Fuzzy Options with Application to Default Risk Analysis for Municipal Bonds in China[1]

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ABSTRACT: Contrast to identical rationality, non-identical rationality is laid as the basis for option pricing, realized by a family of non-additive fuzzy measures mathematically, which naturally induces the new concept and analyzing methods, fuzzy options. Then as demonstration, the paper applies the fuzzy options to default risk analysis for municipal bonds in China.

Keywords: fuzzy options; non-identical rationalities; default risk; municipal bonds
JEL: G28, G13, G31

1. Introduction

Options are among the most important kinds of financial derivatives because of their abundant functions and properties, and they have been bringing profound influence to the global financial market. Good performance of options in financial market must be based on appropriately pricing for them. Option pricing theory has become the scientific theory since Black, Scholes and Merton accomplished their great work in 1973. Their classical B-S option pricing method and the succeeding martingale methods inherit the basic economics paradigm--“representative agent”, in which “identical rationality” is assumed. Rubinstein(1976), Breeden and Litzenberger (1978), Brennan (1979), Stapleton and Subrahmanyam (1984), Bick (1987,1990), He and Leland(1993) have pointed out this character, respectively. It assumed that preference, expectation and investment strategy of all investors are homogeneous, so one agent can represent all the investors to maximize their interest rationally. Based on this sort of identical rationality, financial models naturally deviate from the reality and their implement in practice is awful because it ignored heterogeneity inherited in human behaviors.

According to identical rationality, all the investors know the unique probabilistic distribution of

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the uncertain future return perfectly, and their interests can be maximized by a representative agent, so they share the same expected option price; and the pricing model can reflect this price precisely. But empirical research finds out that there are systematic error between real market price and the classical model based price. Namely, there exists mispricing! Black and Scholes, Galai, Klemkosky, and Resnicky, MacBeth and Merville, Rubinstein, Bates etc have pointed out that phenomenon like violation for Call-Put parity, volatility smile, volatility term structure, Black's hole and so on, are the evidence of mispricing, and It has the unsolvable difficulty for classical option pricing models. Trying to correct these defects, many scholars proposed many new models, such as stochastic volatility model, compound option model, displaced diffusion model, pure jump model, jump diffusion model and so on, but they failed without exception, because they do not touch the basic economic assumption. Additionally, there is a reconcilable paradox concerning "identical rationality" in classical option pricing models. That is, as a kind of redundant assets, options have a property of "net zero supply", namely, long and short poison in the market are the same and trades are very active. But according to "identical rationality", because investors in long and short poison are homogeneous, it will result in no trades and nobody needs option! (Leland，1980)

Economists paid attention to the above-mentioned assumption about human behavior in economics early. In 1930s, Keynes pointed out that agents can not obtain enough knowledge to form right expectation possessed by all the agents. Herbert Simon（1955）doubted about unlimited information processing ability of human in economic model. Psychologists deem that human are inherent heterogeneous and they deal with all information they owned on their own, and their decision-making process related to their own mental activities. And economists acknowledged that heterogeneity is the main drive to Pareto improvement, and "identical rationality" assumption is not for reality but just for convenience. So behavioral economists and financers stressed that financial theory must pay more attention to human behavior and related heterogeneity, and "non-identical rationalities" must be really taken account into economics and finance. Mankiw and Zeldes（1991）,Kennickell and Woodburn(1992), De Bondt（1993）, Welch(2000) presented empirical evidences for heterogeneity. Brock and Hommes (1997,1998), Chiarella and He(2000,2001), and Galleyer(2000) have founded that heterogeneity will bring large difference to trading pattern and significant impact on equilibrium price and allocation. Recently, some scholars such as Rubinstein(1994), Franke(1996), Benninga and Mayshar（1997）begin to consider option pricing from heterogeneous perspective. And they found that "non-identical rationalities" might be the only way to solve the difficulties in classical option pricing models.

A part of literature about heterogeneity for option pricing are :Rubinstein(1973)，Jaffee and Winkler(1976), Figlewski(1978,1983), Feiger (1978), Mayshar(1983), Shefrin(1984), Dumas(1989), Harris and Raviv(1993), Benninga and Mayshar(1993,2000), Detemple and Murthy(1994), Shefrin and Statman（1994）, Huang(1996), Huang(2000) and Basak(2000), Kurz(1997), Carr and Madan(1997), Shefrin（2000）. From those literatures, the ways to treat heterogeneity can be classified to three kinds. The first one is to consider aggregating function of different heterogeneous utilities. The second one is to add another measure or indicator --such as "cautious measure" proposed by Huang(2000), "sentiment measure" proposed by Shefrin(2000). And the last one is to substitute the traditional probability measure with a new measure and alter the expectation utility rules. This method express heterogeneity with subjective probability, which is represented by a kind of non-additive measures--fuzzy measures. Based on fuzzy measure, option then can be priced through Choquet expectation (integral) in stead of Savage's subjective expectation. Some scholars like Jouuiny &

Kallal(1995), Wang(1996，2000), Muzzioli & Torricelli(2001), Cherubini(1997，2001), more or less touched the fuzzy measure and integral, but they did not view this problem from economics, and did not realize the “non-identical rationalities” concept obviously.

Corresponding to the consideration above, the paper gives a new thinking to use the family of fuzzy measures to exploit diversification of rationality. We build up the new concept of “non-identical rationalities”, which break through the “identical rationality” assumption. And non-identical rationalities will be modeled by the family of fuzzy measures, in which each fuzzy measure represents the unique rationality of an investor. And based on this, fuzzy option pricing method is proposed.

