Asset Pricing Under Asymmetric Information About Distribution of Risk Aversion

Qi Zeng†

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Abstract

We study a dynamic competitive equilibrium model with asymmetric information about time-variant aggregate risk aversion. We show that there still exists a linear price function in our model, a nice result in the other asymmetric information competitive equilibrium models. Furthermore, we show that in our model, asymmetric information play a role in the long run risk premium. When the proportion of informed agents is large, the risk premium will monotonically decrease as more agents become informed because of less uncertainty. When the proportion of informed agents is small, the adverse selection brought by the asymmetric information will increase the long-run risk premium. Furthermore, an increase in the noise about aggregate risk aversion may decrease risk premium when there is enough asymmetric information. We also show that our model predicts positive correlation between volume and absolute dividend change, volume and absolute price change. Expected future excess returns are negatively related to current volume and current dividend change, current volume and price change, current volume and current excess returns. All these are consistent with empirical facts in the literature.

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†Finance Department, Faculty of Economics and Commerce, University of Melbourne, VIC 3010, Australia. Email: qzeng@unimelb.edu.au
1 Introduction

Competitive equilibrium models with asymmetric information have seen a long history ever since the work by Grossman (1976). These models have shown that asymmetric information plays a significant role in stock price and trading volumes among others\textsuperscript{1}.

In this type of models, one usually assumes some additional additive noises as well as the stock payoff. This way the price function will not be fully revealing\textsuperscript{2}. The noises can be uncertainties about total shares available because of liquidity trading from noise traders\textsuperscript{3}, private investment opportunities\textsuperscript{4}, and additional stock payoffs\textsuperscript{5}. The additive form of noise makes the models much simpler because of the existence of linear price functionals in equilibrium. Our first contribution in this paper is to introduce uncertainty about aggregate risk aversion in a dynamic setting and show that it is still solvable.

Uncertainty about risk aversion is not new\textsuperscript{6}. But we are the first to study it in a dynamic setting. We build a model in which there are informed agents and uninformed agents. The aggregate risk aversion is time variant as well as the mean payoff of the stocks. Informed agents know about the true state of the world while uninformed agents can only infer about aggregate risk aversion and mean stock payoff from stock price and dividend. We show that in this framework a linear price function still exists, which holds the original merits of this type of models.

To make price function not fully revealing, we must introduce time-variant aggregate risk aversion. That aggregate risk aversion is time-variant has been shown in many works in literature\textsuperscript{7}. In our model, we use the simplest possible case, namely the aggregate risk aversion follows a AR(1) process. While more complicated and realistic assumptions can be made about this setup, our choice makes the solution of the model much simpler, and we will try to show that our intuitions are quite general.

One of the main results from asymmetric competitive equilibrium models is that risk premium is larger than symmetric information case (See Wang (1993)). This is quite intuitive since introducing additional noise will lower the stock price: agents require additional premium when holding stocks with more uncertainty in the economy. When agents are more

\textsuperscript{1}The earliest literature includes among others Grossman (1976)(1977)(1981), Admati (1985), De Long et al. (1990) etc.
\textsuperscript{3}See for examples, Grossman (1976), Admati (1984), Wang (1993) etc.
\textsuperscript{4}Wang (1994).
\textsuperscript{5}Llorente, Michaely, Saar, and Wang(2001)
\textsuperscript{6}See for example, the original paper by Grossman (1977).
\textsuperscript{7}See for examples, Campbell and Cochrane (1999) and the literature cited therein.
informed, uncertainty decreases and risk premium decreases as well. Asymmetric information itself does not play roles in the long-run risk premium. The second contribution of our paper is to show that although we still have linear price function, asymmetric information does play a role in the long-run risk premium. With nonzero mean aggregate risk aversion, asymmetric information will bring some nonlinear relationship between long-run risk premium and asymmetric information. When information asymmetry is large, uninformed agents fear that they will be taken advantage of the trading by informed agents. They may have less demand in risky stocks. The stock price decreases. The result is that risk premium may first increase then decrease when agents become more informed. Furthermore, increase noise in aggregate risk aversion may decrease risk premium when there is enough information asymmetry, a result not straight forwardly seen.

A most significant contribution from asymmetric information competitive equilibrium models is the study on stock trading volumes. In additional to the usual trading volumes from agents’ rebalancing their portfolios because of hedging demand, asymmetric information will introduce excess trading volume, sometimes not seen in the symmetric information case. An extreme situation is when agents have CARA utility function in which the stock holdings are fixed. As shown, for example, in Wang (1994) and Llorente et. al. (2001), asymmetric information have significant effects on trading volumes. Specifically, Wang (1994) shows that trading volumes are positively correlated to absolute dividend change and price change. Furthermore, expected future excess returns is positively related to current volume and current dividend change. It may be positive or negative related to current volume and current excess returns. Empirically, Karpoff (1987) surveys the literature showing that trading volume is positively correlated with the absolute value of contemporary price changes. Campbell, Grossman, and Wang (1993) shows that there exists reverse effect on the future expected excess returns and current volume and current excess returns.

Our third contribution is to show that in our model, we also obtain the results that trading volume is positively correlated with absolute dividend change and price change. Furthermore, we show that expected future excess returns is negatively related to current volume and current dividend change, current volume and current price change, current volume and current excess returns. This is consistent with all of the above empirical facts.

The literature for asymmetric information is vast, though the basic setup is presented in the above citations. There is another trend of literature on heterogeneous beliefs, as in Varian (1987), Abel (1989) and Basak (1998), which is also a motivation for this paper. We believe similar method can also be applied in this type of models and this amounts to another paper to explore.
The remainder of this paper is organized as follows. We study, as an illustration, the two period case in section 2. Then in section 3 we establish the results in a multi-period setting. In section 4 we conclude and have some discussions. All the proof details are in the appendix.

2 Two-Period Model

There are two periods \( t = 0 \) and \( t = 1 \). Agents do not consume at the first period and only maximize expected second period consumption. We have two assets in the economy. One stock and one bond. The price of the bond is taken as numeraire and its return is given exogenously as \( R \). The stock has current price \( p \) and will pay \( \tilde{y} \) in the second period. The price of the stock is determined endogenously.

Each agent \( a \) maximizes the expected utility of the second period \( U_a(W_1) \), where \( W_1 \) is the second period wealth and \( U_a(\cdot) \) is a CARA utility function

\[
U_a(W) = -e^{-\gamma_a W}.
\]  

(2.1)

Each agent has current wealth \( W_0 \). He buys \( \theta_a \) shares of stock at price \( p \). So next period wealth is

\[
\tilde{W}_1 = (W_0 - \theta_a p)R + \theta_a \tilde{y}
\]

\[
= W_0 R + \theta_a (\tilde{y} - pR).
\]  

(2.2)(2.3)

We will normalize the total amount of the stock to be \( K \). Since the bond return is given, we will not model the supply of the bond here.

Now we set the information structure of the agents. Each agent will have prior beliefs about the next period stock payoff:

\[
\tilde{y}_a \sim \mathcal{N}(m_a, s^2).
\]

That is, agents know the payoff is normal and they know the volatility of the stock payoff. The only difference is the heterogenous beliefs about the mean.

If that is all the uncertainty in the economy, we know that the rational equilibrium price will be fully revealing. So here we introduce another uncertainty. Instead of the usual assumption about some “noise” supply of the stock, we will assume that agents do not know the distribution of the risk aversion across the population. So we can imagine
the following scenario: before everything begins, nature will assign each agent with a mean and a risk aversion coefficient, which draw from the corresponding distributions. These two distributions (the mean of the stock returns and the risk aversion coefficients) are independent to each other across the population.

Agents maximize their expected utility function, $E(U_a(W_1)|F_a)$, where $F_a$ is the information set of agent $a$. Explicitly, this is

$$E(U_a(W_1)|F_a) = -E(e^{-\gamma_a(W_1)}|F_a)$$

(2.4)

$$= -e^{-\gamma_aW_0R}E(e^{-\gamma_a\theta_a(\tilde{y} - pR)}|F_a)$$

(2.5)

$$= -e^{-\gamma_a(W_0R - \theta_apR)}E(e^{-\gamma_a\theta_a\tilde{y}}|F_a)$$

(2.6)

Agents solve the maximization problem to get the stock demand $\theta_a$. As one can see, the critical step is how to get the expectations under the agents’ information set.

2.1 Fully-revealing Equilibrium

There are several possible equilibria definitions here depending on the information structure, namely on what agents’ information set $F_a$ will be. The equilibrium concept will be that (1) each agent maximizes his utility function given the information set; (2) markets clear, i.e. $\sum_a \theta_a = K$.

2.1.1 Equilibrium: No Learning

Here agents will ignore the information contents of the price and the only information they use is their own priors. Thus the perceived payoff the stocks will be a normal random variable with mean $m_a$ and volatility $s^2$. Specifically, we have

$$E(e^{\gamma_a\theta_a\tilde{y}}|F_a) = -e^{\gamma_a\theta_am_a + 1/2\gamma^2a\theta^2as^2}$$

So

$$E(U_a(W_1)|F_a) = -e^{-\gamma_aW_0R}e^{\gamma_a\theta_a(pR - m_a) + 1/2\gamma^2a\theta^2as^2}$$

Agents choose $\theta_a$ to maximize the above expected utility function. The result is

$$\theta_a = \frac{m_a - pR}{\gamma_a s^2}.$$ 

The market clearing condition is
\[
\sum_a \frac{m_a - pR}{\gamma_a s^2} = K.
\]

Define

\[
m \equiv \frac{1}{N} \sum_a m_a
\]

and \(\Gamma\) is the harmonic mean of individual risk aversion:

\[
\frac{1}{\Gamma} \equiv \frac{1}{N} \sum_a \frac{1}{\gamma_a},
\]

where \(N\) is the size of the population.

If we assume that \(\gamma_a\) and \(m_a\) are independent to each other across the population, we will have

**Proposition 1.** The equilibrium price function in the equilibrium without learning is

\[
p = \frac{m}{R} - \frac{K\Gamma s^2}{NR}.
\]

This result is similar to that in Abel (1989) (see equation (14) in that paper)\(^8\). It is actually quite intuitive:

- When \(m\) increases or \(s^2\) decreases, \(p\) increases. This is just the payoff and risk effect of the stocks.

