Mean Reversion Level Extensions of Time-Homogeneous Affine Term Structure Models

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Abstract. It is well-known that time-homogeneous affine term structure models are not compatible with initial forward rate curves in general. For the Vasicek (1977) and Cox, Ingersoll and Ross (1985) models, time-inhomogeneous extensions compatible with any given initial forward rate curve were introduced in Hull and White (1990), and similar extensions, for short rate models in general, were introduced in Bjork and Hyll (2000), Brigo and Mercurio (2001), and Kwon (2004). This paper introduces mean-reversion level extensions of time-homogeneous affine term structure models that are compatible with any given initial forward rate curve. These extensions are minimal in the sense that the system of Riccati equations determining the bond prices remain essentially unchanged under the extension. Moreover, the extensions considered in Bjork and Hyll (2000), Brigo and Mercurio (2001), and Kwon (2004), for time-homogeneous affine term structure models, are all special cases of the extensions introduced in this paper.

Key words: Affine term structure model, mean reversion level, initial forward rate curve, time-inhomogeneous extension

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1. Introduction

Many tractable term structure models, including those introduced in Vasicek (1977), Cox, Ingersoll and Ross (1985), and Chen and Scott (1992) are time-homogeneous finite factor models. Moreover, such models are in general affine,
which means that the forward rate curve is an affine function of the factors, or equivalently the zero coupon bond prices are exponential affine in the factors. A systematic study of these models is given in Duffie and Kan (1996), where the structure of the factor dynamics and the set of Ricatti equations determining the bond prices are determined.

One of the key deficiencies of time-homogeneous affine term structure models is their incompatibility with all but a limited set of initial forward rate curves. An undesirable consequence of this is that it is not possible to recover the prices of traded securities from the initial forward rate curves implied by these models. The important role played by the initial curve also emerges in the general term structure modeling framework of Heath, Jarrow and Morton (1992), henceforth HJM, where the initial curve is an input in the forward rate dynamics. Hence, given an initial forward rate curve, a natural problem that arises is whether or not it is possible to extend the time-homogeneous affine term structure models so that they are compatible with the given curve while maintaining the tractability of these models.

Another reason for focusing on affine term structure models is the result from Filipović and Teichmann (2002), which establishes that a model from the HJM framework, if it admits a finite factor realization and can accommodate any initial forward rate curve, must necessarily be affine. In particular, it may be concluded that the affine models are the only tractable models from the HJM framework. Since there does not exist a systematic way to generate exhaustive list of affine models within the HJM framework, the extensions of the time-homogeneous affine models provide a useful subset of affine models in this framework.

The mean-reversion level extensions of time-homogeneous affine term structure models introduced in this paper result in affine models within the HJM framework. Moreover, since the Ricatti equations determining the zero coupon bond prices in the extensions are essentially the same as those in the underlying model, these extensions retain much of the tractability of the underlying model. The Hull-White extensions of the Vasicek (1977) and the Cox, Ingersoll and Ross (1985) models, the deterministic shift extensions of short rate models considered in Björk and Hyll (2000) and Brigo and Mercurio (2001), the short rate extensions introduced in Kwon (2004) are special cases of the extensions considered in this paper.

The structure of the remainder of this paper is as follows. A brief review of time-homogeneous affine term structure models is first given in Section 2, and the mean-reversion level extensions of these models are then introduced in Section 3. It is also shown there that extensions introduced in Björk and Hyll (2000), Brigo and Mercurio (2001), and Kwon (2004) are special cases of the mean-reversion level extensions. Some applications of these extensions, in particular to futures prices, are given in Section 4, and Section 5 concludes the paper.

2. Affine Term Structure Models

Fix $n \in \mathbb{N}$ and $\tau \in \mathbb{R}_+$, and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \tau]}, \mathbb{P})$ be a probability basis satisfying the usual conditions, where $(\mathcal{F}_t)_{t \in [0, \tau]}$ is the filtration generated by
a standard $n$-dimensional $\mathbb{P}$-Wiener process $w_t = (w^1_t, \ldots, w^n_t)$ augmented with $\mathbb{P}$-nullsets. Assume given an $n$-dimensional time-homogeneous $(\mathcal{F}_t, \mathbb{P})$-Markov process, $x_t$, with values in some subset $\mathcal{D} \subset \mathbb{R}^n$ such that the interior $\mathcal{D}^0 \neq \emptyset$. If $x^*_t$ are the factors for an affine term structure model, then provided that the drift and the volatility of $x_t$ meet some mild restrictions, the results from Duffie and Kan (1996) imply that $x_t$ can be assumed to satisfy a stochastic differential equation of the form

$$
 dx_t = \kappa (\theta - x_t) \, dt + \sigma \sqrt{d(x_t)} \, dw_t \tag{1}
$$

subject to $x_0 = x^*_0 \in \mathcal{D}$, where $\theta \in \mathbb{R}^n$, $\kappa \in \text{GL}(n, \mathbb{R})$, $\sigma \in \text{GL}(n, \mathbb{R})$, and

$$
 d(x) = \text{diag}(a_1 + b'_1 x, \ldots, a_n + b'_n x), \tag{2}
$$

for some $a_i \in \mathbb{R}$, $b_i \in \mathbb{R}^n$, and for all $x \in \mathcal{D}$. Here, $\text{GL}(n, \mathbb{R})$ denotes the set of all invertible $n \times n$ matrices with entry values in $\mathbb{R}$.