2. Fuzzy Options

In our economy of non-identical rationalities, although sharing the same information, investors foresee the coming state of nature and evaluate the same asset on their own, which can be quite different from each others. They understand the whole market from their own perspectives and then build their own very unique markets, called “individual markets”. In the individual market, either no arbitrage chance exists there and the equilibrium is maintained, or one will buy the assets underpriced and sell the assets overpriced. At the state of individual market equilibrium, he prices all the assets in the marketplace, including underlying assets and their derivatives. Because the information possessed by different investors can be different, and even they have the same information, the methods by which they analyze and judge could not be the same, so the price for the same asset evaluated by different investors can be very different. These differences are the original drive for trade. These prices can be different from the real market price ex post. But the real market price ex post is surely among those prices, because the individual market equilibrium is constructed rationally through micro-analysis based on macro-analysis, which means that the prices dispersed in a limited interval. The real market equilibrium price ex post can stay at any position in the interval, which means multi-equilibriums.

This kind of non-identical rationalities can be presented by subjective probabilities assigned by investor to her own average weighted price. The subjective probability can be transformed from an identical mathematical expression based on fuzzy measures. Here we applied a kind of specific fuzzy measures named λ-additive fuzzy measures，following Cherubini (1997，2001).

Write Ω a nonempty set, and let Σ denote σ-algebra on Ω，(Ω，Σ) is a measurable space. Some very basic concepts are here borrowed for our introducing fuzzy options.

Concept 1. A finite, monotone set function $\mu : \Sigma \to [0,\infty)$ is a fuzzy measure on Σ if the following conditions are satisfied:

(1) $\mu(\emptyset) = 0;$

(2) if $A, B \in \Sigma$， and $A \subset B$， then $\mu(A) \le \mu(B)$;

(3) if { A_n } is a monotone set sequence in Σ， then $\lim_{n\to\infty} \mu(A_n) = \mu(\bigcup_{n=1}^{\infty} A_n)$ or $\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap_{n=1}^{\infty} A_n)$.

If $\mu(\Omega) = 1$, then μ is called the regular fuzzy measure.

Concept 2. A λ-additive fuzzy measure is defined on （Ω，Σ）, denoted by μ, which possessing the property that for arbitrary $\lambda \in (-1,+\infty)$, $A, B \in \Sigma$, and $A \cap B = \varnothing$,

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B)$$

Furthermore, for $A_n \in \Sigma$ （$n = 1,2\cdots$) being disjoint to each other, we have

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \begin{cases} \frac{1}{\lambda}[\prod_{n=1}^{\infty}(1+\lambda\mu(A_n)) - 1], & \lambda \neq 0 \\ \sum_{n=1}^{\infty}\mu(A_n), & \lambda = 0 \end{cases} \tag{1}$$

Obviously, when $\lambda = 0$, μ is the ordinary probability measure.

When $\lambda \neq 0$, there is a correspondence between λ-additive fuzzy measures and probability measures. If ν is a λ-additive fuzzy measure on Σ, then $P = \ln_{1+\lambda}(1+\lambda\nu)$ is a probability measure on Σ; and if P is a probability measure，then $\nu = \frac{1}{\lambda}[(1+\lambda)^P - 1]$ is a λ-additive fuzzy measure.(Zhang and Ye, 1997)

Concept 3. Generally speaking, for a given fuzzy measure μ on Σ, if it satisfies:

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B), \quad A, B \in \Sigma$$

then μ is called the **likelihood measure**, which possesses the **sub-additive property**; per contra, if "$\geq$"takes palace of "$\leq$" in the inequality above, then μ is called the **belief measure**, which possesses the **super-additive property**.

By choosing the parameter λ, each investor assigns his own subjective measure when pricing asset. This method is similar to distortion probability of Wang (2000). When the fuzzy measure was applied in option pricing, fuzzy integral must be utilized. Here we use Choquet integral, which is a kind of rank-depended expectation and generalized form of ordinary expectation.

Concept 4. (Choquet integral) $X = X(\omega)$ is measurable real function on Ω, then for $X \geq 0$ Choquet integral based on the fuzzy measure μ is:

$$(c)\int X d\mu = \int_0^{\infty} \mu\{\omega \in \Omega : X \geq x\} dx$$

in which the right-hand side is Riemann integral.

Now denote underlying asset $X_{t+1} = X_{t+1}(X_t)$ at time $t+1$, X_t is the market value at time t, which has been the historical realization. So in the classical identical rationality economy, the price is derived by the risk-free interest rate, r

$$p_t = \frac{1}{1+r} CE_t^*[X_{t+1}(X_t)] = X_t \tag{2}$$

in which $CE^*[\cdot]$ is Choquet expectation, and for any nonnegative function *f*, Choquet integral is defined as $(c)\int f d\mu = \int_0^\infty \mu\{x \in \Omega : f(x) \geq a\} da$. But in the non-identical rationality economy,

$$p_t^\lambda = \frac{1}{1+r} CE_t^*[X_{t+1}(X_t)] \begin{cases} = X_t & \lambda = 0 \\ \neq X_t & \lambda \neq 0 \end{cases} \tag{3}$$

When an investor gets back from her individual market to the real market, she will compare her own price p_t^λ with the market price realized as $p_0 = X_t$, which is the reference point. If $p_0^\lambda = p_0$, she will stay still, if $p_0^\lambda > p_0$, she will long the asset, else if $p_0^\lambda < p_0$, she will short the asset. p_t^λ is her reserved price. And the final market price ex post can stay between the two reserved prices with long or short position. So the final market price ex post will be expected collectively as an interval. In the market with enough investors, λ may vary continuously, which make the final market price ex post form a fuzzy sets naturally.

