the explanatory variable).

Proof:
We begin by proving that \( \tilde{\alpha} = \bar{r} - \tilde{\beta}\bar{L} \) results with unbiased predictions. Let \( u_i \) denote the error that corresponds to \( \tilde{\alpha} \) and \( \tilde{\beta} \)
\[
u_i = r_i - (\tilde{\alpha} + \tilde{\beta}\bar{L}_{t-1,N}) \tag{38}\]
Averaging both sides of (38), we have
\[
\bar{u} = \bar{r} - (\tilde{\alpha} + \tilde{\beta}\bar{L}) \tag{39}\]
Substituting the suggested estimator of \( \tilde{\alpha} \) we have \( \bar{u} = 0 \), implying unbiased predictions.

We next show that \( \tilde{\alpha} = \bar{r} - \tilde{\beta}\bar{L} \) results with minimum sum of squared errors. The sum of squared errors is
\[
\sum_{t=1}^{T} (u_i - \bar{u})^2 \tag{40}\]
Since the suggested estimator of the intercept results with \( \bar{u} = 0 \), (40) reduces to
\[
\sum_{t=1}^{T} u_i^2 \tag{41}\]
Substituting \( u_i = r_i - (\tilde{\alpha} + \tilde{\beta}\bar{L}_{t-1,N}) \) in (41), differentiating the resulting expression with respect to \( \tilde{\alpha} \) and equating to zero, we get \( \tilde{\alpha} = \bar{r} - \tilde{\beta}\bar{L} \).

It follows that \( \tilde{\alpha} = \bar{r} - \tilde{\beta}\bar{L} \) results with unbiased predictions and minimum sum of squared errors.
Q.E.D.
Appendix D: data description

We took commodity prices and the Consumer Price Index from the site http://www.globalfindata.com/ Dr. Bryan Taylor, who runs the site, provided us with descriptions of these indices, given below. The sources for the NYSE index are also described below.

The NYSE index

We use annual value weighted returns on the NYSE index including dividend payments. The data are spliced from two resources. For the 1926-2002 period, we use the CRSP data. For the 1825-1925 period, data are taken from a CRSP compatible index provided by the Yale International Centre for Finance in a paper by Goetzmann, Ibboston and Peng (2000).

Goetzmann, Ibboston and Peng use two different sources for dividend yields. For the period 1871-1925 they use the Cowles dividend data. For the earlier 1825-1870 period they suggest two different alternatives: the “low dividend yield” and the “high dividend yield”. We make several adjustments to these two alternatives in order to generate one dividend series for the 1825-1870 period. First, because dividend yields (both high and low) for the year 1868 are missing, we estimate them by averaging dividend yields for the years 1867 and 1869. Second, we adjust the level of the two dividend series. The high and low dividend yields for 1870 are 9.12% and 4.2%, respectively. Both values seem too extreme compared to subsequent years’ yields of about 5.5% to 7.0% (Cowles data), so we adjust the two series (high and low) by multiplying each by a constant equal to the ratio of the 1871 yield to the 1870 yield. Finally, both series (high and low) have standard deviations significantly larger than in subsequent periods. The standard deviations of the high and low dividend yields are 1.98% and 2.51% respectively, compared with 1.03% for the subsequent 1871-1925 period and 1.57% for the 1926-2002 period. This excess volatility may be due to missing data. A possible approach to reduce the variance is to find a combination of the two series that has a lower variance. We find that the combination with the lowest variance assigns a weight of 0.72 to the low dividend yield and 0.28 to the high dividend yield. The standard deviation of the combined dividend series is 1.87%, an estimate more compatible with that of the subsequent periods. We use this series of dividend yields for the period 1825-1870.
Coffee
The coffee series uses Santos Coffee in New York through 1984, and Brazilian coffee thereafter. These data are taken from the Commodity Research Bureau, Commodity Yearbook (Chicago: CRB), except for 1874 through 1896, taken from Bezanson (1954).

Silver
For the period 1825 through 1933, data are taken from Warren and Pearson (1937). For the period 1934 through 2000, data are taken from the Commodity Research Bureau, Commodity Yearbook (Chicago: CRB).

Cotton
Prices from 1825 through 1869 are taken from Bezanson, Gray and Hussay (1936-1937) and Bezanson (1954). Data from September 1870 through 1945 are taken from the NBER. From 1946 on, the data are taken from the Commodity Research Bureau, Commodity Yearbook (Chicago: CRB).