Given a term structure model $m$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \tau]}, \mathbb{P})$, let $P_t(T)$ be the price of the $T$-maturity zero coupon bond for any $T \in [0, \tau]$, let $f_t(T)$ be the $T$-maturity instantaneous forward rate, and let $r_t$ be the short rate at time $t$. Then by definition

$$
 f_t(T) = -\partial_T \ln P_t(T), \tag{3}
$$

$$
 r_t = f_t(t) = -\partial_t \ln P(t, T)|_{T=t}. \tag{4}
$$

**Lemma 1.** (Duffie and Kan (1996)) Let $x_t$ be as given in (1), and let $A \in C^1[0, \tau]$ and $\beta \in C^1([0, \tau], \mathbb{R}^n)$ such that $A(0) = 0$ and $\beta(0) = 0$. Then setting

$$
 P_t(T) = e^{-A(T-t) - \beta(T-t)' x}, \tag{5}
$$

defines a term structure model for which $\mathbb{P}$ is a martingale measure if and only if $A$ and $\beta$ satisfy the system of Riccati equations

$$
 -\dot{A}(t) = -\theta' \kappa' \beta(t) + \frac{1}{2} \sum_{i=1}^n a_i |\sigma' \beta(t)|_i^2 - \dot{A}(0), \tag{6}
$$

$$
 -\dot{\beta}(t) = \kappa' \beta(t) + \frac{1}{2} \sum_{i=1}^n |\sigma' \beta(t)|_i^2 b_i - \dot{\beta}(0) \tag{7}
$$

for all $t \in [0, \tau]$.

**Proof:** See Duffie and Kan (1996) or Kwon (2004). \hfill \Box

Any term structure model for which $P_t(T)$ given by (5) for some $A$ and $\beta$, and $x_t$ satisfying (1) is called a time-homogeneous affine term structure model. For notational convenience, the set of all $n$-factor time-homogeneous ATSMs with $\mathbb{P}$ as a martingale measure will be denoted $\mathcal{M}_{n, \beta}$.

It is possible to identify the time-homogeneous ATSMs with a subset of 10-tuples of the form, $(\kappa, \theta, \sigma, a_i, b_i, A, \beta, x_t, x^*_0)$, where the components are as
defined in (1), (2) and (5). If \((\kappa, \theta, \sigma, S, a_i, b_i, A, \beta, x_t, x^*_0)\) are associated with \(m \in \mathcal{A}_{n,P}\), then we will identify \(m\) with the 10-tuple, and write

\[ m = (\kappa, \theta, \sigma, d, a_i, b_i, A, \beta, x_t, x^*_0). \tag{8} \]

There is a certain level of redundancy in the 10 components. For example, \(A, \beta\) and \(x_t\), if they exist, are determined by the other 7 components. Modulo this redundancy, \(\mathcal{A}_{n,P}\) consists of those \((\kappa, \theta, \sigma, d, a, b, A, \beta, x_t, x^*_0)\) for which \(x_t\) is the solution to (1) and \(P\) is a martingale measure for the bond price defined by (5).

Note also that if \((\kappa, \theta, \sigma, d, a, b, A, \beta, x_t, x^*_0)\) \(\in \mathcal{A}_{n,P}\), then \(A(0) = 0\) and \(\beta(0) = 0\).

3. Mean Reversion Level Extensions of ATSMs

In considering the extensions of \(m \in \mathcal{A}_{n,P}\) to the time-inhomogeneous setting, we need to consider stochastic differential equations of the form

\[ d\bar{x}_t = \kappa(\bar{\theta}(t) - \bar{x}_t)\, dt + \sigma\, \sqrt{d(\bar{x}_t)}\, dw_t \tag{9} \]

subject to \(\bar{x}_0 = \bar{x}^*_0 \in \mathcal{D} \subset \mathbb{R}^n\), where \(\bar{\theta} \in C^0([0, \tau], \mathbb{R}^n)\) and \(\kappa, \sigma\), and \(d\) are as given in (1).