The λ-interval of the fuzzy price will be constructed though the dual measure as follow:

$$\overline{\mu}(A) = \mu(\Omega) - \mu(A^C) \tag{4}$$

in which $A^C = \Omega - A$.Apparently, if μ is λ-additive measure on Σ, then $\overline{\mu}$ is λ^*-additive measure on Σ, where $\lambda^* = -\lambda/(\lambda+1)$, called the dual parameter of λ, because $\lambda > -1$, so $\lambda^* < 0$. Upon careful discussion, we could get the conclusion that λ=0 means being absolutely confident, and with λ more and more leaving 0, the degree of confidence become smaller and smaller. Due to the duality of λ and λ^*, we could only consider the situation being λ>0. Let $\theta = 1 - \lambda/\lambda_{\max}$, where $\lambda_{\max}$ is the maximum value of all λ, interpreting the most pessimism, in reality there usually exists the upper boundary for pessimism. With this setting, it is convenient to induce the membership of a fuzzy subset, which is the fuzzy option.

So the λ-interval of the fuzzy price is:

$$[p_t^\lambda, p_t^{\lambda^*}] = \left[CE_t^{*\lambda}[e^{-r(T-t)}X_T], CE_t^{*\lambda^*}[e^{-r(T-t)}X_T]\right] \tag{5}$$

When we apply this method to options pricing, we get fuzzy options. Considering European options pricing, we have underlying asset process of stock options with non-dividend as follows, $dS_t = S_t(rdt + \sigma dB_t)$, where r being risk-free interest rate. The payment function is $X_T = (S_T - K)^+$.

For having the clear relation with risk-neutral probability measure, we choose the form of the λ-additive fuzzy measure as

$$\mu_\lambda = \mu_\lambda(Q) = \begin{cases} \dfrac{1}{\lambda}[(1+\lambda)^Q - 1] & ,\lambda \neq 0 \\ Q & ,\lambda = 0 \end{cases} \tag{6}$$

where Q is the risk-natural probability measure in consideration. So the λ-interval of the fuzzy price is

$$[c_t^\lambda, c_t^{\lambda^*}] = [CE_t^{Q\lambda}[e^{-r(T-t)}X_T], CE_t^{Q\lambda^*}[e^{-r(T-t)}X_T]] \tag{7}$$

in which,

$$\begin{aligned} c_t^\lambda &= e^{-r(T-t)} \bullet (c)\int (S_T - K)1_{S_T>K}\, d\nu \\ &= e^{-r(T-t)}\int_0^\infty \nu[(S_T - K)1_{S_T>K} > x]dx \\ &= \begin{cases} e^{-r(T-t)}\int_K^\infty (\dfrac{1}{\lambda}[(1+\lambda)^{N[d_1]} - 1])dx, & \lambda > 0 \\ e^{-r(T-t)}\int_K^\infty N[d_1]dx, & \lambda = 0 \end{cases} \end{aligned} \tag{8}$$

$$\begin{aligned} c_t^{\lambda^*} &= e^{-r(T-t)} \bullet (c)\int (S_T - K)1_{S_T>K}\, d\nu^* \\ &= \begin{cases} e^{-r(T-t)}\int_K^\infty (\dfrac{1}{\lambda^*}[(1+\lambda^*)^{N[d_1]} - 1])dx, & \lambda > 0, \lambda^* = -\dfrac{\lambda}{1+\lambda} \\ e^{-r(T-t)}\int_K^\infty N[d_1]dx, & \lambda^* = 0 \end{cases} \end{aligned} \tag{9}$$

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)\sqrt{T-t}}{\sigma\sqrt{T-t}}$$

where $N[\cdot]$ denoting cumulative probability function of standard normal distribution. In the equation above, if $\lambda = 0$, then *Black-Scholes* model can be got.

Let $\theta = 1 - \lambda/\lambda_{\max}$, then comes the following set-valued mapping

$$\theta \to [c_t^\lambda, c_t^{\lambda^*}] = \left[CE_t^{Q\lambda}[e^{-r(T-t)}X_T], CE_t^{Q\lambda^*}[e^{-r(T-t)}X_T]\right], \quad \theta \in [0,1] \tag{10}$$

It is a set nesting, hence constructing a fuzzy set by the expression theorem in fuzzy set theory. That is the fuzzy options.

Theorem (Expression for fuzzy options)

The fuzzy set of fuzzy options is expressed as

$$FO = \bigcup_{\theta\in[0,1]} \theta\left[CE_t^{Q\lambda}[e^{-r(T-t)}X_T], CE_t^{Q\lambda^*}[e^{-r(T-t)}X_T]\right]$$

Proof: For any given $\lambda \in [0, \lambda_{\max}]$, the option prices form an interval as follows

$$[p_t^\lambda, p_t^{\lambda^*}] = [CE_t^{*\lambda}[e^{-r(T-t)}X_T], CE_t^{*\lambda^*}[e^{-r(T-t)}X_T]]$$

Let

$$\varepsilon = 1 - \frac{\lambda}{\lambda_{\max}}, \text{ then } \varepsilon \in [0,1]$$

Note $H(\varepsilon) = [p_\varepsilon, p_\varepsilon^*] = [p_t^\lambda, p_t^{\lambda^*}]$, therefore a set-valued mapping is conducted as

$$H : [0,1] \to \wp(R^+), \quad \varepsilon \mapsto H(\varepsilon) = [p_t^\lambda, p_t^{\lambda^*}]$$

Then

$$\varepsilon_1 < \varepsilon_2 \Rightarrow \lambda_1 > \lambda_2,$$

for $\lambda^* = -\frac{\lambda}{1+\lambda}$, so $\lambda_1^* < \lambda_2^*$,

According to the properties of λ-expectation derived by Choquet integrals, we obtain

$$p_t^{\lambda_1} < p_t^{\lambda_2}, \quad p_t^{\lambda_1^*} > p_t^{\lambda_2^*}$$

That is $[p_t^{\lambda_1}, p_t^{\lambda_1^*}] \supseteq [p_t^{\lambda_2}, p_t^{\lambda_2^*}]$, or $[p_{\varepsilon_1}, p_{\varepsilon_1}^*] \supseteq [p_{\varepsilon_2}, p_{\varepsilon_2}^*]$, and then

$$\varepsilon_1 < \varepsilon_2 \Rightarrow H(\varepsilon_1) \supseteq H(\varepsilon_2)$$

The mapping is a set nesting, therefore $\bigcup_{\varepsilon\in[0,1]} \varepsilon H(\varepsilon)$ is a fuzzy set on.