- When \(\Gamma\) increases, \(p\) decreases. This says when on average agents become more risk averse, they will demand less stocks. The price of the stock goes down, equity premium goes up.

- When \(R\) goes up, \(p\) goes down. This is just the substitution effect between the stock and the riskless asset.

There is another way to interpretate this equilibrium concept. As in Varian (1989), one can distinguish between information and belief: When an agent takes expectation, his belief is the part which he will not change even if he knows others’ beliefs, while information is the part which he will update his if he knows others’ information. This definition induces the following case.

\(^8\)However, in that paper the riskless return is endogenous determined, so just the above equation will not be enough to pin down both the stock price and the riskfree rate. One needs the supply of the riskless assets.
In the usual asymmetric information literature, agents will have a common prior and the price will aggregate additional information of the market. So an agent will update his expectation about the payoff of the asset when he knows that price has this function. But what if he does not know the joint distribution of the price and the payoff? Say he only knows that the price is an aggregation of the market beliefs about the future payoff, but it may also be an aggregation of, say risk aversion. In our case, $\Gamma$ is a random variable agents know nothing about, except that it is random. So price will carry no information at all. Thus agents who observe the price will not update their beliefs about the mean of the payoff. Then the above equation is still the equilibrium price function.

### 2.1.2 Rational Expectation Equilibrium (REE) with Perfect Information about Distribution of Risk Aversion

Here agents’ risk aversion will be random distributed. However, now we let the agents be all informed. Namely that the price will be a function of aggregate risk aversion as well as the mean expected payoff, and agents all know this distribution.\(^9\)

To solve the equilibrium price, one begins by assuming that the current price $p$ is of the linear form about the average beliefs of the mean, $m \equiv E_a(m_a)$ and the aggregate risk aversion $\Gamma$ defined before. Namely

$$p = \alpha_0 + \alpha_1 m + \alpha_2 \Gamma.$$  

Imposing rational expectation requires that the average belief of the mean will be the “true” mean $m$. So another way to model it is to assume

$$m_a = m + \epsilon^m_a,$$

where $\epsilon^m_a$ is the deviation term. We will assume that the distribution of $\epsilon^m_a$ is the same for each agent and independent across agents.

Given that agents know $\Gamma$, if $m_a$ and $\epsilon^m_a$ are normally distributed, then under the above linear assumption of price and $m$, the three variables $(p, m_a, m)$ will be jointly normal. Thus given the information $(p, m_a)$ the agent will, by Bayes’ rule, get the distribution of $m$. Furthermore, one can see that $p$ is a sufficient statistics of $m$. Namely given $p$, one will have all the information about $m$ without any reference to his own prior. So this equilibrium is

\(^9\)There is one difference between our setting and the usual setting such as that in Grossman (1977), which comes from Abel (1989). In our setup, the uncertainty is the mean of the stock payoff. There is no uncertainty about the volatility of the stock payoff. So the price will be an aggregation of the mean. This is actually the extension in the dynamic case as in Wang (1993).
fully revealing. Given $p$,

$$m = \frac{p - \alpha_0 - \alpha_2 \Gamma}{\alpha_1}.$$ 

Thus from agents’ point of view, the stock payoff will be normally distributed with mean $\frac{p - \alpha_0 - \alpha_2 \Gamma}{\alpha_1}$ and variance $s^2$.

So we have

$$\tilde{y} - pR = \tilde{y} - (\alpha_0 + \alpha_1 m + \alpha_2 \Gamma)R$$

$$= -\alpha_0 R + (1 - \alpha_1 R)m - \alpha_2 R \Gamma + \epsilon_y,$$

where $\epsilon_y \sim \mathcal{N}(0, s^2)$.

The result of maximization will give us

$$\theta_a = \frac{-\alpha_0 R + (1 - \alpha_1 R)m - \alpha_2 R \Gamma}{\gamma_a s^2}$$

So using the market clearing condition, we have

$$\sum_a -\alpha_0 R + (1 - \alpha_1 R)m - \alpha_2 R \Gamma = K.$$ 

This has to be satisfied by any equilibrium price $p$. So by equating the constant term and the coefficient of $m$ and $\Gamma$, one will solve the three unknowns $\alpha_0, \alpha_1, \alpha_2$. The results are

$$\alpha_0^* = 0 \quad \text{(2.8)}$$

$$\alpha_1^* = \frac{1}{R} \quad \text{(2.9)}$$

$$\alpha_2^* = -\frac{K s^2}{NR}. \quad \text{(2.10)}$$

From the above discussion, we have the following

**Proposition 2.** The equilibrium price function in REE with fully-revealing price is

$$p^* = \frac{m}{R} \frac{K s^2}{NR}. \quad \text{(2.11)}$$
This result is the same as that in the previous section. The reason for this is the following: in REE with fully-revealing prices, each agent will not use his own prior belief because the price will reveal the “true” payoff \( m \). Thus each agent will act like the “average” investor in the previous section, who, by rational expectation requirement, knows the “true” payoff.

There are two different aspects between the two equilibrium though:

1. Even though they have the same price function, the demand of each agent for risky asset is different.

2. The price variable \( p \) and the mean payoff \( m \) are different: In the first case, they are just two numbers determined in equilibrium. In the second case, however, they are two random variables as shown in the above discussion. Actually, this is one of main departing points of REE with asymmetric information from those without learning: all the equilibrium prices are random variables.

### 2.2 REE with Uncertainty About Risk Aversion

The above fully revealing equilibrium makes the economy trivial. So we will introduce some additional uncertainty to make the economy not fully-revealing. The standard exercise is to use so-called “noise” supply of stocks. Here we will make the risk aversion uncertain.

To start with, let us recall the set up of the model. The next period payoff of the stock is distributed as normal \( \mathcal{N}(m, s^2) \).

As we mentioned before, there are two types of agents in the economy: informed and uninformed. The informed agents know the value \( m \) and uninformed agents only know that \( m \) is distributed as normal \( \mathcal{N}(\bar{m}, s^2_m) \). Let the fraction of informed agents to be \( \omega \in [0, 1] \). To prevent uninformed agents infer the value \( m \) from equilibrium price \( p \), there is uncertainty about the realization of \( \Gamma \). Informed agents know \( \Gamma \) while the uninformed agent only know that \( \Gamma \) is distributed as normal \( \mathcal{N}(\bar{\Gamma}, s^2_\Gamma) \).

To facilitate the presentation below, we write as follows:

\[
\begin{align*}
y & = m + \epsilon_y, \quad \epsilon_y \sim \mathcal{N}(0, s^2) \\
m_a & = m + \epsilon_a, \quad \epsilon_a \sim \mathcal{N}(0, s^2_a) \\
m & = \bar{m} + \epsilon_m, \quad \epsilon_m \sim \mathcal{N}(0, s^2_m) \\
\Gamma & = \bar{\Gamma} + \epsilon_\Gamma, \quad \epsilon_\Gamma \sim \mathcal{N}(0, s^2_\Gamma),
\end{align*}
\]

where we assume \((\epsilon_y, \epsilon_a, \epsilon_m, \epsilon_\Gamma)\) are jointly normal and uncorrelated to each other.
The uninformed agents only know their own prior about mean $m$, namely $m_a$, and the equilibrium price $p$. They try to infer $m$ (and $\Gamma$, for that matter) from their knowledge about the economy. Define $\hat{m} = E(m|\mathcal{F}^u) \equiv E^u(m)$ as the expectation about the $m$ of uninformed agents, where $E^u(\cdot)$ denotes the expectation operator of uninformed agents. We also denote $\hat{\Gamma} = E^u(\Gamma)$.

The plan for solving the problem is the following:

- We first conjecture that the equilibrium price is a linear function of $m$, $\Gamma$ and the expectations of uninformed agent.
- Note that from equation (2.5) the critical step is to get the share excess return, $\tilde{y} - pR$ so that we can calculate the expectation under both informed and uninformed agents information structure. To do this, we determine the estimation $\hat{m}$ given the price function. Then we calculate the expectations for both informed and uninformed agents;
- Maximizing utility function for both informed and uninformed agents will give the demand function.
- Markets clearing condition will give us the necessary equations to determine the coefficients in the price function.

### 2.2.1 Price Function

We start by making the following assumption and we will clarify later on why this is needed:

**Assumption 3.** For both informed and uninformed agents, their information and belief structure are independent of their risk aversion across population. Moreover, the population mean of risk aversion across informed and uninformed agents are the same $\Gamma$, namely

$$ \frac{1}{\Gamma} = \sum_i \frac{1}{\gamma_i^a} = \sum_u \frac{1}{\gamma_u^a}. \quad (2.16) $$

Then we conjecture that the equilibrium price is a linear function of $m$, $\Gamma$ and uninformed agents’ expectations.

**Proposition 4.** Under the above assumption, the equilibrium price is of the form

$$ p = \alpha_0 + \alpha_1 m + \alpha_2 \Gamma + \alpha_3 \hat{m}, \quad (2.17) $$

where $\alpha_0, \alpha_1, \alpha_2$ are constants to be determined, and

$$ \alpha_3 = \frac{1}{R} - \alpha_1. \quad (2.18) $$
Proof: See Appendix.

There are some observations about this proposition.

• We do not claim that this is the unique price function available for the model. But this is the unique linear price function.

• One might want to argue that the price function should also include the expectation about risk aversion (\(\hat{\Gamma}\)), which is correct. But we will show in the following that \(\hat{\Gamma}\) is not needed.

• The coefficient in front of \(\hat{m}\) is such that in the share excess return we only have two state variables. To see this, we substitute the price function into the share excess return to get the following:

\[
y - Rp = y - R(\alpha_0 + \alpha_1 m + \alpha_2 \Gamma + \alpha_3 \hat{m}) \\
= -R\alpha_0 + (1 - R\alpha_1)m - R\alpha_2 \Gamma - \alpha_3 R\hat{m} + \epsilon_y.
\]

For future reference, define \(\eta \equiv m - \hat{m}\), which is the difference between the knowledge about the future payoff of informed agents and that of uninformed agents. In fully-revealing economy, both informed and uninformed agents will have same information, while now there will be difference between them. With this, the above expression is then

\[
y - pR = -\alpha_0 R - \alpha_2 R\Gamma + (1 - \alpha_1 R)\eta + \epsilon_y. \tag{2.19}
\]

The term of \(m\) is canceled because of the form \(\alpha_3\) takes.