**Definition 1.** For any \(m = (\kappa, \theta, \sigma, d, a_i, b_i, A, \beta, x_t, x^*_0) \in \mathcal{A}_n\), define

\[ \Theta(m) = \{ \bar{\theta} \in C^0([0, \tau], \mathbb{R}^n) : \text{there exist a unique} \]

\[ \text{solution to (9) with mean reversion level } \bar{\theta}, \]

\[ \text{initial value } \bar{x}_0 = x^*_0 \in m \text{ and } d(\bar{x}_t) > 0 \}, \tag{10} \]

where by \(d(x) > 0\) we mean \(a_i + b_i'x > 0\) for all \(1 \leq i \leq n\) and \(x \in \mathcal{D}\), and by \(x^*_0 \in m\) we mean \(x^*_0\) is the final component of \(m\).

The reason for defining \(\Theta(m)\) this way is to avoid having to consider the existence of solutions to (9) for which the volatility remains positive. The problem of determining the necessary and sufficient conditions under which such solutions exist and are unique for a given \(\bar{\theta} \in C^0([0, \tau], \mathbb{R}^n)\) is difficult, and we leave it for future research. Note, however, the fact that (9) has a unique solution can be established by arguments similar Duffie and Kan (1996) p403, since the introduction of a deterministic term in the drift does not affect its Lipschitz property. The difficulty lies in establishing the conditions that ensure the factor volatilities remain positive.

For notational convenience, let \(\Delta = \{(t, T) \in [0, \tau]^2 : t \leq T\}\).

**Lemma 2.** Let \(m \in \mathcal{A}_{n,p}, \bar{\theta} \in \Theta(m)\), and let \(\alpha \in C^{1,1}(\Delta)\) such that \(\alpha(T, T) = 0\) for all \(T \in [0, \tau]\). Then for each \((t, T) \in \Delta\), setting

\[ P_t(T) = e^{-\alpha(t, T) - \beta(T-t)'x_t} \tag{11} \]
for the T-maturity bond price defines a term structure model for which $\mathbb{P}$ is a martingale measure if and only $A$ and $\beta$ satisfy

$$\partial_t \alpha(t, T) = -\bar{\theta}(t) \kappa' \beta(T - t) + \frac{1}{2} \sum_{i=1}^{n} \alpha_i [\sigma'(\beta(T - t))]^2_i - \partial_T \alpha(t, T)_{|T=t}$$

(12)

for all $(t, T) \in \Delta$, and

$$-\dot{\beta}(t) = \kappa' \beta(t) + \frac{1}{2} \sum_{i=1}^{n} [\sigma'(\beta(t))]^2_i b_i - \dot{\beta}(0)$$

(13)

for all $t \in [0, \tau]$.

**Proof:** Same arguments to the proof of Lemma 1 apply with $\bar{\theta}$ in place of $\theta$. \(\square\)

**Definition 2.** Let $m \in \mathcal{M}_{n, \bar{\theta}}$, $\bar{\theta} \in \Theta(m)$, and $f_0^* \in C^1[0, \tau]$. Then a term structure model, $m(f_0^*, \bar{\theta})$, is called a mean reversion level extension of $m$ with initial curve $f_0^*$ and mean reversion level $\bar{\theta}$ if the following conditions are satisfied:

(a) There exists $\alpha \in C^{1,1}(\Delta)$ such that the bond price in $m(f_0^*, \bar{\theta})$ is given by

$$P_t(T) = e^{-\alpha(t, T) - \beta(T-t)'x_t},$$

(14)

where $x_t$ is the solution to (9) with mean reversion level $\bar{\theta}$ and initial value $x_0^* \in m$, and $\mathbb{P}$ is a martingale measure for $m(f_0^*, \bar{\theta})$.

(b) The forward rate curve, $f^*_t(\cdot)$, in $m(f_0^*, \bar{\theta})$ satisfies $f_0^*(T) = f^*_0(T)$.

**Proposition 1.** Let $m \in \mathcal{M}_{n, \bar{\theta}}$ and $f^*_0 \in C^1[0, \tau]$. Then for any $\bar{\theta} \in \Theta(m)$ there exists a unique mean reversion level extension $m(f^*_0, \bar{\theta})$ of $m$. The corresponding $\alpha^\theta \in C^{1,1}(\Delta)$ is given by

$$\alpha^\theta(t, T) = \int_t^T \zeta^\theta(s) ds + \int_t^T \bar{\theta}(s)' \kappa' \beta(T - s) ds - \frac{1}{2} \sum_{i=1}^{n} a_i \int_t^T [\sigma'(\beta(t - s))]^2_i ds,$$

(15)

where

$$\zeta^\theta(T) = f_0^*(T) - \int_0^T \bar{\theta}(s)' \kappa' \beta(T - s) ds$$

$$+ \frac{1}{2} \sum_{i=1}^{n} a_i \int_0^T \partial_T [\sigma'(\beta(T - s))]^2_i ds - \dot{\beta}(T)'x_0.$$

(16)