Here we present the relationship between option value (call or put) and parameter λ and fuzzy option sets with diagram, as showing in figure 1 and figure 2.

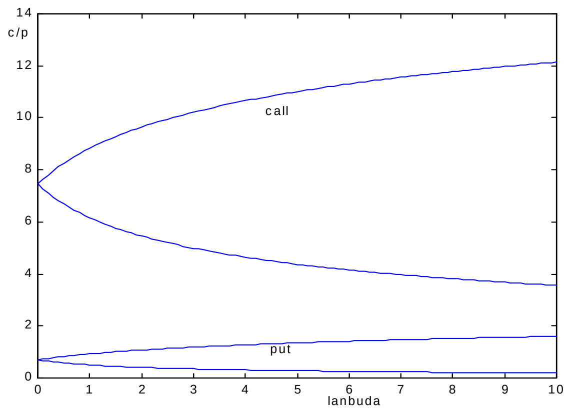

Figure 1 Relationship between option value (call or put) and λ

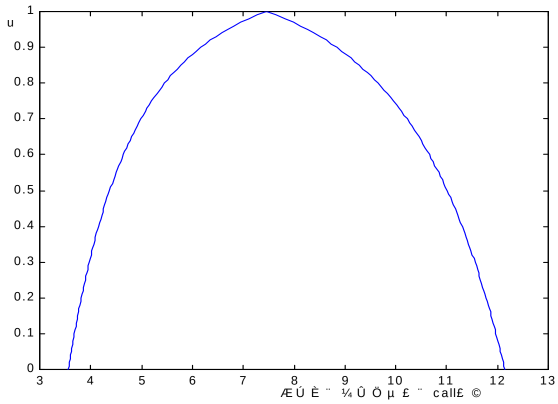

Figure 2 Membership function of the call fuzzy option

3. Delta Hedge Ratio of Fuzzy Options

Delta ratio is an essential property of options, which has profound and broad application in risk management and speculative portfolio selection. From the frame of the fuzzy options, the delta ratio is naturally induced as a fuzzy set.

Fuzzy option Delta is defined as the sensitivity of fuzzy option value, f, with underlined asset, $S_{0,}$ as follows,

$$\Delta = \frac{\partial f}{\partial S_0},$$

In discrete case it is used as $\Delta = \dfrac{\Delta f}{\Delta S_0}$.

Corresponding to any given λ, the λ-interval of the fuzzy option is expressed as

$$[f_t^{\lambda}, f_t^{\lambda^*}] = [CE_t^{*\lambda}[e^{-r(T-t)}X_T], CE_t^{*\lambda^*}[e^{-r(T-t)}X_T]]$$

Therefore the λ-interval of the Fuzzy option Delta is implied as following,

$$\Delta = [\Delta_*, \Delta^*] = [\min(\frac{\partial f_t^{\lambda}}{\partial S_0}, \frac{\partial f_t^{\lambda^*}}{\partial S_0}), \max(\frac{\partial f_t^{\lambda}}{\partial S_0}, \frac{\partial f_t^{\lambda^*}}{\partial S_0})]$$

and it could be proved to form a set nesting, furthermore the fuzzy option Delta is achieved.

Specifically, we consider a non-dividend European call option, its Delta hedge ratio can computed as follows,

For any given λ, the extreme point of the corresponding interval could be expressed as

$$c = S_0 e^{-r(T-t)} \int_{K/S_0}^{\infty} \mu(e_T > a)da$$

where

$$\mu(x) = \frac{1}{\lambda}[(1+\lambda)^{P(x)} - 1], \quad e_T = e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}x}.$$

Then

$$\frac{\partial c}{\partial S_0} = \frac{1}{S_0}[c + Ke^{-r(T-t)}\mu(N(d))]$$

there hence ，

$$\frac{\partial c_t^{\lambda}}{\partial S_0} = \frac{1}{S_0}[c_t^{\lambda} + Ke^{-r(T-t)}\mu^{\lambda}(N(d))],$$

$$\frac{\partial c_t^{\lambda^*}}{\partial S_0} = \frac{1}{S_0}[c_t^{\lambda^*} + Ke^{-r(T-t)}\mu^{\lambda^*}(N(d))]$$

$$\Delta = [\Delta_*, \Delta^*] = [\min(\frac{\partial c_t^{\lambda}}{\partial S_0}, \frac{\partial c_t^{\lambda^*}}{\partial S_0}), \max(\frac{\partial c_t^{\lambda}}{\partial S_0}, \frac{\partial c_t^{\lambda^*}}{\partial S_0})].$$

Similarly，for a put option, we obtain

$$p = S_0 e^{-r(T-t)} \int_0^{K/S_0} \mu(e_T < a)da$$

therefore

$$\frac{\partial p}{\partial S_0} = \frac{1}{S_0}[p - Ke^{-r(T-t)}\mu(N(-d))]$$

$$\frac{\partial p_t^{\lambda}}{\partial S_0} = \frac{1}{S_0}[p_t^{\lambda} - Ke^{-r(T-t)}\mu^{\lambda}(N(-d))],$$

$$\frac{\partial p_t^{\lambda^*}}{\partial S_0} = \frac{1}{S_0}[p_t^{\lambda^*} - Ke^{-r(T-t)}\mu^{\lambda^*}(N(-d))]$$

$$\Delta = [\Delta_*, \Delta^*] = [\min(\frac{\partial p_t^{\lambda}}{\partial S_0}, \frac{\partial p_t^{\lambda^*}}{\partial S_0}), \max(\frac{\partial p_t^{\lambda}}{\partial S_0}, \frac{\partial p_t^{\lambda^*}}{\partial S_0})]$$

For an intuitionistic view, we present all graphs for all Delta mentioned above, by taking specific parameter values. And through numerical simulation we get different values of all those Delta, which says the concepts given above are feasible. But still need some practical cases to test the definitions.