Inflation
The Consumer Price Index is based on a combination of three indices:

a. The Federal Reserve Bank’s annual cost-of-living index (for 1820 through 1874).

b. The monthly Index of General Prices calculated by the Federal Reserve Bank of New York, a weighted average of commodity prices (20%), wage payments (35%), the cost of living (35%) and rents (10%) (for 1875 through 1912). For more information on this index, see Snyder (1924).

c. The Bureau of Labor statistics Consumer Price Index (for 1913 on).
References


Table 1: OLS estimation of CARₜ models for the NYSE index

Table 1 presents estimation results for the candidate CARₜ models. In this table, $\hat{\alpha}$ is the sample estimator of $\alpha$ and $\hat{\beta}$ is the sample estimator of $\beta$. The model with the lowest RMSE is CARₜ(2, 90) with RMSE of 0.1864.

<table>
<thead>
<tr>
<th>model</th>
<th>intercept $\hat{\alpha}$</th>
<th>Slope $\hat{\beta}$</th>
<th>RMSE</th>
<th>DF</th>
</tr>
</thead>
<tbody>
<tr>
<td>CARₜ(1, 10)</td>
<td>0.0925</td>
<td>-0.0686</td>
<td>0.1919</td>
<td>82</td>
</tr>
<tr>
<td>CARₜ(1, 20)</td>
<td>0.1091</td>
<td>-0.0644</td>
<td>0.1912</td>
<td>82</td>
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<tr>
<td>CARₜ(1, 30)</td>
<td>0.1439</td>
<td>-0.0836</td>
<td>0.1896</td>
<td>82</td>
</tr>
<tr>
<td>CARₜ(1, 40)</td>
<td>0.1677</td>
<td>-0.0839</td>
<td>0.1895</td>
<td>82</td>
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<tr>
<td>CARₜ(1, 50)</td>
<td>0.1925</td>
<td>-0.0840</td>
<td>0.1894</td>
<td>82</td>
</tr>
<tr>
<td>CARₜ(1, 60)</td>
<td>0.2333</td>
<td>-0.0933</td>
<td>0.1887</td>
<td>82</td>
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<tr>
<td>CARₜ(1, 70)</td>
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<td>-0.0964</td>
<td>0.1886</td>
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<tr>
<td>CARₜ(1, 80)</td>
<td>0.3010</td>
<td>-0.0993</td>
<td>0.1885</td>
<td>82</td>
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<tr>
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<td>0.3486</td>
<td>-0.1065</td>
<td>0.1881</td>
<td>82</td>
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<tr>
<td>CARₜ(2, 10)</td>
<td>0.1008</td>
<td>-0.0941</td>
<td>0.1905</td>
<td>82</td>
</tr>
<tr>
<td>CARₜ(2, 20)</td>
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<td>-0.0808</td>
<td>0.1899</td>
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<tr>
<td>CARₜ(2, 30)</td>
<td>0.1551</td>
<td>-0.0973</td>
<td>0.1882</td>
<td>82</td>
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<tr>
<td>CARₜ(2, 40)</td>
<td>0.1814</td>
<td>-0.0965</td>
<td>0.1882</td>
<td>82</td>
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<tr>
<td>CARₜ(2, 50)</td>
<td>0.2096</td>
<td>-0.0963</td>
<td>0.1881</td>
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<tr>
<td>CARₜ(2, 60)</td>
<td>0.2547</td>
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<tr>
<td>CARₜ(2, 70)</td>
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<td>CARₜ(2, 80)</td>
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<td>CARₜ(2, 90)</td>
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<td>CARₜ(3, 10)</td>
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<tr>
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<tr>
<td>CARₜ(3, 30)</td>
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<td>-0.0745</td>
<td>0.1904</td>
<td>82</td>
</tr>
<tr>
<td>CARₜ(3, 40)</td>
<td>0.1535</td>
<td>-0.0733</td>
<td>0.1904</td>
<td>82</td>
</tr>
<tr>
<td>CARₜ(3, 50)</td>
<td>0.1772</td>
<td>-0.0748</td>
<td>0.1902</td>
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<td>0.2125</td>
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<tr>
<td>CARₜ(3, 80)</td>
<td>0.2756</td>
<td>-0.0893</td>
<td>0.1896</td>
<td>82</td>
</tr>
<tr>
<td>CARₜ(3, 90)</td>
<td>0.3187</td>
<td>-0.0960</td>
<td>0.1894</td>
<td>82</td>
</tr>
</tbody>
</table>
Figure 1: Natural log of the nominal NYSE index

Figure 1 presents the natural log of the NYSE index from the end of 1824 to the end of 2002. From the end of 1925 to the end of 2002, data are taken from the CRSP database. From the end of 1824 to the end of 1925 the index is taken from Goetzmann, Ibboston and Peng (2000).
Figure 2: Variance ratios for the NYSE index and three commodity price series in the United States