**Proof:** Let $m \in \mathcal{M}_{n, \bar{\theta}}$ and $\bar{\theta} \in \Theta(m)$. Then by definition of $\Theta(m)$, there exists a unique solution, $x_t$, to (9) for which $x_0^* \in m$ and $d(x_t) > 0$. Now, for any $(t, T) \in \Delta$, define $\alpha^\theta(t, T)$ according to (15). Then it is clear that $\alpha^\theta \in C^{1,1}(\Delta)$
and \( \alpha^\theta(T, T) = 0 \) for all \( T \in [0, \tau] \). Next, to verify property (a) it suffices to show, in view of Lemma 2, that \( \alpha^\theta \) satisfies (12), viz.

\[
\partial_t \alpha^\theta(t, T) = -\zeta^\theta(t) - \bar{\theta}(t)\kappa\beta(T-t) + \frac{1}{2} \sum_{i=1}^{n} a_i [\sigma^i \beta(T-t)]^2.
\]

So it remains to check that \( \zeta^\theta(t) = \partial_T \alpha(t, T) \mid_{T=t} \). But this follows immediately from the fact that \( \beta(0) = 0 \). Hence, setting the bond prices according to (14) defines a term structure model for which \( \mathbb{P} \) is a martingale measure. To establish the existence of \( m(\bar{\theta}, f^0) \), it remains to verify property (b). In view of (14), the forward rates, \( \bar{f}(\cdot) \), must be given by

\[
\bar{f}_0(T) = \zeta^\theta(T) + \int_0^T \bar{\theta}(s)\kappa\beta(T-s) \, ds
\]

\[
- \frac{1}{2} \sum_{i=1}^{n} a_i \int_0^T \partial_T [\sigma^i \beta(T-s)]^2 \, ds + \dot{\beta}(T)\bar{x}_0,
\]

since \( \bar{x}_0 = x_0^\theta \in \mathfrak{m} \). Substituting the expression for \( \zeta^\theta \) from (16) in the above equation gives \( \bar{f}_0(T) = f_0^\theta(T) \) as required. To establish uniqueness, suppose there exists \( \alpha \in C^{1,1}(\Delta) \) with the required properties. Then by Lemma 2, \( \alpha \) must satisfy (12). Define \( \varsigma \in C^0[0, \tau] \) by \( \varsigma(t) = \partial_T \alpha(t, T) \mid_{T=t} \). Then integrating (12) from \( t \) to \( T \), and using \( \alpha(T, T) = 0 \) gives

\[
\alpha(t, T) = \int_t^T \bar{\theta}(s)\kappa\beta(T-s) \, ds - \frac{1}{2} \sum_{i=1}^{n} a_i \int_t^T [\sigma^i \beta(T-s)]^2 \, ds + \int_t^T \varsigma(s) \, ds.
\]

Since the forward rates must be given by \( \bar{f}_t(T) = \partial_T \alpha(t, T) + \dot{\beta}(T-t)\bar{x}_t \), and \( \bar{f}_0(T) = f_0^\theta(T) \), setting \( t = 0 \) in the first equation gives

\[
\bar{f}_0^\theta(T) = \int_0^T \bar{\theta}(s)\kappa\beta(T-s) \, ds - \frac{1}{2} \sum_{i=1}^{n} a_i \int_0^T \partial_T [\sigma^i \beta(T-s)]^2 \, ds + \varsigma(T) + \dot{\beta}(T)\bar{x}_0,
\]

where we have used \( \beta(0) = 0 \). It follows that \( \varsigma = \zeta^\theta \) and \( \alpha = \alpha^\theta \). \( \square \)

**Proposition 2.** Let \( m, f^\theta_0, \bar{\theta}, \) and \( \varsigma^\theta \) be as given in Proposition 1. If \( f^\theta_0(\cdot) \) is the forward rate curve in \( m \) and \( f_\varsigma(\cdot) \) is the forward rate curve in \( m(f^\theta_0, \bar{\theta}) \), then

\[
\varsigma^\theta(t) = f_\varsigma^\theta(T) - f^\theta_0(T) - \int_0^T (\bar{\theta}(s) - \theta)'\kappa\dot{\beta}(T-s) \, ds + \varsigma^\circ,
\]

where \( \varsigma^\circ = \bar{A}(0) \).
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Proof: Calculations similar to those in the proof of Proposition 1 for \( \mathbf{m} \) shows that

\[
\begin{align*}
f^0_\varphi(T) &= \varsigma^0 + \int_0^T \theta'(T-s) \sigma'(T-s) \beta'(T-s) ds \\
&\quad - \frac{1}{2} \sum_{i=1}^n a_i \int_0^T \partial_T \left[ \sigma'(T-s) \beta'(T-s) \right]_i^2 ds + \beta'(T-s)x_0^i,
\end{align*}
\]

which can be written equivalently as

\[
\frac{1}{2} \sum_{i=1}^n a_i \int_0^T \partial_T \left[ \sigma'(T-s) \beta'(T-s) \right]_i^2 ds - \beta'(T-s)x_0^i.
\]