• There are three coefficients to be determined. We expect to have three equations after the markets clearing condition. Thinking ahead, if we can have three equations from equating the coefficients in front of 1 (constant term), \(\Gamma\) and \(\eta\), it is then solved. As we will show, it is indeed the case.

The next several sections will derive the solutions for \((\alpha_0, \alpha_1, \alpha_2)\). We now derive some analysis for this price function taken for granted for the properties of \((\alpha_0, \alpha_1, \alpha_2)\). The intuition studied here will carry over to the dynamic setting.

Using the results from previous section in equation (2.8) to (2.10), we can rewrite the price function as follows:

\[
p = p^* + \left[\alpha_0 + \left(\frac{1}{R} - \alpha_1\right)(\hat{m} - m) + \alpha_2 + \frac{Ks^2}{NR}\right]\Gamma. \tag{2.20}
\]
\( p^* \) is the stock price when agents are all informed. In other words, when \( \omega = 1 \), \( p = p^* \). This amounts to say that when \( \omega = 1 \), \( \alpha_0 = \alpha_0^* = 0 \), \( \alpha_1 = \alpha_1^* \) and \( \alpha_2 = \alpha_2^* \).

So whatever in the bracket \([\square]\) will be the deviation when \( \omega < 1 \), namely when there is uninformed agents in the economy. Let us study it in more detail.

- First for the second term: Note that \( \eta \equiv \hat{m} - m \) is the deviation of expected mean payoff from the true value for the uninformed agents. When uninformed agents are optimistic, namely \( \eta > 0 \), we will expect that they will buy more of stocks thus drive up the stock price. So it must be that \( \alpha_1 < 1/R \) when \( \omega < 1 \).

- Furthermore, for the price function in the proposition, we will expect that \( \alpha_1 > 0 \) and \( \alpha_2 < 0 \), simply for the fact that price should increase with the expected payoff and decrease with the average risk aversion.

- When \( \omega = 0 \), namely when all agents are uninformed, we expect \( \alpha_1 = 0 \) since now \( m \) will not be in agents’ information set. So when \( \omega \) goes from 0 to 1, \( \alpha_1 \) will increase from 0 to \( 1/R \).

- \( \alpha_2 \) represents the sensitivity of the change in price for the change in mean risk aversion. When \( \omega \) goes from 0 to 1, on the one hand, agents on average are more informed, thus the price should go up (need less premium for the uncertainty), on the other hand, uninformed agents will be taken advantage by the informed agents, the price might go down (the premium for the uninformed agents will be higher). These two forces will make the change of \( \alpha_2 \) nonmonotonic. When \( s_\Gamma^2 \) is larger, the first effect will dominate since agents on average really want to be informed to be less uncertain about \( \Gamma \).

### 2.2.2 Estimation of Uninformed Agents

Given the price functional form, we know that for an uninformed agents, they will know \( \alpha_1 m + \alpha_2 \Gamma \) when they observe price since \( \hat{m} \) are their own estimations. Let us define:

\[
\Lambda \equiv \alpha_1 m + \alpha_2 \Gamma. \tag{2.21}
\]

Also the uninformed agents know that \( (m, \Lambda, m_a) \) are jointly normal. So from the properties of multi-variate normal distribution, we have

\[
\hat{m} = E^u(m|p, m_a) \tag{2.22}
\]

\[
= E^u(m|\Lambda, m_a) \tag{2.23}
\]

\[
= \beta_0 + \beta_1 \Lambda + \beta_2 m_a, \tag{2.24}
\]
where $\beta_0, \beta_1, \beta_2$ are functions of parameters and $\alpha$’s.

Also we define

$$v_a^2 \equiv \text{var}^u(m|\Lambda, m_a).$$  \hspace{1cm} (2.25)

We have the following:

**Proposition 5.** The above $\beta$’s and $v_a$ have the following form

\[
\begin{align*}
\beta_{0a} &= \frac{1}{|\Sigma_{22}|}((\alpha_1^2 s_m^2 s_a^2 - \alpha_2^2 s_\Gamma^2 s_a^2)m - \alpha_1 s_m^2 s_a^2 \Lambda) \hspace{1cm} (2.26) \\
\beta_{1a} &= \frac{\alpha_1 s_m^2 s_a^2}{|\Sigma_{22}|} \hspace{1cm} (2.27) \\
\beta_{2a} &= \frac{\alpha_2^2 s_\Gamma^2 s_a^2}{|\Sigma_{22}|} \hspace{1cm} (2.28) \\
v_a^2 &= \frac{\alpha_2^2 s_\Gamma^2 s_m^2 s_a^2}{|\Sigma_{22}|}, \hspace{1cm} (2.29)
\end{align*}
\]

where

$$|\Sigma_{22}| = \alpha_1^2 s_m^2 s_a^2 + \alpha_2^2 s_\Gamma^2 (s_m^2 + s_a^2). \hspace{1cm} (2.30)$$

Proof: See appendix.

So the coefficients ($\beta_1, \beta_2$) and $v_a^2$ are just functions of parameters and $\alpha$’s. And given these estimations we have

\[
\eta = m - \hat{m}
\]

\[
\begin{align*}
&= m - (\beta_0 + \beta_1 \Lambda + \beta_2 m_a) \\
&= -\beta_0 - \beta_1 \alpha_2 \Gamma + (1 - \beta_2 - \beta_1 \alpha_1)m - \beta_2 \epsilon_a, \\
\end{align*}
\]

and

\[
\text{var}^u(\eta) = (\beta_1 \alpha_2)^2 s_\Gamma^2 + (1 - \beta_2 - \beta_1 \alpha_1)^2 s_m^2 + b_2^2 s_a^2. \hspace{1cm} (2.32)
\]

Before we leave this section, we note that since uninformed agents know $\Lambda = \alpha_1 m + \alpha_2 \Gamma$, we have, by taking expectation with respect to uninformed agents’ information set:

$$\alpha_1 m + \alpha_2 \Gamma = \alpha_1 \hat{m} + \alpha_2 \hat{\Gamma}. \hspace{1cm} (2.33)$$

Or put the other way:

$$\alpha_2 (\hat{\Gamma} - \Gamma) = \alpha_1 \eta. \hspace{1cm} (2.34)$$
2.2.3 Portfolio Choice for Informed and Uninformed Agents

Now that we have the estimation of uninformed agents, we proceed to calculate the portfolio choice for both informed and uninformed agents.

Let us first recall the share excess return (2.19):

\[ y - pR = -\alpha_0 R - \alpha_2 R\Gamma + (1 - \alpha_1 R)\eta + \epsilon_y. \]

For informed agents, they know the value of \( \Gamma \) and \( m \), thus \( \eta \). So the only uncertainty will come from \( \epsilon_y \). Define the state variables for informed agents

\[ \psi^i \equiv (1, \Gamma, \eta)^T. \] (2.35)

Specifically, we have

\[ E^i(y - pR) = -R\alpha_0 - \alpha_2 R\Gamma + (1 - \alpha_1 R)\eta, \] (2.36)

and

\[ \text{var}^i(y - pR) = s^2. \] (2.37)

We have the following

**Proposition 6.** The optimal portfolio choice of an informed agent is

\[ \theta^i = \frac{X^i\psi^i}{s^2\gamma_a^i}, \] (2.38)

where

\[ X^i \equiv (-R\alpha_0, -\alpha_2 R, 1 - \alpha_1 R). \] (2.39)

Proof: See appendix.

Note that \( X^i \) is a constant matrix function only of \( \alpha \)'s. And the critical part is that \( X^i \) does not depend on \( \gamma_a^i \) of the agent, which only appears in the denominator of \( \theta^i \). As we will immediately see, similar properties hold for uninformed agents too. And when we aggregate the demand for stocks, these properties will guarantee the results are linear in \( (\Gamma, \eta) \) so we can solve for the unknown \( \alpha \)'s.

For uninformed agents, they will not observe \( (\Gamma, m) \). If we take the expectations, we have

\[ E^u(y - pR) = -R\alpha_0 - \alpha_2 R\dot{\Gamma}. \] (2.40)
The state variable for uninformed agents is thus
\[ \psi^u \equiv (1, \hat{\Gamma})^T. \] (2.41)

The uncertainty term will be \((1 - \alpha_1 R)\eta + \epsilon_y\). Note that \(\eta\) and \(\epsilon_y\) are uncorrelated, we obtain
\[ \text{var}^u(y - pR) = (1 - \alpha_1 R)^2 \text{var}(\eta) + s^2 \equiv v^2_u. \] (2.42)

We have the following:

**Proposition 7.** The optimal portfolio choice of an uninformed agent is
\[ \theta^u = \frac{X^u \psi^u}{v^2_u \gamma_a}, \] (2.43)
\[ \text{where} \]
\[ X^u \equiv (-R\alpha_0, -\alpha_2 R). \] (2.44)

Proof: See appendix.

Here again, \(X^u\) and \(v^2_u\) do not depend on \(\gamma_a\), which only appears in the denominator of demand function.

### 2.2.4 Equilibrium

Given the above demand for both informed and uninformed agents, we now use the market clearing condition to get the undetermined coefficients \(\alpha\)'s.