Figure 2 presents variance ratios for the following adjusted for inflation series: the NYSE index, coffee prices, silver prices and cotton prices. All 4 variance ratios suggest the presence of a mean-reverting component.
Figure 3: Variance ratio for the Consumer Price Index in the United States for the period 1820-2002

Figure 3 presents the U.S. Consumer Price Index variance ratio for the years 1820 to 2002. The variance ratio suggests a component of persistence in the Consumer Price Index, implying positive autocorrelations in inflation levels.
Figure 4: An example of the price level measure for the real NYSE index for the years 1919-2002 with aggregation of N=90

Figure 4 presents results for the following price level measure:

\[ L_{t,90} = p_t - \frac{1}{90} \sum_{j=1}^{90} p_{t-j} \]

where: \( p_t = \ln \left( \frac{P_t^N}{CPI_t} \right) \), \( P_t^N \) is the nominal NYSE index at time t, and CPI_t is the Consumer Price Index at time t.
Figure 5: The autocorrelation function generated by different CAR\(_d\) models

Figure 5 presents the autocorrelation function generated by the following 5 CAR\(_d\) models: CAR\(_d\)(1,6), CAR\(_d\)(2,6), CAR\(_d\)(4,6), CAR\(_d\)(6,6) and CAR\(_d\)(8,6). In all these models, we consider a negative slope equal to -0.1. Figure 5 demonstrates that CAR\(_d\) models with a negative slope generate a series of positive autocorrelations at low orders and a series of negative autocorrelations at higher orders, with the exception of the CAR\(_d\)(1,6) model which does not generate positive autocorrelations at low orders.
Figure 6: The WACF\textsubscript{e} tool for the real NYSE index for the period 1825-2002

Figure 6 presents results of the WACF\textsubscript{e} tool for the real NYSE index for the period 1825-2002. The unbiased sample estimator of the WACF\textsubscript{e} tool is defined as

\[ \hat{\text{WACF}}_e(k) = \sum_{j=1}^{k} \hat{\rho}_{ub,j} \]

where \( \hat{\rho}_{ub,j} \) is the unbiased estimator of the j-th order autocorrelation. The WACF\textsubscript{e} tool reveals a maximum at order 1 and a minimum at order 52.
Figure 7: Diagnostic check for the CAR\(_d(2,90)\) model with the WACF\(_e\) tool

Figure 7 compares the sample WACF\(_e\) tool of the NYSE index, to that produced in simulations by the CAR\(_d(2,90)\) model corrected for bias (equation (27)). The WACF\(_e\) tool is a simple sum of autocorrelations. The bold line in this figure is the WACF\(_e\) produced by the CAR\(_d(2,90)\) model, and the thin line is the WACF\(_e\) of the NYSE index.

WACF\(_e\) generated from the CAR\(_d(2,90)\) model compared with sample WACF\(_e\)

![Graph showing WACF\(_e\) values for the CAR\(_d(2,90)\) model and the NYSE index.](image-url)
Figure 8: Diagnostic check for the CAR_d(2,90) model with the WACF_d tool

Figure 8 compares the sample WACF_d tool of the NYSE index, to that produced in simulations by the CAR_d(2,90) model corrected for bias (equation (27)). The WACF_d tool is a weighted sum of autocorrelations where the weights assigned to the autocorrelations are linearly decreasing with the order of the autocorrelation. The bold line in this figure is the WACF_d produced by the CAR_d(2,90) model, and the thin line is the WACF_d of the NYSE index.
Footnotes

1 Positive autocorrelations in inflation levels have two interesting implications. First, positive autocorrelations in inflation levels are prone to drift into other nominal variables, through indexation mechanisms (whether economic or contractual). As a result, nominal series might have a higher degree of persistence than real series. Second, positive autocorrelations in inflation levels might be of some interest to monetary decision makers: the higher the positive autocorrelation of successive inflation levels, the harder it is to direct inflation to its desired target level.

2 The WACF$e$ tool is closely related to what is known in spectral analysis as the spectrum at frequency zero.

3 An approximation for the autocorrelation bias under the null can be found in Campbell, Lo and MacKinlay (1997, p. 46).

4 While our data start from 1825, the dependent variable in the CAR regression equation ranges from 1919 to 2002 only. The data from 1825 to 1918 are not part of the dependent variable due to the long tail (N=90) needed for the estimation of the price level measure and due to the lag of the price level measure (i=2). Because the RMSE is generated from returns over the period 1919-2002, and because the unconditional (and the conditional) returns might have different variances over these two sub-periods, one must be careful not to mix observations from the 1919-2002 period with observations from the 1825-1918 period during the shuffling process. Therefore, the shuffling is done in two blocks. The 1825-1918 returns are shuffled separately from the 1919-2002 returns.