Substituting this into (16) gives

\[
\begin{align*}
\varsigma^\delta(T) &= \int_0^T \theta'(T-s) \sigma'(T-s) \beta'(T-s) ds \\
&\quad - \int_0^T \partial_T \left[ \sigma'(T-s) \beta'(T-s) \right]_i^2 ds + \beta'(T-s)x_0^i.
\end{align*}
\]

Substituting this into (16) gives

\[
\varsigma^\delta(T) = \int_0^T \theta'(T-s) \sigma'(T-s) \beta'(T-s) ds \\
&\quad - \int_0^T \partial_T \left[ \sigma'(T-s) \beta'(T-s) \right]_i^2 ds + \beta'(T-s)x_0^i.
\]

which is the required expression (17).

In Brigo and Mercurio (2001), extensions are considered for which the factor dynamics are left unchanged so that \( \bar{\theta} = \theta \) and the short rate, \( \bar{r}_t \), in \( \mathbf{m}(\theta, f^0_\varphi) \) is related to the short rate, \( r_t \), in \( \mathbf{m} \) by the equation

\[
\bar{r}_t = \varphi(t) + r_t,
\]

for some \( \varphi \in C^0[0, \tau] \).

**Corollary 1.** The deterministic short rate extension in Brigo and Mercurio (2001) is a special case corresponding to \( \bar{\theta} \equiv \theta \) and \( \varphi(t) = \varsigma^\delta(t) - \varsigma^0 \), where \( \varsigma^0 = \bar{A}(0) \).

**Proof:** Note that the short rate in \( \mathbf{m}(\theta, f^0_\varphi) \) is given \( \bar{r}_t = \varsigma^\delta(t) + \beta(0)'x_0 \), where \( \varsigma^\delta \) is as defined in (16), while the short rate in \( \mathbf{m} \) is given by \( r_t = \varsigma^0 + \beta(0)'x_0 \). Taking the difference of the two short rates gives \( \bar{r}_t = (\varsigma^\delta(t) - \varsigma^0) + r_t \), which is (18) with \( \varphi(t) = \varsigma^\delta(t) - \varsigma^0 \).

**Corollary 2.** If the short rate is the \( n \)-th factor in \( \mathbf{m} \) and \( \mathbf{m}(\bar{\theta}, f^0_\varphi) \), and \( [\kappa'(\beta(0))]_n \neq 0 \), then the short rate extension in Kwon (2004), and hence the extension in Björk and Hyll (2000) for ATSMs, is a special case corresponding to \( \bar{\theta}(t) = (\theta_1, \ldots, \theta_{n-1}, \bar{\theta}_n(t)) \), where \( \bar{\theta}_n \in C^0([0, \tau], \mathbb{R}^n) \) is the unique solution to the Volterra integral equation of the second kind

\[
\bar{\theta}_n(T) = \xi(T) + \int_0^T \bar{\theta}_n(s) \left[ \frac{\kappa'(\beta(T-s))}{[\kappa'(\beta(0))]_n} \right]_n ds,
\]
and where

\[
\xi(T) = \frac{1}{\kappa'\beta(0)} \left[ f^*_0(T) - \sum_{i=1}^{n-1} \theta_i \kappa' \beta(0) - \sum_{i=1}^{n-1} \theta_i \int_0^T \kappa' \beta(T - s) \, ds \right. \\
+ \left. \frac{1}{2} \sum_{i=1}^n a_i \int_0^T \sigma^2 \beta(T - s)^2 \, ds - \bar{\beta}(T)'x_0 \right].
\]

**Proof:** See Kwon (2004) Proposition 2. \(\square\)

4. Applications

This section considers some of the applications of the mean reversion level extensions introduced in the previous section.

4.1. Futures on Zero Coupon Bonds

Let \(m \in \mathcal{A}_{\mathcal{N}}\), \(P\), and denote by \(F_t(T, U)\) the price of a \(T\)-maturity futures on a \(U\)-maturity zero coupon bond where \(t \leq T \leq U \leq \tau\). Then \(F_t(T, U)\) can be written in the form

\[
F_t(T, U) = e^{-G(t,T,U) - \zeta(t,T,U)'x_t},
\]

where \(\eta\) and \(\zeta\) satisfy the equations

\[
\dot{\eta}(t, T, U) = -\theta' \kappa \zeta(t, T, U) + \frac{1}{2} \sum_{i=1}^n a_i |\sigma' \zeta(t, T, U)|^2_i,
\]

\[
\dot{\zeta}(t, T, U) = \kappa' \zeta(t, T, U) + \frac{1}{2} \sum_{i=1}^n |\sigma' \zeta(t, T, U)|^2_i b_i,
\]

and satisfy the conditions \(G(T, T, U) = A(U - T)\) and \(\zeta(T, T, U) = \beta(U - T)\).