The market clearing condition is \(\sum_a \theta^a = K\). Or put the other way, we have
\[ \sum_i \theta^i + \sum_u \theta^u = K. \] (2.45)

Using the results from the previous section, we have
\[ \theta^i = \frac{1}{\gamma} \left( -\frac{\alpha_0 R}{s^2} - \frac{\alpha_2 R}{s^2} \Gamma + \frac{(1 - \alpha_1 R)}{s^2} \eta \right) \] (2.46)
\[ \theta^u = \frac{1}{\gamma} \left( -\frac{\alpha_0 R}{v^2_u} - \frac{\alpha_2 R}{v^2_u} \hat{\Gamma} \right). \] (2.47)

Note we have shown that \(\alpha_1 \hat{m} + \alpha_2 \hat{\Gamma} = \alpha_1 \hat{m} + \alpha_2 \hat{\Gamma}\), or \(\hat{\Gamma} = \Gamma + \alpha_1/\alpha_2 \eta\) So we can substitute \(\hat{\Gamma}\) using \((\Gamma, \eta)\). The market clearing condition will have only three independent variables \((1, \Gamma, \eta)\). By letting the coefficients of these three variables to equal zero, we will
have three equations. But in order for us to be able to do this, we need the following assumption:

**Assumption 8.** For both informed and uninformed agents, their information and belief structure are independent of their risk aversion across population. Moreover, the population mean of risk aversion across informed and uninformed agents are the same $\Gamma$, namely

$$\frac{1}{\Gamma} = \sum_i \frac{1}{\gamma_i} = \sum_u \frac{1}{\gamma_u}.$$  \hfill (2.48)

The first part of this assumption says that agents' risk aversions are independent of what their prior beliefs or private information about the future payoff of the stocks, which, we think, is a reasonable assumption. The purpose of this assumption is, as in the case no learning, to separate the information part from the risk aversion part in aggregation across population.

The second part of this assumptions says the aggregation of risk aversion for both informed and uninformed agents are the same as that in the whole population aggregation. This is a little stronger than the first part. We think, when the numbers of both informed and uninformed agents are large, and since the draw of risk aversion are independent of information and belief structure, this will represent a reasonable assumption. The purpose of this assumption is just to simplify the aggregation of informed and uninformed agents’ risk aversion to be just that in the whole population.

Of course, one may also make some other assumptions on the aggregation of informed and uninformed agents’ risk aversion. And as long as they are linear functions of $\Gamma$. The method still applies.

So we have three equations with three unknown $\alpha$’s. We can solve for $\alpha$’s as functions of the parameters in the economy.

Instead of studying more on the static model, we now turn our attention to the dynamic model from the next section on.

### 3 Multi-period Model

The Model setup here is similar to that in Wang (1994). Informed agents know all the state variables while uninformed agents do not, they have to infer the value of unobservables from the price. And the price is a function of both state variables and uninformed agents’
expectations\textsuperscript{10}. The process is similar as what we have done in two period model.

3.1 Model

The model is as follows:

- Infinite horizon, $t = \ldots, -1, 0, 1, \ldots$. Note that we only discuss the steady state economy so we let the economy starts with $-\infty$;

- Two assets, one stock and one bond. Again, we will assume the gross return of bond is constant and denoted as $R$. The stock will pay dividend $D_t$. Let us assume that as in one-period model, $D_t$ follows the process

$$D_t = \bar{m} + m_t + \epsilon_t^D,$$

(3.1)

where $\bar{m}$ is the long run average dividend, and $m_t$ follows

$$m_t = a_m m_{t-1} + \epsilon_t^m.$$  

(3.2)

Here $\epsilon_t^D$ and $\epsilon_t^m$ are i.i.d shocks to $D_t$ and $m_t$ respectively. So here $m_t$ follows AR(1) process.

- Preferences: All investors have CARA utility, they maximize expected utility of the following form

$$E_t \left\{ -\sum_{s=0}^{\infty} (1-\lambda)^s \beta^s e^{-\gamma_a c_{t+s}} \right\}.$$  

(3.3)

Here $E_t$ is the expectations operator conditional on the investors’ information at time $t$. $\beta$ is the time discount factor. $\gamma_a$ is the risk aversion parameter for investor $a$. To make sure that risk aversion is time variant, we assume that at each time period, with probability $\lambda$ the investor will leave the market forever\textsuperscript{11}. And there will be same amount of investors come into the market. Thus the risk aversion of the market is time variant. We assume that it follows the process

$$\Gamma_t = a_{\Gamma} \Gamma_{t-1} + \epsilon_t^\Gamma.$$  

(3.4)

\textsuperscript{10}The fact that we use discrete time setup in the main text is completely coincidental. Actually continuous time version is preferred in some cases because of its simplicity. We will give a compact version of continuous time model at the last part of appendix.

\textsuperscript{11}Assuming $\lambda$ is the probability that investor will die at each time period is another interpretation. But in this case, there maybe the issue of annuity market.
where \(1/(\bar{\Gamma} + \Gamma) = \sum_a 1/\gamma_a\). Here \(\bar{\Gamma}\) is the unconditional mean of the aggregate risk aversion and \(\Gamma_t\) is the deviation, which follows AR(1) process.

If one wants to connect \(a_\Gamma\) with \(\lambda\), then the following argument will give \(a_\Gamma = 1 - \lambda\): With the leaving of \(\lambda\) population, one will have \((1 - \lambda)\Gamma_{t-1}\), and that of the added \(\lambda\) portion will be \(\epsilon_t^\Gamma\).

- **Information structure:** There are two types of agents in the economy. The informed agents observe all the state variables such as \((m_t, \Gamma_t)\) at time \(t\), while uninformed agents receive a signal about the mean of the dividend, \(m_t\):

\[
m_t^u = m_t + \epsilon_t^a.
\]  

(3.5)

So at time \(t\), the information of an uninformed investor is \(\mathcal{F}_t^u = \{D_s, P_s, m_s^u|s \leq t\}\), while the information of an informed investor is \(\mathcal{F}_t^i = \{D_s, P_s, m_s, \Gamma_s|s \leq t\}\).

Let the fraction of informed investors be \(\omega\) and that of uninformed investor is \(1 - \omega\) \((0 \leq \omega \leq 1)\).

- **Distributional Assumptions:** We assume that all shocks \(\epsilon_t^D, \epsilon_t^m, \epsilon_t^\Gamma, \epsilon_t^a\) are jointly normal and i.i.d. over time. For simplicity, we will assume that they are all uncorrelated.

### 3.2 Price Function

As in the previous section, the state variables include \((m, \Gamma)\). But in addition to these, the equilibrium price also depends on the expectation an uninformed agent forms about the true state variables. In the previous section, we ignore this and in fact only consider the true state variables\(^{12}\).

Let \(\bar{m}_t \equiv E_t(m_t|\mathcal{F}_t^u)\), the expectation of uninformed investor about the mean payoff of the stock.

**Proposition 9.** The economy has a steady state rational expectations equilibrium with the equilibrium stock price

\[
P_t = \alpha_0 + (\alpha - \alpha_1)\bar{m}_t + \alpha_1 m_t + \alpha_2 \Gamma_t,
\]  

(3.6)

where \(\alpha_0, \alpha_1, \alpha_2\) are the coefficients to be determined, and \(\alpha = a_m/(R - a_m)\).

\(^{12}\)There are extensively literature in discussing the higher order expectations as state variables, which basically is in the heart of model in which one has to have informed agent to avoid this.
Proof: See appendix.

Again, this proposition says that price is a function of state variables \((\hat{m}_t, m_t, \Gamma_t)\)\(^{13}\). Note however, we still have three coefficients to be determined. We will show later that the coefficient of \(\hat{m}_t\) is necessary.

### 3.3 Estimation of The Uninformed Agents

The task of the uninformed agent is to estimate the distribution of the state variables \(z_t \equiv (m_t, \Gamma_t)\) from the observables, \((D_\tau, P_\tau, m^\tau_a), \tau \leq t\). Again we define

\[
\Lambda_t \equiv \alpha_1 m_t + \alpha_2 \Gamma_t. \quad (3.7)
\]

Then observing \(P_t\) is equivalent to observing \(\Lambda_t\). So we define the observed variable as

\[
s^T_t \equiv (m^a_t, D_t, \Lambda_t)^T.
\]

Similarly, we have

**Proposition 10.** Given \(\mathcal{F}_t = \{D_s, P_s, m^a_s : s \leq t\}\), the conditional expectation of the dividend for agent \(a\), \(\hat{m}_t^a = E^a_t(m_t)\), is given by the following Kalman filter equation:

\[
\begin{pmatrix}
\hat{m}_t^a \\
\hat{\Gamma}_t^a
\end{pmatrix} = \begin{pmatrix}
E_{t-1}(m_t) \\
E_{t-1}(\Gamma_t)
\end{pmatrix} + \kappa(s_t - E_{t-1}(s_t)),
\]

where \(s_t\) are observables, \(\kappa\) are constant, it is a function of \(\alpha\)'s and parameters in the evolution functions.

Proof: See Appendix.

One thing to note here is that since \(\Lambda_t\) is known to uninformed agent from the stock price. One have the following

\[
\alpha_1 m_t + \alpha_2 \Gamma_t = \alpha_1 \hat{m}_t + \alpha_2 \hat{\Gamma}_t. \quad (3.9)
\]

So \(\hat{\Gamma}_t\) is redundant.

### 3.4 Share Excess Return

We recall that the share excess return is

\(^{13}\)One might think that price also should depend on \(\hat{\Gamma}_t\). We will show later that in fact \(\hat{\Gamma}\) is redundant.
\[ Y_{t+1} = P_{t+1} + D_{t+1} - R P_t, \]  
\[(3.10)\]

and \( \eta_t = m_t - \hat{m}_t. \)

With the price function and estimation of the uninformed agent, we can get the expectations of the agents about the share excess return.

**Proposition 11.** The share excess return has the following form:

\[ Y_{t+1} = \bar{m} + (1 - R)\alpha_0 + (a_\Gamma - R)\alpha_2 \Gamma_t + (\alpha - \alpha_1)(R - a_\eta)\eta_t + \epsilon_{Y,t+1}, \]

\[(3.11)\]

where \( \epsilon_{Y,t+1} \) is a linear function in \( \epsilon_{t+1}. \)

Proof: Substitute the price function into the excess return definition and we get the result.

With this expression, we can see that for informed agents the conditional expectation of the return is

\[ E(Y_{t+1}|\mathcal{F}_t) = \bar{m} + (1 - R)\alpha_0 + (a_\Gamma - R)\alpha_2 \Gamma_t + (\alpha - \alpha_1)(R - a_\eta)\eta_t. \]

\[(3.12)\]

And that for uninformed agents is

\[ E(Y_{t+1}|\mathcal{F}_u) = \bar{m} + (1 - R)\alpha_0 + (a_\Gamma - R)\alpha_2 \hat{\Gamma}_t. \]

\[(3.13)\]

Namely the state variables for informed agents \( \psi^i_t \) is \( (1, \Gamma_t, \eta_t) \), and the state variables for uninformed agents \( \psi^u_t \) is \( (1, \hat{\Gamma}_t) \).