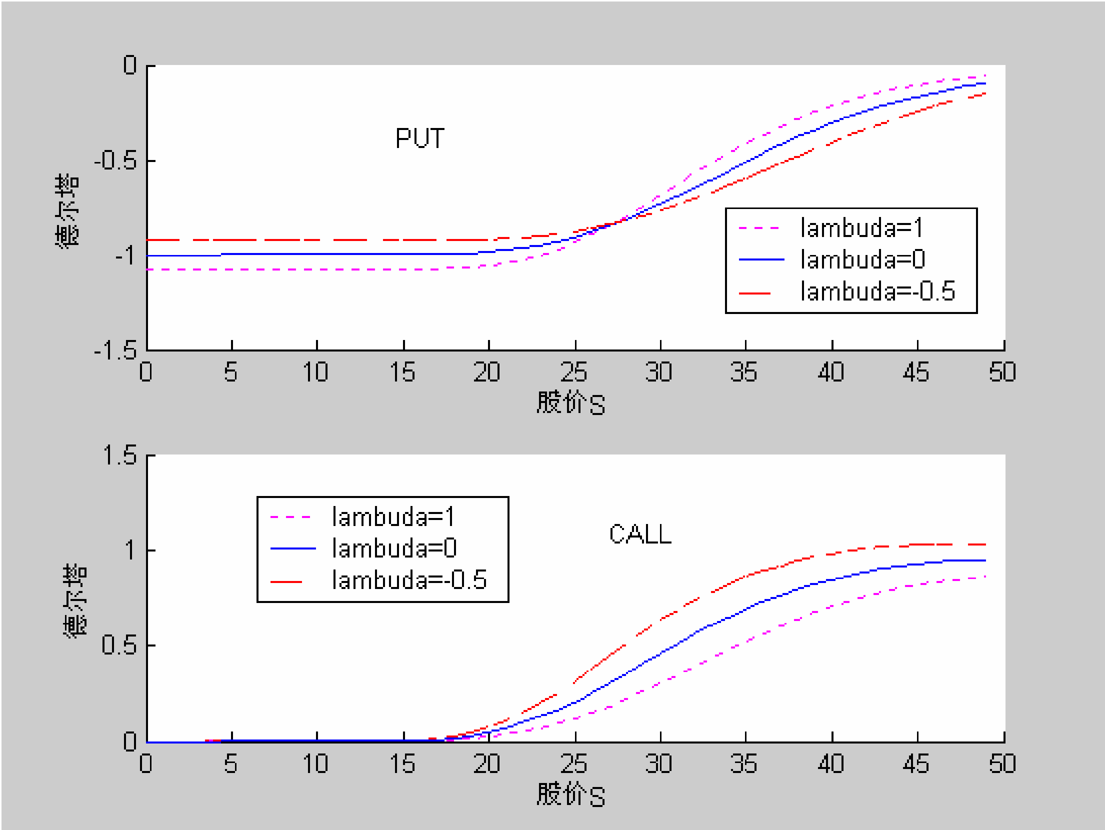

Figure 3 Relation between Delta and option prices of non-dividend European options

($r=0.08$, $\sigma=0.25$, $T-t=1$, $K=40$, $\lambda=1$)