**Proposition 3.** Let \(m \in \mathcal{A}_{\mathcal{N}}, \theta \in \Theta(m)\), and \(f^*_0 \in C^0[0, \tau]\). If the futures price, \(F_t(T, U)\) in \(m\), is given by (19), then the futures price in \(m(f^*_0, \theta)\) is given by

\[
\bar{F}_t(T, U) = e^{-\eta(t,T,U) - \zeta(t,T,U)'x_t},
\]

where \(\zeta\) is given by (21) and \(\eta\) is given by

\[
\eta(t, T, U) = \alpha^\theta(T, U) + \int_t^T \theta(s)' \kappa \zeta(s, T, U) \, ds \\
- \frac{1}{2} \sum_{i=1}^n a_i \int_t^T |\sigma' \zeta(s, T, U)|^2_i \, ds,
\]

with \(\alpha^\theta\) as defined in (15).
Proof: Since \(\bar{F}_t(T, U)\) is a \(\mathbb{P}\)-martingale, it follows from standard arguments that if \(\bar{F}_t(T, U)\) is is expressed in the form (22), then \(\zeta\) must satisfy (21) and \(\eta\) must satisfy

\[
\dot{\eta}(t, T, U) = -\bar{\theta}(t)\kappa\zeta(t, T, U) + \frac{1}{2} \sum_{i=1}^n a_i [\sigma'(t, T, U)]^2
\]

subject to \(\eta(T, T, U) = \alpha\bar{\theta}(U)\). Integrating the differential equation for \(\eta\) and using the terminal condition gives (23) as required. \(\square\)

Example 1. Consider the Vasicek (1977) model in which \(n = 1\), \(a_1 = 1\), \(b_1 = 0\), and \(x_t = r_t\) satisfies the equation

\[
dr_t = \kappa(\theta - r_t) dt + \sigma dw_t
\]

subject to \(r_0 = f^*_0(0)\), where \(f^*_0 \in C^0[0, \tau]\) is the initial forward rate curve. It is well-known that \(\beta\) and \(\zeta\) for this model are given by

\[
\beta(t) = \frac{1}{\kappa} (1 - e^{-\kappa t}), \quad \zeta(t, T, U) = \beta(U - t) - \beta(T - t).
\]

It is also well-known that the mean reversion level extension is uniquely determined and corresponds to \(\bar{\theta} \in C^0[0, \tau]\) given by

\[
\bar{\theta}(T) = f^*_0(T) + \frac{1}{\kappa} f^*_0(T) + \frac{\sigma^2}{2\kappa^2} (1 - e^{-2\kappa T}).
\]

So the bond futures prices in \(m(f^*_0, \bar{\theta})\) are given by

\[
\bar{F}(t, T, U) = e^{-\eta(t, T, U) - \zeta(t, T, U)r_t},
\]

where \(\zeta\) is as given in (26) and

\[
\eta(t, T, U) = \alpha^\bar{\theta}(T, U) + \kappa \int_t^T \bar{\theta}(s)\zeta(s, T, U) ds - \frac{\sigma^2}{2} \int_t^T \zeta(s, T, U)^2 ds
\]

with

\[
\alpha^\bar{\theta}(T, U) = \int_T^U \bar{\zeta}(s) ds + \kappa \int_T^U \bar{\theta}(s)\beta(U - s) ds - \frac{\sigma^2}{2} \int_T^U \beta(U - s) ds,
\]

\[
\bar{\zeta}(T) = f^*_0(T) - \kappa \int_0^T \bar{\theta}(s)\beta(T - s) ds + \frac{\sigma^2}{2} \beta(T)^2 - \beta(T)f^*_0(0).
\]
4.2. Fitting the Term Structure of Futures Prices

It was seen that mean reversion level extensions are not uniquely determined in general. One exception to this rule was for the short rate extensions in the case where \( n = 1 \). By requiring that the extension satisfies additional conditions, it is also possible to obtain uniqueness in higher dimensions.

Let \( m \in \mathcal{A}_2 \) and \( f_0^* \in C^0[0, \tau] \). Suppose we consider extensions for which \( \partial T_0 \alpha(t, T) \big|_{T=t} \equiv 0 \), and rather than just requiring that it be compatible with the initial forward rate curve, we also require that it matches the initial term structure of observed futures prices on zero coupon bonds. For example, suppose we also require that the extended model matches the initial term structure of futures prices \( F_0^*(T, \tau) \), where \( 0 \leq T \leq \tau \). It is shown that if an extension satisfying the above requirements exists, then it is unique.