### 3.5 Optimal Portfolio Choice of Individual Agents

The problem of the agent \( a \) is the following

\[ V^a(W^a_t; z^a_t; t) \equiv \max_{c, \theta_t} E_t^a(- \sum_{s=0}^{\infty} ((1 - \lambda)\beta)^s e^{-\gamma a c_{t+s}}) \]

\[(3.14)\]

subject to the budget constraint

\[ W^a_{t+1} = (W^a_t - c^a_t)R + \theta^a_t(P_{t+1} + D_{t+1} - R P_t), \]

\[(3.15)\]

where \( W_t \) is the wealth, \( \theta_t \) is the share of stock holdings\(^\dagger\).

\(^\dagger\)In this section we will omit superscript \( a \) if there is no confusion.
With the above estimation results, one can solve the optimal portfolio choice problem of the agent. The value function can be written as\(^{15}\)

\[
V(W_t; \psi_t; t) \equiv \max_{c, \theta} (-\beta^t e^{-\gamma c_t} + E_t(V(W_{t+1}; \psi_{t+1}; t+1))).
\] (3.16)

**Proposition 12.** The informed agent’s value function has the following form

\[
V^i(W_t; \psi^i_t, t) = -\beta^t e^{-b^i W_t - 1/2(\psi^i_t)^T v^i_i \psi^i_t},
\] (3.17)

where \(b^i\) is a constant, \(v^i\) is a constant matrix not depending on \(\gamma\). And the optimal shares to buy is

\[
\theta^i_t = \frac{R}{(R - 1) \gamma} (X^i)^T \psi^i_t,
\] (3.18)

where \(X^i\) does not depend on \(\gamma\).

Proof: See appendix.

**Proposition 13.** The uninformed agent’s value function has the following form

\[
V^u(W_t; \psi^u_t, t) = -\beta^t e^{-b^u W_t - 1/2(\psi^u_t)^T v^u_i \psi^u_t},
\] (3.19)

where \(b^u\) is a constant, \(v^u\) is a constant matrix not depending on \(\gamma\). And the optimal shares to buy is

\[
\theta^u_t = \frac{R}{(R - 1) \gamma} (X^u)^T \psi^u_t,
\] (3.20)

where \(X^u\) does not depend on \(\gamma\).

Proof: See appendix.

### 3.6 Equilibrium

Now that we have solved the optimal consumption and portfolio choice problem for individual agent, we will use market clearing condition to get the coefficients of the price function. Note that all the variables so far can be expressed as functions of the coefficients of the price function and the exogeneous constant in the law of motions.

The market clearing condition is

\[
\sum_a \theta^a_t = K.
\] (3.21)

\(^{15}\)We only use \(\beta\) with the understanding that it actually represents \((1 - \lambda)\beta\) in the following equations.
From the portfolio choice results in the previous section, we have

\[
\theta^i_t = \frac{1}{\gamma} \left( a_{i0}^i + a_{i1}^i \Gamma_t + a_{i2}^i \eta_t \right) \quad (3.22)
\]

\[
\theta^u_t = \frac{1}{\gamma} \left( a_{u0}^u + a_{u1}^u \hat{\Gamma}_t \right), \quad (3.23)
\]

where the coefficients are functions of \(\alpha\)'s and parameters, and they do not depend on \(\gamma\).

Now if we assume that the information structure is independent of the distribution of risk aversion, then the summation cross population will separate \(\Gamma\) from those other variables. And we can solve the price function. (See appendix for detail proof).

4 Prices, Volume and Asymmetric Information

In this section, we will study the relationship between asset prices, volume and asymmetric information. Specifically we want to study the differences between asymmetric information about aggregate risk aversion and about other uncertainty such as private investment opportunities (Wang, 1994).

4.1 Asset Pricing and Risk Premium

Here we study the effects of asymmetric information about aggregate risk aversion on the stock prices and long term risk premium. As we soon will see, instead of monotonic relationship between price coefficients, \(\alpha_i, i = 0, 1, 2\) and the degree of asymmetric information, \(\omega\), as in Wang (1993, 1994), the relationships actually depends on \(\omega\).

We first take a look at the situation where all the agents are informed\(^{16}\), namely \(\omega = 1\). In this case, there is no estimation problem. So the price function will be:

\[
P^*_{t} = \alpha_0^* + \alpha m_t + \alpha_2^* \Gamma_t, \quad (4.1)
\]

where \(\alpha_2^* < 0\), and \(\alpha = a_m/(R - a_m)\).

Fig 1-4 show the general situation for the price coefficients \(\alpha_0, \alpha_1, \alpha - \alpha_1, \alpha_2\)\(^{17}\). First let us look at Fig. 2 and 3. This shows the sensitivity of price change to the change in \(m\) and \(\hat{m}\). We can rewrite the price function as follows:

\[
P_t = \alpha_0 + (\alpha - \alpha_1)(\hat{m} - m) + \alpha m_t + \alpha_2 \Gamma_t, \quad (4.2)
\]

---

\(^{16}\)This is a special case of the above general results. We do not give proofs for the results here.

\(^{17}\)Unless otherwise noted in the text, we want to show the qualitative properties of the relationship. The results shown here are robust to different parameters.
This figure plots $\alpha_0$ against $\omega$, the proportion of informed agents in the economy.

Here $\gamma = 2, k = 1, \lambda = 0.1, \beta = 0.98$,
$\sigma_\Gamma = 0.9, a_m = 0.9, \bar{m} = 1, \sigma_a = .2, \sigma_d = .2, \sigma_m = .2$.
$\sigma_\Gamma$: solid line: 0.1, dashed line: 0.2, dotted line: 0.3

**Figure 1** : $\alpha_0$ vs. $\omega$

where $\hat{m} - m$ is the difference between the expectation of uninformed agents about the mean payoff of dividend and the actual value. When $\omega = 1$, namely all agents are informed, $\alpha - \alpha_1 = 0$. When $\omega \in [0,1)$, $\alpha - \alpha_1 > 0$. This means the price is positive correlated with the estimation error of uninformed agents. When uninformed agents are more optimistic about the mean payoff, stock price increases. When uninformed agents are more pessimistic about the mean payoff, stock price decreases. Moreover, $\alpha_1 > 0, \alpha - \alpha_1 > 0$. This means that when $m$ increases, namely when the mean payoff of the stock increases, the stock price increases. This is in consistent with the results in Wang (1993). One thing needs to note is that $\alpha_1$ does not depend on $\sigma_\Gamma$, as one can see that the three different curves overlap on each other. So increasing $\sigma_\Gamma$ only changes the pattern for $\alpha_0$ and $\alpha_2$.

Fig. 4 shows the sensitivity of stock price to the shocks to the aggregate risk aversion, $\alpha_2$. First of all, note that $\alpha_2 < 0$. This is quite intuitive in that when the aggregate risk aversion $\Gamma$ increases, the price of the stock will decrease, and expected return increases. Furthermore, when $\omega = 1$, $\alpha_2$ is larger than when $\omega = 0$. This is again very intuitive. When agents are all informed, $\omega = 1$, there are less uncertainty than then agents are all uninformed, $\omega = 0$. Thus the stock price is higher when $\omega = 1$ than that when $\omega = 0$. This is purely the effect of appearance of more uncertainty. However, in between these two extremes, one can see that
This figure plots $\alpha_1$ against $\omega$, the proportion of informed agents in the economy.

Here $\tau = 2, k = 1, \lambda = 0.1, \beta = 0.98$,
$\alpha_T = 0.9, \alpha_m = 0.9, \bar{m} = 1, \sigma_a = .2, \sigma_d = .2, \sigma_m = .2$.
$\sigma_T$: solid line: 0.1, dashed line: 0.2, dotted line: 0.3

**Figure 2 : $\alpha_1$ vs. $\omega$**

This figure plots $\alpha - \alpha_1$ against $\omega$, the proportion of informed agents in the economy.

Here $\tau = 2, k = 1, \lambda = 0.1, \beta = 0.98$,
$\alpha_T = 0.9, \alpha_m = 0.9, \bar{m} = 1, \sigma_a = .2, \sigma_d = .2, \sigma_m = .2$.
$\sigma_T$: solid line: 0.1, dashed line: 0.2, dotted line: 0.3

**Figure 3 : $\alpha - \alpha_1$ vs. $\omega$**
This figure plots \( \alpha_2 \) against \( \omega \), the proportion of informed agents in the economy.

Here \( \tau = 2, k = 1, \lambda = 0.1, \beta = 0.98, \alpha_T = 0.9, a_m = 0.9, \bar{m} = 1, \sigma_a = .2, \sigma_d = .2, \sigma_m = .2, \)

\( \sigma_\tau \): solid line: 0.1, dashed line: 0.2, dotted line: 0.3

**Figure 4 : \( \alpha_2 \) vs. \( \omega \)**

\( \alpha_2 \) first decreases before it increases with increasing \( \omega \). As pointed out in Wang (1993), this is because there is the effect of adverse selection problem when there is information asymmetry. When informed agents know more about the state of the economy than uninformed agents do, they can take advantage of this fact. Uninformed agents, knowing this, will try to decrease their holdings of stocks in order to prevent this from happening. However, when \( \omega \) is large enough, this effect will be dominated by the effect of decreasing in uncertainty. The effect of changing \( \sigma_\tau \) is also different for different \( \omega \). When \( \omega \) is small, the adverse selection effect is dominating. So increasing \( \sigma_\tau \) will actually increase \( \alpha_2 \). When \( \omega \) is large, the imperfect information is dominating. So increasing \( \sigma_\tau \) will decrease \( \alpha_2 \).

All these results for \( \alpha_2 \) also hold for \( \alpha_0 \), as seen from figure 1. Namely there are non-monotonic relationship between \( \alpha_0 \) and \( \omega \). When \( \omega \) goes from 0 (all uninformed agents) to 1 (all informed agents), \( \alpha_0 \) will first decrease then increase. And the effect of increasing \( \sigma_\tau \) will depend on value \( \omega \).