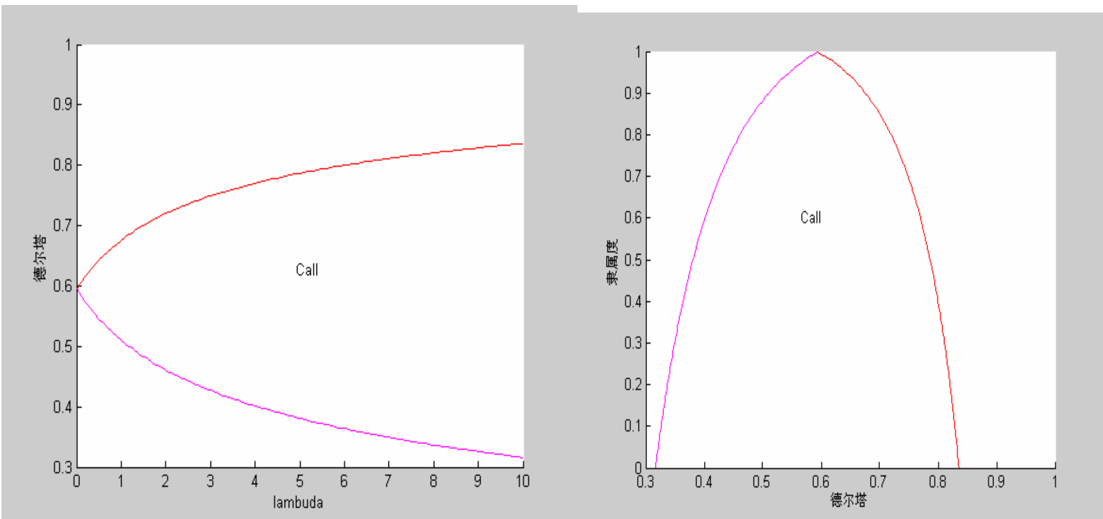

Figure 4 Delta of non-dividend European call options

$r=0.08$, $\sigma=0.25$, $T-t=1$, $K=40$, $S_0=38$

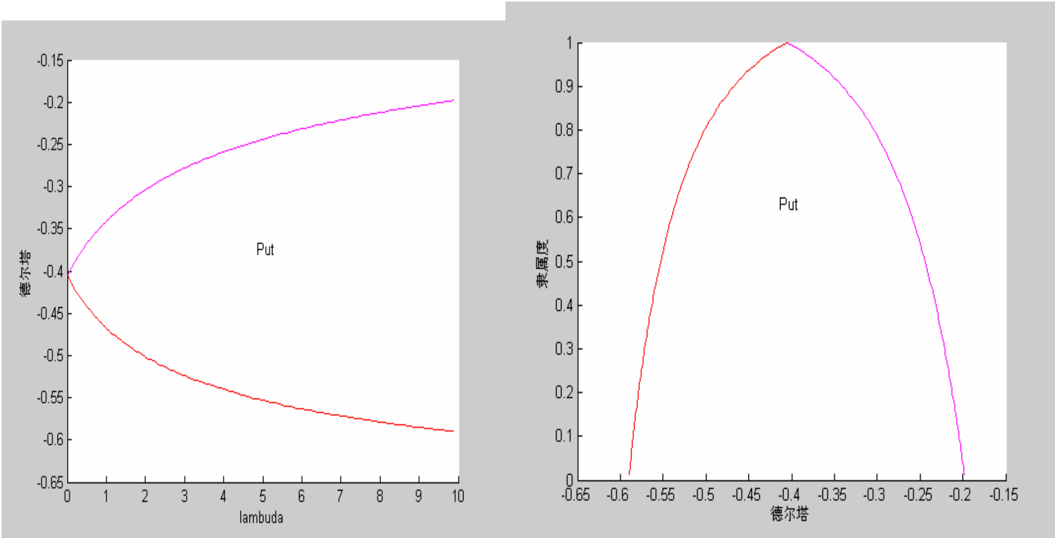

Figure 5 Delta of non-dividend European put options

$r = 0.08$， $\sigma = 0.25$， $T - t = 1$， $K = 40$， $S_0 = 38$

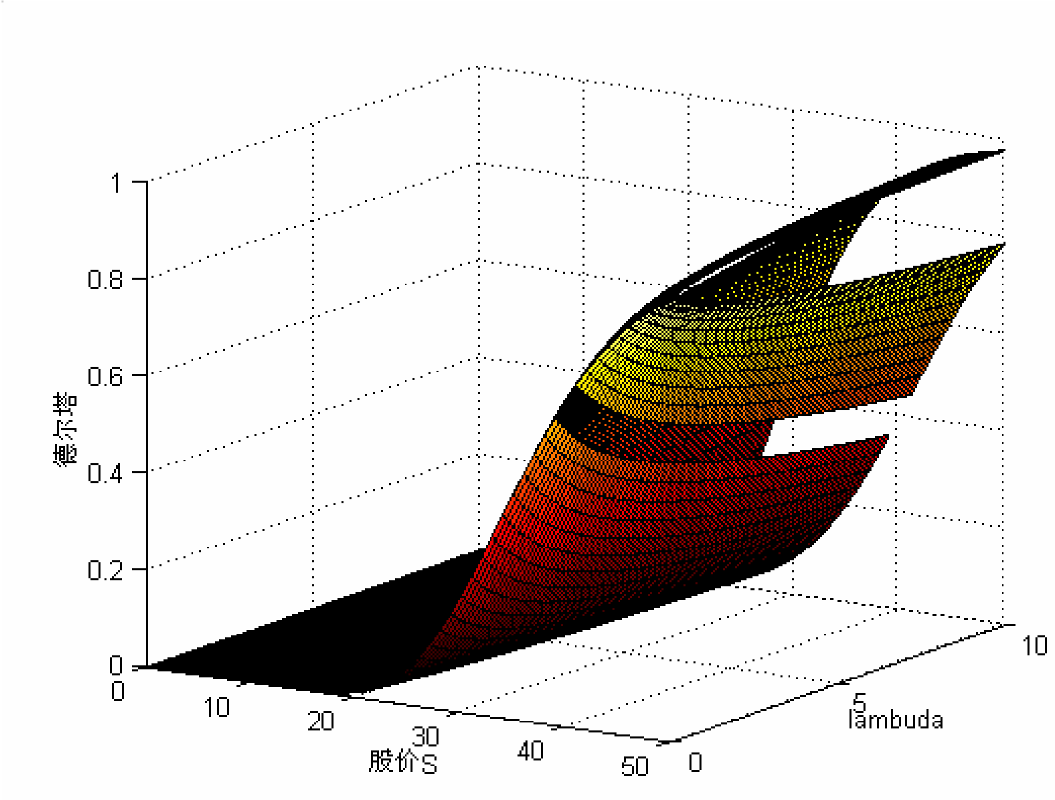

Figure 6 Fuzzy sets for Delta of non-dividend European call options

$r = 0.08$， $\sigma = 0.25$， $T - t = 1$， $K = 40$， $S_0 = 40$ where being sliced