**Proposition 4.** Let \( m \in \mathcal{A}_2 \), \( f_0^* \in C^0[0, \tau] \) and \( F_0^* \in C^0[0, \tau] \), and suppose

\[
M(T) = \begin{bmatrix}
\dot{\beta}(0)' \\
\dot{\zeta}(T, T, \tau)'
\end{bmatrix}
\]  

(32)

is non-singular for all \( T \in [0, \tau] \). If a mean reversion level extension, \( m(f_0^*, \bar{\theta}) \), exists for which \( \partial T_0 \alpha(t, T) \big|_{T=t} \equiv 0 \) and \( F_0(T, \tau) = F_0^*(T) \), then it is unique. If it exists, then the mean reversion level \( \bar{\theta} \in C^0([0, \tau], \mathbb{R}^2) \) is the unique solution to the 2-dimensional Volterra integral equation of the second kind

\[
\bar{\theta}(T) = \xi(T) + \int_0^T K(s, T)\theta(s) \, ds,
\]  

(33)

where

\[
K(s, T) = -\kappa^{-1}M(T)^{-1} \begin{bmatrix}
\dot{\beta}(T-s)' \\
\dot{\zeta}(s, T, \tau)'
\end{bmatrix}
\]  

(34)

\[
\xi(T) = \kappa^{-1}M(T)^{-1} \begin{bmatrix}
\nu_1(T) \\
\nu_2(T)
\end{bmatrix}
\]  

(35)

and

\[
\nu_1(T) = \dot{f}_0^*(T) + \frac{1}{2} \sum_{i=1}^2 a_i \partial_T \sigma_i' \beta(T-s) \big|_{s=T} + \frac{1}{2} \sum_{i=1}^2 a_i \int_0^T \partial_T^2 \sigma_i' \beta(T-s) \big|_{s=T} ds - \dot{\beta}(0)' x_0^*,
\]  

(36)

\[
\nu_2(T) = -\partial_T \ln F_0^*(T) + \frac{1}{2} \sum_{i=1}^2 a_i \partial_T \sigma_i' \zeta(s, T, \tau) \big|_{s=T} + \frac{1}{2} \sum_{i=1}^2 a_i \int_0^T \partial_T^2 \sigma_i' \zeta(s, T, \tau) \big|_{s=T} ds - \dot{\zeta}(0, T, \tau)' x_0^*.
\]  

(37)
Proof: Note firstly that the assumption $\partial_T \alpha(t, T)|_{T=1} = 0$ is equivalent to $\zeta^\theta = 0$. By (16), this is equivalent to

$$f_0^*(T) = \beta(T)' x_0 + \int_0^T \theta(s) \kappa'(\beta(T-s)) ds - \frac{1}{2} \sum_{i=1}^n a_i \int_0^T \partial_T |\sigma'(\beta(T-s))|^2_1 ds.$$

Differentiating with respect to $T$ and using $\beta(0) = 0$ gives

$$\dot{f}_0^*(T) = \beta(T)' x_0^* + \dot{\beta}(0)' \kappa\dot{\theta}(T) - \frac{1}{2} \sum_{i=1}^n a_i \partial_T [\sigma'(\beta(T-s))^2]_{s=T}^T \theta(s) ds - \frac{1}{2} \sum_{i=1}^n a_i \int_0^T \partial_T^2 [\sigma'(\beta(T-s))]^2_1.$$

(39)

Next, setting $U = \tau$ and $t = 0$ in (22) and equating to $F_0^*(T)$ gives

$$-\ln F_0^*(T) = \zeta(0, T, \tau)' x_0^* + \eta(0, T, \tau).$$

Substituting for $\eta(0, T, \tau)$ from (23) gives

$$-\ln F_0^*(T) = \zeta(0, T, \tau)' x_0^* + \int_0^T \theta(s) \kappa'(\tau-s) ds - \frac{1}{2} \sum_{i=1}^n a_i \int_0^T [\sigma'(\tau-s)]^2_1 ds$$

$$+ \int_0^T \theta(s) \kappa' \zeta(s, T, \tau) ds - \frac{1}{2} \sum_{i=1}^n a_i \int_0^T [\sigma'(s, T, \tau)]^2_1 ds.$$