In the long run, the price of the stock is \( E(P) = \alpha_0 + \alpha \bar{m} + \alpha_2 \bar{\Gamma} \). And the excess return is \( E(Y) = \bar{m} - (R - 1) \alpha_0 - (R - a_T) \alpha_2 \bar{\Gamma} \) The risk premium is

\[
E(Y)/E(P) = \frac{((R - 1)\alpha + 1)\bar{m} - \alpha_2(1 - a_T)\bar{\Gamma}}{\alpha_0 + \alpha \bar{m} + \alpha_2 \bar{\Gamma}} - (R - 1). \tag{4.3}
\]
This figure plots long run risk premium against $\omega$, the proportion of informed agents in the economy.

Here $\tau = 2, k = 1, \lambda = 0.1, \beta = 0.98$

$\alpha = 0.9, a_m = 0.9, \hat{m} = 1, \sigma_a = .2, \sigma_d = .2, \sigma_m = .2$

$\sigma_\Gamma$: solid line: 0.1, dashed line: 0.2, dotted line: 0.3

**Figure 5 : Risk Premium vs. $\omega$**

From the above discussion about $\alpha_0$ and $\alpha_2$, we can see that when $\omega$ goes from 0 to 1, $\alpha_0$ and $\alpha_2$ first decrease then increase. Furthermore, increasing $\sigma_\Gamma$ has different effects on different value of $\omega$: when $\omega$ is small, it actually increases stock price, while when $\omega$ is large, it decreases stock price as one would expect. The fact that increasing uncertainty has different effects on stock price is worth emphasizing since it is quite different from the usual linear models. It means asymmetric information has long term effect on the stock prices. When $\omega$ is small, adverse selection effect is dominating. When $\sigma_\Gamma$ increases, the trading of informed agents will more likely come from hedging against risk rather than taking advantage of uninformed agents, making the uninformed agents more like to trade with informed agents. The stock thus becomes more attractive. So asymmetric information will play an important role not only in the volume study we are going to look into in a moment, but also in the asset pricing study.

### 4.2 Volume, Dividend and Excess Returns

In this section, we study the relationship between volume and dividend, volume and excess return.

Let us recall the trading strategy of uninformed agents. When we sum up the stocks
holdings of uninformed agents, we obtain

\[ X^u_t = \frac{\delta^u_0 + \delta^u_1 \hat{\Gamma}_t}{\Gamma_t + \bar{\Gamma}}, \]  

(4.4)

where \( \delta^u_0 > 0, \delta^u_1 > 0 \). So the trading volume is:

\[ V_t = |X^u_t - X^u_{t-1}| = \left| \frac{\delta^u_0 + \delta^u_1 \hat{\Gamma}_t - \delta^u_0 + \delta^u_1 \hat{\Gamma}_{t-1}}{\Gamma_t + \Gamma} \right| = \left| \frac{\delta^u_0 + \delta^u_1 \hat{\Gamma}_t}{\Gamma_t + \Gamma} - \frac{\delta^u_0 + \delta^u_1 \hat{\Gamma}_{t-1}}{\Gamma_{t-1} + \Gamma} \right| = \left| \frac{\delta^u_1 (\hat{\Gamma}_t - \hat{\Gamma}_{t-1})}{\Gamma_t + \Gamma} \left( \frac{1}{\Gamma_t + \Gamma} - \frac{1}{\Gamma_{t-1} + \Gamma} \right) \right| (\delta^u_0 + \delta^u_1 \hat{\Gamma}_{t-1}) \right| . \]  

(4.5)

We can further decompose the first term as follows:

\[ \hat{\Gamma}_t - \hat{\Gamma}_{t-1} = E_t(\Gamma_t) - E_{t-1}(\Gamma_{t-1}) = (E_t(\Gamma_t) - E_t(\Gamma_{t-1})) + (E_{t-1}(\Gamma_{t-1}) - E_{t-1}(\Gamma_{t-1})). \]  

(4.6)

Here the first term is the trading of uninformed agents for the hedging purpose, namely uninformed agents trade the stock regarding the change of the new shocks to the uncertainty about aggregate risk aversion \( \Gamma \). The second term is the trading of uninformed agents against the superior information of the informed agents. Returning to the previous equation, there is an additional term against the shocks to the aggregate risk aversion. This comes from the nonlinear representation of the trading strategy with respect to aggregate risk aversion.

Because of this nonlinear term, the distribution of the volume is not normal anymore. So in the following we will use numerical examples to show the relationship between volume and dividend, volume and excess returns. Specifically, for each set of parameters we first obtain the price function and trading strategies for informed and uninformed agents as specified above. Then we simulate 300 series of stocks with four series shocks to \((D, m, \Gamma, m^a)\) of length 5000. Then we delete the first 1000 points to get rid of effects from initial condition. The results are the average across series.
This figure plots correlation between volume and absolute change of dividend against $\omega$, the proportion of informed agents in the economy.

Here $\tau = 2, k = 1, \lambda = 0.1, \beta = 0.98,$
$a_T = 0.9, a_m = 0.9, m = 1, \sigma_a = .2, \sigma_d = .2, \sigma_m = .2.$
$\sigma_T$: solid line: 0.1, dashed line: 0.2, dotted line: 0.3

Figure 6: Cor$(V, |\Delta D|)$ vs. $\omega$

4.2.1 Correlation between Volume and Absolute Change of Dividend and Stock Price

We first study the correlations between volume and absolute changes of dividend and prices. Namely we want to study $\text{cor}(V_t, |D_t - D_{t-1}|)$ and $\text{cor}(V_t, |P_t - P_{t-1}|)$.

Fig. 6 and 7 are the results from the simulation. First of all, note that they are all positive. This is consistent with the results from Wang (1994). As pointed out in Wang (1994), this relationships are mainly caused by the existence of information asymmetry.

For the volume and price relationship, recall in the symmetric case in which every agent is informed the difference of price is correlated with volume. When there is information asymmetry, there is adverse selection effect. Uninformed agents require extra premium when buying stock against informed agents’ superior information. So given a trading size, the price has to adjust more than in the case of symmetric information. Thus the correlation will increase with the increase of asymmetric information.

For the volume and dividend change, things become a little bit different. In the symmetric case in which every agent knows about the true state of the world, a shock in dividend will change the price but will not change the stock holdings in the CARA case. Only asymmetric
This figure plots correlation between volume and absolute change of stock against $\omega$, the proportion of informed agents in the economy.

Here $\tau = 2$, $k = 1$, $\lambda = 0.1$, $\beta = 0.98$,

$\begin{align*}
a_T &= 0.9, a_m = 0.9, \bar{m} = 1, \sigma_u = .2, \sigma_d = .2, \sigma_m = .2.
\end{align*}$

$\sigma_T$: solid line: 0.1, dashed line: 0.2, dotted line: 0.3

**Figure 7**: Cor$(V, |\Delta P|)$ vs. $\omega$

Information will generate trade. This intuition of trading under public information from asymmetric information is the main attractive point of using asymmetric information to analyze trading volume issue. The trading volume from rebalancing portfolio holdings under symmetric case (for example, CRRA case) will not generate this type of excess trading volume.

### 4.2.2 Expected Excess Return condition on Volume and Pricing Variables

In this section we study the relationship between expected future return and current volume and other variables. Specifically we want to study the conditions on the joint effects of volume and other variables such as dividend change, price change, and current excess returns.

Since we can not obtain analytic results, we use the following regression in the simulations. For dividend, we run the following regression:

$$
r^e_{t+1} = \beta_0 + \beta_v V_t + \beta_d \Delta D_t + \beta_{vd} V_t \Delta D_t + \epsilon, \tag{4.7}
$$

where $r^e = R^S - R$ is the excess return of the stocks.

Fig. 8 shows the regression coefficient $\beta_{vd}$. First of all, note that $\beta_{vd} < 0$. This means
This figure plots regression coefficient (Expected excess return on volume and current dividend change) against $\omega$, the proportion of informed agents in the economy.

Here $\tau = 2, k = 1, \lambda = 0.1, \beta = 0.98,$ $a_T = 0.9, a_m = 0.9, m = 1, \sigma_a = .2, \sigma_d = .2, \sigma_m = .2$. $\sigma_T$: solid line: 0.1, dashed line: 0.2, dotted line: 0.3

**Figure 8 :** $\beta_{vd}$ vs. $\omega$

that given a high trading volume, a current increase in dividend ($\Delta D_t > 0$) will more likely lower the future excess return. The intuition behind this is the following. In the symmetric case in which every agent knows the true state, a shock to the dividend, namely a public signal, will not change the future expected excess return. In the asymmetric case however, uninformed agent will realize that they underestimate the dividend. They will begin to buy the stock and push up the current price. Thus the future expected return will decrease. This is purely from the asymmetric information. With more information asymmetry, this effect becomes stronger. In Wang (1994), the relationship between future expected excess return and current volume and dividend change is positive because to the first order approximation, the dividend change is persistent, thus uninformed agents will adjust their underestimation in the future. This pushes up future stock price also. The net effect is positive. In our case, the nonlinearity makes relationship opposite. In general in our model, a shock in public signal associated with high volume today will tends to lower future expected return.

For the price volume variables, we run the following regression:

$$r_{t+1} = \beta_0 + \beta_v V_t + \beta_p \Delta P_t + \beta_{vp} V_t \Delta D_t + \epsilon.$$  \hspace{1cm} (4.8)
This figure plots regression coefficient (Expected excess return on volume and current price change) against \( \omega \), the proportion of informed agents in the economy.