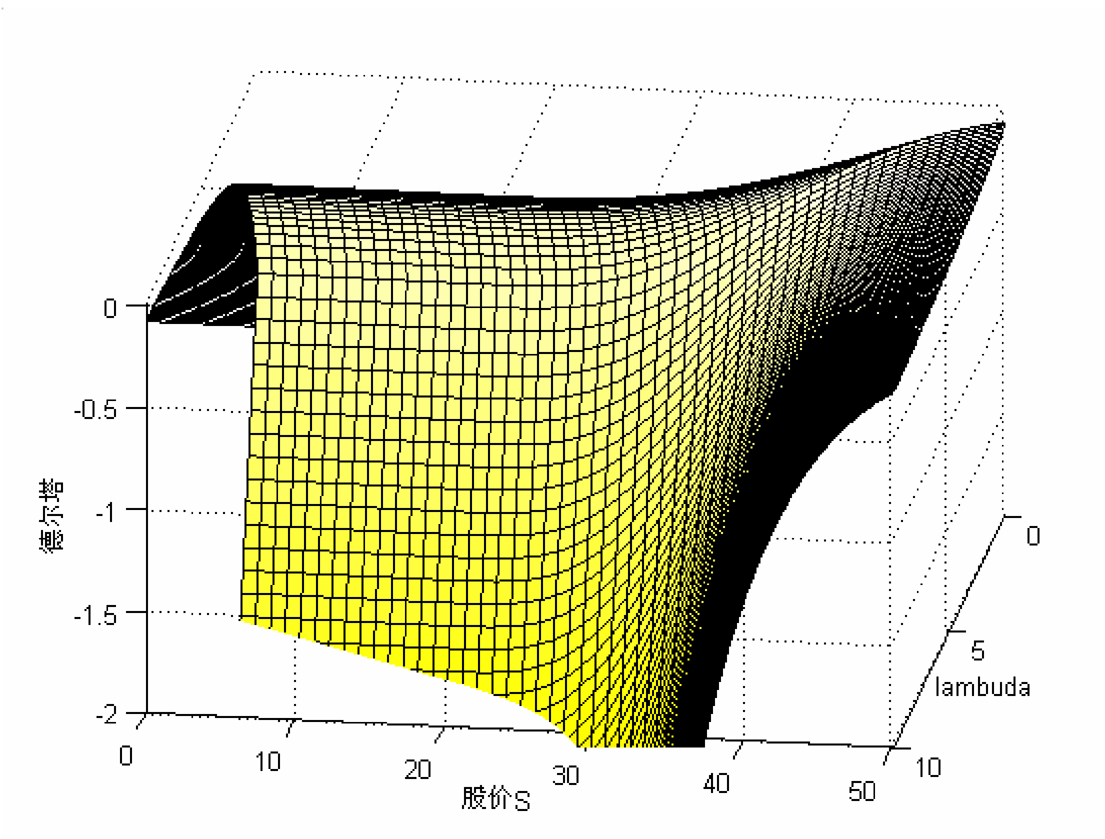

Figure 7 Fuzzy sets for Delta of non-dividend European put options

$r = 0.08$， $\sigma = 0.25$， $T - t = 1$， $K = 40$

For the convenience of calculation, we take an approximate expansion for a fuzzy option, achieved as

for fuzzy call options：

$$c = S_0 e^{-r(T-t)} \int_{K/S_0}^{\infty} \mu(e_T > a) da$$

$$= S_0 \mu[N(d + \sigma\sqrt{T-t})](1+\lambda)^{N(d) - N(d+\sigma\sqrt{T-t})} - Ke^{-r(T-t)} \mu[N(d)]$$

for fuzzy put options：

$$p = S_0 e^{-r(T-t)} \int_0^{K/S_0} \mu(e_T < a) da$$

$$= -S_0 \mu[N(-d - \sigma\sqrt{T-t})](1+\lambda)^{N(-d) - N(-d-\sigma\sqrt{T-t})} + Ke^{-r(T-t)} \mu[N(-d)]$$

Furthermore, we have approximation for fuzzy Delta, as follows

$$\frac{\partial c}{\partial S_0} = \mu[N(d + \sigma\sqrt{T-t})](1+\lambda)^{N(d) - N(d+\sigma\sqrt{T-t})}$$

$$\frac{\partial p}{\partial S_0} = -\mu[N(-d - \sigma\sqrt{T-t})](1+\lambda)^{N(-d) - N(-d-\sigma\sqrt{T-t})}$$

We have also calculated the Gamma ratio, Rho ratio and Theta ratio for the fuzzy options, respectively, omitted here.

4. Default Risk Analysis for Municipal Bonds in China Based on Fuzzy Options

The municipal bounds are still in the stage of academic research owing to law limit. One of the most important obstacles is the unclear default risk control. Having the same market conditions and the same administrative institution, all agents involved including policy makers, investors and the municipal authorities, do not share the same understanding of the risk. In another words, they are endowed with quite different rationalities. Therefore it is worth trying the fuzzy options method, and making a comparative study with traditional methods.

Upon the base of KMV model, we calculate fuzzy default risk by using the family of fuzzy measures. We take the λ-additive fuzzy measures as shown in formula (6), $\mu_\lambda, \lambda \in (-1, +\infty)$. When λ is restricted in [0, +∞)，λ* will be in (-1, 0].

As usual, the municipal fiscal revenue at time t, A_t, follows $dA_t = \mu A_t dt + \sigma A_t dZ_t, t > 0$, there hence the municipal fiscal revenue subjects to a stochastic process, $A_t = f(Z_t | A_0)$ where is the municipal fiscal revenue in year t, t=0,1,…,T, Z_t as a stochastic process which stands for all other factors determining A_t besides A_0. At maturity T, the municipal government will be default if A_t is smaller than the total face volume of the issued municipal bound, F, denoted by $A_T < F$.

Let p stand for the default probability, i.e. default risk, then

$$p = P[A_T < F] = P[f(Z_T | A_0) < F] \tag{11}$$

here it actually does not assign any type of distributions to Z_T, which is quite different from the KMV model. However if it is assumed that $Z_T \sim N(0,1)$, then (11) could be rewritten as:

$$p = P[Z_T < f^{-1}(F)] = N[f^{-1}(F)] \tag{12}$$

Let $DD = -f^{-1}(F)$, which is called the default distance, hence

$$p = N(-DD) \tag{13}$$

From this point, we can have the same result as KMV model.