Differentiating with respect to $T$, and using $\beta(0) = 0$ and $\zeta(T, T, \tau) = \beta(\tau - T)$ gives

$$-\partial_T \ln F_0^*(T) = \dot{\zeta}(0, T, \tau)' x_0^* + \int_0^T \dot{\zeta}(s, T, \tau)' \kappa \dot{\theta}(s) ds$$

$$- \frac{1}{2} \sum_{i=1}^n a_i \int_0^T \partial_T [\sigma'(\zeta(s, T, \tau))]^2_1 ds.$$

and differentiating again with respect to $T$ gives

$$-\partial_T^2 \ln F_0^*(T) = \ddot{\zeta}(0, T, \tau)' x_0^* + \dot{\zeta}(T, T, \tau)' \kappa \ddot{\theta}(T) + \int_0^T \dot{\zeta}(s, T, \tau)' \kappa \ddot{\theta}(s) ds$$

$$- \frac{1}{2} \sum_{i=1}^n a_i \partial_T [\sigma'(\zeta(s, T, \tau))]^2_1|_{s=T} - \frac{1}{2} \sum_{i=1}^n a_i \int_0^T \partial_T^2 [\sigma'(\zeta(s, T, \tau))]^2_1 ds.$$ (40)

Equations (39) and (40) imply that $\ddot{\theta}$ must satisfy (33).
4.3. Ensuring Positivity of Factor Volatilities

Let \( f^{*}_{0} \in [0, \tau] \) be the initial forward rate curve, and consider the Cox, Ingersoll and Ross (1985) model, \( m_{\text{cir}} \), corresponding to \( n = 1 \), where \( r_{t} = x_{t} \) satisfies

\[
dr_{t} = \kappa(\theta - r_{t}) \, dt + \sigma \sqrt{r_{t}} \, dw_{t},
\]

with \( r_{0} = f^{*}_{0}(0) \). It is well-known that provided \( 2\kappa\theta \geq \sigma^{2} \) the short rate remains positive and the above equation remains valid. The zero coupon bond prices in \( m_{\text{cir}} \) are given by

\[
P_{t}(T) = e^{-A(T-t) - \beta(T-t)r_{t}}, \quad (41)
\]

where \( \dot{A}(0) = 0 \) and

\[
\beta(t) = \frac{2(e^{\gamma t} - 1)}{(\kappa + \gamma)e^{\gamma t} + 2\gamma} \quad (42)
\]

where \( \gamma = \sqrt{\kappa^{2} + 2\sigma^{2}} \).

However, suppose in implementing this time-homogeneous model, the parameters calibrated to market data fail to satisfy the required condition \( 2\kappa\theta \geq \sigma^{2} \). Then \( r_{t} \) may become negative and lead to implementation problems. This can be overcome by considering, for example, the mean reversion extension, \( m_{\text{cir}}(\bar{\theta}, f^{*}_{0}) \), where we choose \( \bar{\theta} \in \mathbb{R} \) so that \( 2\kappa\bar{\theta} \geq \sigma^{2} \). The model \( m_{\text{cir}}(\bar{\theta}, f^{*}_{0}) \) is driven by a factor \( \bar{x}_{t} \) that satisfies the equation

\[
d\bar{x}_{t} = \kappa(\bar{\theta} - \bar{x}_{t}) \, dt + \sigma \sqrt{\bar{x}_{t}} \, dw_{t}, \quad (43)
\]

with \( \bar{x}_{0} = f^{*}_{0}(0) \). Since \( 2\kappa\bar{\theta} \geq \sigma^{2} \), it follows that \( \bar{x}_{t} \) remains positive. In this case, we have from (16) that

\[
\varphi^{\bar{\theta}}(T) = f^{*}_{0}(T) - \bar{\theta}\kappa\beta(T) - \bar{\beta}(T)f^{*}_{0}(0), \quad (44)
\]

so that from (15) we have

\[
\alpha^{\bar{\theta}}(t, T) = \int_{t}^{T} \left( f^{*}_{0}(s) + \bar{\theta}\kappa[\beta(T - s) - \beta(s)] - \bar{\beta}(s)f^{*}_{0}(0) \right) ds. \quad (45)
\]

So provided that \( P_{t}(T) = e^{-\alpha^{\bar{\theta}}(t, T) - \beta(T-t)\bar{x}_{t}} \) is a strictly decreasing function of \( T \) for all \( t \) and \( \bar{x}_{t} \), the extension results in a valid term structure model with well-defined factor dynamics and exact fit to the initial forward rate curve.
5. Conclusion

This paper introduced mean-reversion level extensions of time-homogeneous affine term structure models to the time-inhomogeneous setting to allow compatibility of these models with any given initial forward rate curve. These extensions are affine models within the HJM framework and retain much of the structure of the underlying models. The Hull-White extensions of the Vasiček (1977) and the Cox, Ingersoll and Ross (1985) models, the deterministic shift extensions in Björk and Hyll (2000) and Brigo and Mercurio (2001), and the short rate extensions in Kwon (2004) are all special cases of the mean-reversion level extensions. It was established that although such extensions are not unique in general, the uniqueness can be achieved by imposing additional conditions such as the fit to the initial term structure of futures prices. Since it follows from Filipović and Teichmann (2002) that tractable models from the Heath, Jarrow and Morton (1992) framework are affine, the extensions introduced in this paper provide a convenient subset of such models for which bond and futures prices are readily available.

References