Here \( \tau = 2, k = 1, \lambda = 0.1, \beta = 0.98 \),
\( a_T = 0.9, a_m = 0.9, \bar{m} = 1, \sigma_a = .2, \sigma_d = .2, \sigma_m = .2 \).
\( \sigma_T \): solid line: 0.1, dashed line: 0.2, dotted line: 0.3

Figure 9: \( \beta_{vp} \) vs. \( \omega \)

Fig. 9 shows the regression coefficient \( \beta_{vp} \) against \( \omega \). Here again we have \( \beta_{vp} < 0 \), namely a high volume associated with a positive price change will lower future expected excess return. The intuition behind this is the following. Unlike the shocks to the public signal such as dividend, a change in current price may come from two sources: a change in mean dividend or a change in uncertainty. A change in mean dividend will lower the future expected return as stated above. A positive shock to price from shocks to uncertainty in aggregate risk aversion will be more likely to be induced by informed agents who want to buy more stocks because of the change in uncertainty rather than future dividend. So a higher increase in price today associated with high volume (buying from say less risk aversion) will lower the future expected return. Since both a shock to current dividend and to current aggregate risk aversion will lower the future excess return, the aggregate effect is lowering future excess returns.

Lastly, we study the future excess return conditional on current volume and current excess return. We run the following regression:

\[
r_{t+1} = \beta_0 + \beta_v V_t + \beta_r r_t + \beta_{vr} V_t r_t^\epsilon + \epsilon.
\] (4.9)
This figure plots regression coefficient (Expected excess return on volume and current excess return) against \( \omega \), the proportion of informed agents in the economy.

Here \( \tau = 2, k = 1, \lambda = 0.1, \beta = 0.98, \alpha = 0.9, \alpha_m = 0.9, m = 1, \sigma_a = .2, \sigma_d = .2, \sigma_m = .2. \)
\( \sigma_\Gamma \): solid line: 0.1, dashed line: 0.2, dotted line: 0.3.

**Figure 10 : \( \beta_{vr} \) vs. \( \omega \)**

Fig. 10 shows the regression coefficients \( \beta_{vr} \) against \( \omega \). Again, the regression coefficient \( \beta_{vr} < 0 \). This is clear from the above discussion of \( \beta_{vp} \) so we do not want to say more about this.

The fact that our model obtains reverse effect on the excess return conditional on current volume, dividend change, price change and current excess return is consistent with the empirical findings in, say, Campbell, Grossman and Wang (1993).

### 4.3 Further Remark

In this section we want to discuss several issues about our model. First of all, the fact that we are using AR(1) process to model the evolvement of aggregate risk aversion means that the distribution of aggregate risk aversion maybe positive (in fact, it can be positive infinity). This is purely an assumption for simplification, similar to the usual asymmetric competitive literature in which one assumes that the stock dividend follows an AR(1) process. The main reason we can still obtain linear price function is because of this. One can make the argument that \( \bar{\Gamma} \) large enough or \( \sigma_\Gamma \) small enough to make the possibility that \( \Gamma < 0 \) as small as possible.
A more serious issue is the fact that there are informed agents who know the true $\Gamma$. The original informed/uninformed agents models are mainly to avoid the infinite regress expectation problem (Townsend, 1983). But then that agents are more informed about total shares trading (liquidity trader), private investment opportunities, and future stock payoff seem to be approximated by some real world examples. In our case, this assumption seems more ad hoc. A more realistic assumption would be for agents to have differential information. But this needs another paper to explore.

5 Conclusion

We study a model in which agents are uncertain about the distribution of risk aversion in the economy and study the effects on the risk premium and stock trading volume. Asymmetric information plays significant roles both in long-run risk premium and in stock trading volume. All our results are qualitatively consistent with the empirical findings in the literature. One direction of the future work is to use more realistic time-variant risk aversion function and actually calibrate the model with the data to see quantitatively the effects of asymmetric information. Another possible extension of the model is to look at cross-sectional stock returns to see what the effects will be. Finally, a much harder problem is to do all this work in a differential information setting (He and Wang, 1995).
Appendix

Proofs For Section 2

Proposition 5

Given that \((m, \Lambda, m_a)\) are jointly normal. Their variance-covariance matrix is

\[
V = \begin{pmatrix}
    s_m^2 & \alpha_1 s_m^2 & s_m^2 \\
    \alpha_1 s_m^2 & \alpha_1^2 s_m^2 + \alpha_2^2 s_m^2 & \alpha_1 s_m^2 \\
    s_m^2 & \alpha_1 s_m^2 & s_m^2 + s_a^2
\end{pmatrix}
\]  \hspace{1cm} (5.1)

If we define

\[
\Sigma_{22} \equiv \begin{pmatrix}
    \alpha_1^2 s_m^2 + \alpha_2^2 s_m^2 & \alpha_1 s_m^2 \\
    \alpha_1 s_m^2 & s_m^2 + s_a^2
\end{pmatrix}
\]  \hspace{1cm} (5.2)

and

\[
\Sigma_{12} \equiv (\alpha_1 s_m^2, s_m^2) = \Sigma_{21}^T
\]  \hspace{1cm} (5.3)

we have (from the properties of multi-variate normal distribution)

\[
(\beta_{1a}, \beta_{2a}) = \Sigma_{12} \Sigma_{22}^{-1} \hspace{1cm} (5.4)
\]

\[
v_a^2 = s_m^2 - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \hspace{1cm} (5.5)
\]

\[
\beta_{0a} = \overline{m} - \Sigma_{12} \Sigma_{22}^{-1} (\Lambda, \overline{m})^T \hspace{1cm} (5.6)
\]

where \(\overline{\cdot}\) are the unconditional mean.

Simplifying, we obtain

\[
\beta_{0a} = \frac{1}{|\Sigma_{22}|} \left( (\alpha_1^2 s_m^2 s_a^2 - \alpha_2^2 s_m^2 s_a^2) \overline{m} - \alpha_1 s_m^2 s_a^2 \overline{\Lambda} \right)
\]

\[
\beta_{1a} = \frac{\alpha_1 s_m^2 s_a^2}{|\Sigma_{22}|}
\]

\[
\beta_{2a} = \frac{\alpha_2^2 s_m^2 s_a^2}{|\Sigma_{22}|}
\]

\[
v_a^2 = \frac{\alpha_2 s_m^2 s_a^2}{|\Sigma_{22}|} \Gamma
\]

where

\[
|\Sigma_{22}| = \alpha_1^2 s_m^2 s_a^2 + \alpha_2^2 s_m^2 (s_m^2 + s_a^2).
\]

Q.E.D.
**Proposition 6 and 7**

The proofs for these two propositions are identical except for the labels. So we only prove proposition 6 in the following.

Given the mean and variance of share excess return, if we let

\[ \xi \equiv y - pR \]

and

\[ \bar{\xi} \equiv X^i \psi^i, \]

we have:

\[
E^i(U) = -1 \gamma a E^i e^{-\gamma a W} \\
= -1 \gamma a \int -\infty \infty e^{-\gamma a \theta} e^{-\frac{\xi^2}{2\sigma^2}} d\xi \\
= \frac{1}{2} \gamma a e^{\gamma a W} e^{-\frac{2\bar{\xi}^2}{2\sigma^2}}.
\]

Maximizing will give the results in the text. QED.

---

**Proofs For Section 3**

**Proposition 10**

The state variables in the economy are \( z_t^T \equiv (m_t, \Gamma_t)^T \), which follows the process

\[ z_t = a_z z_{t-1} + b_z \epsilon_t, \quad (5.1) \]

where \( \epsilon_t \equiv (\epsilon^D_t, \epsilon^m_t, \epsilon^\gamma_t, \epsilon^a_t)^T \) and

\[
a_z = \begin{pmatrix} a_m & 0 \\ 0 & a_\Gamma \end{pmatrix}, \quad b_z = \begin{pmatrix} b_m \\ b_\Gamma \end{pmatrix}.
\quad (5.2)
\]

Here \( b_m = (0, 1, 0, 0) \) and \( b_\Gamma = (0, 0, 1, 0) \).

The observables \( s_t \) follows that

\[ s_t = a_s z_t + b_s \epsilon_t, \quad (5.3) \]

where

\[
a_s = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ \alpha_1 & \alpha_2 \end{pmatrix}, \quad b_s = \begin{pmatrix} b_a \\ b_d \end{pmatrix}.
\quad (5.4)
\]
Here \( b_a = (0, 0, 0, 1) \) and \( b_D = (1, 0, 0, 0) \).

This is a standard Kalman filter problem. The results follows immediately. Moreover, we have the following results.

Let \( \hat{z}_t = E_t(z_t) \) and \( \Omega_t = E_t((\hat{z}_t - z_t)(\hat{z}_t - z_t)^T) \), we have

\[
\begin{align*}
\hat{z}_t &= a_z \hat{z}_{t-1} + \kappa_t (s_t - E_{t-1}^a(s_t)), \\
\Omega_t &= (a_z \Omega_{t-1} a_z^T + b_z \Sigma b_z^T) - \kappa_t a_s (a_z \Omega_{t-1} a_z^T + b_z \Sigma b_z^T),
\end{align*}
\]

(5.5)

(5.6)

where \( \Sigma = E(\epsilon \epsilon^T) \) and

\[
\kappa_t = (a_z \Omega_{t-1} a_z^T + b_z \Sigma b_z^T) a_s (a_z \Omega_{t-1} a_z^T + b_z \Sigma b_z^T) a_s^T + b_s \Sigma b_s^T)^{-1}.
\]

(5.7)

We only consider the steady state solution. So in the above equation, we substitute \( \Omega \) for \( \Omega_t \) and \( \Omega_{t-1} \) to get a Riccati equation with a constant \( \kappa \) for \( \kappa_t \). The steady-state solutions are given by

\[
\hat{z}_t = a_z \hat{z}_{t-1} + k (s_t - a_s a_z \hat{z}_{t-1})
\]

(5.8)

If we define \( \tilde{z}_t = z_t - \hat{z}_t \), we have

\[
\tilde{z}_t = a_z \tilde{z}_{t-1} + b_z \epsilon_t,
\]

(5.9)

where \( a_z = a_z - \kappa a_s a_z \) and \( b_z = b_z - \kappa a_s b_z - \kappa b_s \).

Specifically from this, let us define \( \eta_t = m_t - \hat{m}_t \), we have

\[
\eta_t = a_{\eta} \eta_{t-1} + b_{\eta} \epsilon_t,
\]

(5.10)

where \( a_{\eta}, b_{\eta} \) can be calculated using the above coefficients.

**Proposition 12 and 13**

From the evolution equations of \( \Gamma_t \) and \( \eta_t \), we can easily calculate the coefficients in the equation

\[
\psi^i_t = a_i^i \psi^i_{t-1} + b_i^i \epsilon^i_t,
\]

(5.11)

where \( \epsilon^i_t \) is normal \((0, g^i)\).

As for the uninformed agent, note that

\[
\hat{\Gamma}_t = \Gamma_t + \frac{\alpha_1}{\alpha_2} \eta_t
\]

\[
= a_\Gamma \Gamma_{t-1} + b_\Gamma \epsilon_t + \frac{\alpha_1}{\alpha_2} (a_\eta \eta_{t-1} + b_\eta \epsilon_t)
\]

\[
= a_\Gamma \hat{\Gamma}_{t-1} + \frac{\alpha_1}{\alpha_2} (a_\eta - a_\Gamma) \eta_{t-1} + (b_\Gamma + \frac{\alpha_1}{\alpha_2} b_\eta) \epsilon_t.
\]

So if we let \( \epsilon^u_t = (\eta_{t-1}, \epsilon_t) \), it is again normal \((0, g^u)\), where \( g^u \) can be calculated from \( \alpha \)'s and parameters.

The state variable \( \psi^u \) follows
\[ \psi_t^n = a^n_{\psi} \psi_{t-1} + b^n_{\psi} e_t^n, \]  
(5.12)

where \(a^n_{\psi}, b^n_{\psi}\) can be calculated from the above discussion.

In both cases, we can write agents’ state evolution equation as
\[ \psi_t^a = a^a_{\psi} \psi_{t-1} + b^a_{\psi} e_t^a, \quad a = i, u, \]  
(5.13)

and the excess return can be represented by
\[ Y_{t+1} = a^a_{\psi} \psi_{t}^a + b^a_{\psi} e_{t+1}^a, \quad a = i, u. \]  
(5.14)

In the following, we will omit superscript \(a = i, u\) with the understanding that it applies to both cases, specifically \(\epsilon\) are the corresponding error terms in the above equations instead of the \(\epsilon\) defined in the main text.

With the estimation results and budget constraint, we have
\[ V(W_{t+1}; \psi_{t+1}, t + 1) = -\beta^{t+1} e^{-bW_{t+1} - 1/2 |\psi_{t+1}|^2} \]
\[ = -\beta^{t+1} e^{-b(W_{t+1} - \epsilon_t)(R + \theta_t Y_{t+1})} \left[ (a^v_{\psi} \psi_t + b^v_{\psi} \epsilon_t + 1) \right] \]
\[ = -\beta^{t+1} e^{-b(W_{t+1} - \epsilon_t)(R - b\theta_t a^v_{\psi} \epsilon_t) \tilde{V}_{t+1}}, \]
where
\[ \tilde{V}_{t+1} = e^{-b\theta_t b^v_{\epsilon_t + 1} - \frac{1}{2} (a^v_{\psi} \psi_t + b^v_{\epsilon_t + 1})^T v (a^v_{\psi} \psi_t + b^v_{\epsilon_t + 1})} \]  
(5.15)

To calculate \(E_t V(W_{t+1}; \psi_{t+1}, t + 1)\), we need to calculate \(E_t \tilde{V}_{t+1}\). Note that \(\epsilon_{t+1}\) is \(\mathcal{N}(0, g)\), so
\[ E_t \tilde{V}_{t+1} = \frac{1}{2^{n/2}|g|^{1/2}} \int e^{-\frac{1}{2} \epsilon_{t+1}^T g^{-1} \epsilon_{t+1}} \tilde{V}_{t+1} d\epsilon_{t+1}, \]  
(5.16)

where \(n\) is the dimension of \(\epsilon\) \((n = 4\) for informed agents, and \(n = 5\) for uninformed agents). The exponential term is just
\[ -\frac{1}{2} \epsilon_{t+1}^T (g^{-1} + b^T_{\psi} v \psi_{\psi}) \epsilon_{t+1} - (a \theta_t b_{\psi} + \psi_{t}^T a^T_{\psi} v \psi_{\psi}) \epsilon_{t+1} - \frac{1}{2} \psi_{t}^T a^T_{\psi} v a_{\psi} \psi_{t}. \]

Define
\[ A_1 = (g^{-1} + b^T_{\psi} v \psi_{\psi})^{-1}, \]
\[ A_2 = b \theta_t b_{\psi} + \psi_{t}^T a^T_{\psi} v \psi_{\psi}, \]
\[ A^T = -A_2 * A_1. \]

We have
\[ E_t \tilde{V}_{t+1} = \frac{1}{2^{n/2}|g|^{1/2}} \int e^{-\frac{1}{2} \epsilon_{t+1}^T A_1^{-1} (\epsilon_{t+1} - A_2) - \frac{1}{2} (\psi_{t}^T a^T_{\psi} v a_{\psi} \psi_t - A^T A_1^{-1} A) d\epsilon_{t+1} \]
\[ = \frac{|A_1|^{1/2}}{|g|^{1/2}} e^{-\frac{1}{2} (\psi_{t}^T a^T_{\psi} v a_{\psi} \psi_t - A^T A_1^{-1} A).} \]
So now we represent the expectation of next period value function as a function of current state variables. Thus we can get the first order condition:

- For $\theta_t$:
  \[-a_Y \psi_t + b_Y A_1 (b \theta_t b_Y + b_T^T v \psi_t) = 0.\]

  Thus
  \[\theta_t^* = \frac{1}{b} X^T \psi_t,\] (5.17)
  where
  \[X^T = (b_Y A_1 b_Y^T)^{-1} (a_Y - b_Y A_1 b_Y^T v \psi_t).\] (5.18)

- With the above $\theta_t$, we substitute it back to the value function. Then the maximization term becomes
  \[-\beta_t e^{-\gamma c_t} + \beta^{t+1} \delta e^{-\gamma c_t} R - \frac{1}{2} \psi_t J \psi_t,\]
  where
  \[J = a_T^T v \psi_t + 2 X a_Y - X b_Y A_1 b_Y^T X - 2 X b_Y A_1 b_Y^T v \psi_t - a_T^T v b_Y A_1 b_Y^T v \psi_t,
  = a_T^T v a \psi_t - a_T^T v b_Y A_1 b_Y^T v a \psi_t + (a_Y - b_Y A_1 b_Y^T v \psi_t)^T (b_Y A_1 b_Y^T)^{-1} (a_Y - b_Y A_1 b_Y^T v \psi_t),\]
  and $\delta = \frac{|A_1|^2}{b}. \quad (5.19)$

  Taking derivatives with respect to $c_t$, we have
  \[\ln \frac{\gamma}{\beta b a R} - \gamma c_t = -b(W_t - c_t) R - \frac{1}{2} \psi_t^T J \psi_t, \quad (5.20)\]
  where $c_t^* = c + \frac{b R}{\gamma + b R} W_t + \frac{1}{2(\gamma + b R)} \psi_t^T J \psi_t$.

Now with the optimal portfolio choice and consumption, we substitute them back to the value function, and equate the two sides to get the value for the initial guessing $b, v$. Namely we have
\[-\beta e^{-b(W_t - \frac{1}{2} \psi_t^T v \psi_t)} = -\beta e^{-\gamma c_t^*} - \beta^{t+1} \delta e^{-\gamma c_t} R - \frac{1}{2} \psi_t^T J \psi_t. \quad (5.21)\]

Substitute $c_t^*$ in and equate the coefficient of $W_t$ and $\psi_t$, we have
\[b = \frac{(R - 1) \gamma}{R} \quad (5.22)\]

and
\[\ln \frac{R - 1}{R} + \gamma c + \frac{1}{2} \psi_t^T (J - v) \psi_t = 0, \quad (5.23)\]
where \(\tilde{c} = -\frac{1}{R^2} \ln(\beta \delta (R - 1))\). So the second equation becomes

\[
\frac{1}{2} \psi_t^T (2(\ln \frac{R - 1}{R} - \frac{1}{R} \ln \beta \delta (R - 1)) \mathbf{1}_{11} + \frac{J}{R} - v) \psi_t = 0, \tag{5.24}
\]

where \(\mathbf{1}_{ij}\) is index matrix. Since this holds for all \(\psi_t\), we have

\[
2(\ln \frac{R - 1}{R} - \frac{1}{R} \ln \beta \delta (R - 1)) \mathbf{1}_{11} + \frac{J}{R} - v = 0. \tag{5.25}
\]

So \(v\) is the solution of the above systems of linear equations and it does not depend on \(\gamma\).

Substitute \(b\) back to the portfolio choice we have

\[
\theta_t = \frac{R}{(R - 1)\gamma} X^t \psi_t. \tag{5.26}
\]

Note that \(X\) does not depend on \(\gamma\) either.

**Proposition 9**

By the assumption in the main text, we have

\[
\frac{N}{\Gamma + \Gamma_t} (\omega(a_{00}^t + a_{10}^t \Gamma_t + a_{20}^t \eta_t) + (1 - \omega)(a_{00}^u + a_{10}^u \hat{\Gamma}_t)) = K. \tag{5.27}
\]

Note that \(\hat{\Gamma}_t = \Gamma_t + \frac{\alpha_1}{\alpha_2} \eta_t\). Substitute it back to the above equation, we have a linear equation as follows:

\[
\omega a_{00}^t + (1 - \omega)a_{00}^u + \omega a_{10}^t \Gamma_t + \omega a_{20}^t \eta_t + (1 - \omega)a_{10}^u (\Gamma_t + \frac{\alpha_1}{\alpha_2} \eta_t) = \frac{K}{N}(\Gamma + \Gamma_t). \tag{5.28}
\]

It is only valid if the constant term and the two coefficients in front of \(\Gamma_t\) and \(\eta_t\) are zero. This will give us three equations, exactly solving the three \(\alpha\)'s. Thus solve the price function!

\[
\omega a_{00}^t + (1 - \omega)a_{00}^u = \frac{K}{N} \tilde{\Gamma} \tag{5.29}
\]

\[
\omega a_{10}^t + (1 - \omega)a_{10}^u = \frac{K}{N} \tag{5.30}
\]

\[
\omega a_{20}^t + (1 - \omega)a_{10}^u \frac{\alpha_1}{\alpha_2} = 0. \tag{5.31}
\]

**References**