Based on formula (11), we could set the fuzzy default risk model for municipal bonds by applying the fuzzy options methods.

As the specified municipal bonds compared with normal stocks, we can not calculate the fiscal revenue and its volatility as in KMV model; and therefore forecast the fiscal revenue by forming an econometric model with appropriately choosing factors.

For clearness and simplicity, we build the model of default risk for municipal bonds based on fuzzy options, step by step as follows:

1. Take the maturity T;

2. Determine the default point F, and choose the safe issue size (i.e. liability size) by the criterion of KMV or Moody. Here for simplicity, it is assumed that being as zero-coupon bonds, the municipal bonds issued will be clearly paid off at the maturity, T=1, therefore the default point is taken as the issue size each year.

3. Estimate fiscal revenue and its volatility. Usually calculation could be achieved through iteration by option pricing formula. However, due to the well done relationship between GDP and the fiscal revenue, an econometric model could be developed for estimating the fiscal revenue at time t, denoted by A_t^0, further then given A_t^0, the real fiscal revenue could be defined as

$$A_t = A_t^0 \exp(\sigma_A w_t) = A_t^0 \exp(\sigma_A \sqrt{t} Z_t) \tag{14}$$

where w_t is a Wiener process, $Z_t \sim N(0,1)$; and σ_A is the volatility of the fiscal revenue, which could be estimated by formula (15) setting τ=1,

$$\ln \frac{A_{t+\tau}}{A_t} = \ln \frac{A_{t+\tau}^0}{A_t^0} + \sigma_A (w_{t+\tau} - w_t) = \ln \frac{A_{t+\tau}^0}{A_t^0} + \sigma_A \sqrt{\tau} Z \tag{15}$$

4. Determine the scale of hypothecation, *k*.

5. Calculate the default distance and fuzzy default risk.

Let DD stand for default distance, from formula (14), we have

$$DD = -f^{-1}(F) = \frac{\ln\left(kA_T^0\right) - \ln F}{\sigma_A \sqrt{T}} \tag{16}$$

Then calculate the fuzzy default risk, which is induced from the λ-set valued mapping, whose value denoted $\left(EDF\right)_\lambda$, is called the λ-default risk interval

$$\left(EDF\right)_\lambda = [\left(EDF\right)_\lambda^L, \left(EDF\right)_\lambda^U] \tag{17}$$

where in detail

$$\left(EDF\right)_\lambda^L = \mu_\lambda(A_T < F) = \mu_\lambda[P(A_T < F)] = \mu_\lambda[N(-DD)] \tag{18}$$

$$\left(EDF\right)_\lambda^U = \overline{\mu}_\lambda\left(A_T < F\right) = \mu_{\lambda^*}[P(A_T < F)] = \mu_{\lambda^*}[N(-DD)] \tag{19}$$

Let T=1 (year), then for computing fiscal revenue, an econometric model is developed by adapting Beijing data from the year 1971 to the year 2000, resulting in

$$A_t^0 = 29.874 + 0.095\left(GDP\right)_t + [\mathrm{AR}(1) = 2.246, \mathrm{AR}(2) = -1.499, \mathrm{MA}(1) = -0.990] \quad (20)$$

(11.982) (10.897) (R^2=0.994 SE=7.433 DW=1.877 F=1019.289)

With taking the average of the real economic growth of Beijing for recent five years as 12.2%, we imply $A_{2001}^0 = 47.614$ (billion RMB), $\sigma_A = 0.109343$. Then after getting rid of the autonomous expenditure can we have the amount of fiscal revenue for hypothecating the issued municipal bonds. Based on statistical analysis for historical data from 1971 to 2000, we have found that the year of 1993 through the year of 2000 forming a same class, so the average of government expending across these years can be taken as the autonomous expending nowadays, which possesses 50% of the fiscal revenue in 2000. Therefore the scale, k, is defined as 0.5.

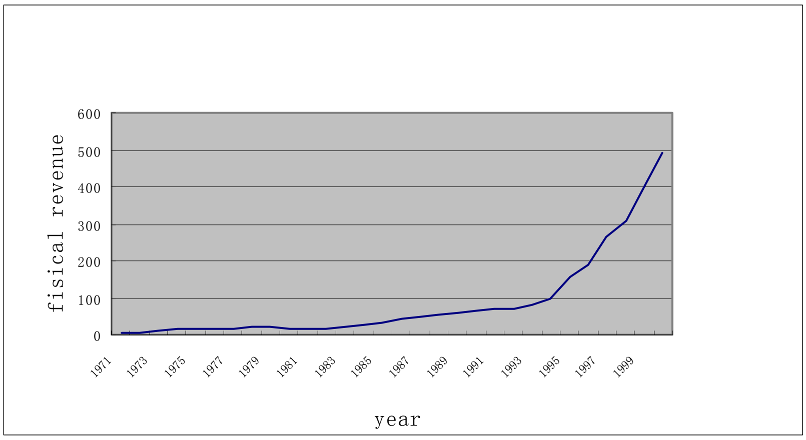

Figure 8 Beijing fiscal revenue (Source : Beijing Statistics Yearbook 2002)

In the research we have had a concrete discussion on the type of the distribution of fiscal revenue. First, the fiscal revenue is assumed to follow the logarithm normal distribution, and then the real distribution is introduced into consideration.

Under the lognormal assumption, we have achieved in the map from the size to the default distance and the default probability when the membership being 1, shown in figure 1, and the correspondence from the membership level to the λ-interval of the fuzzy default risk, as in figure 2, by applying formulas 14 through 19.

Table 1 Default distance and default risk with the size given membership being 1 （unit：billion RMB）

Issue size	7.5	10.0	12.5	15.0	16.0	17.5	20.0	22.5
Default distance	10.5638	7.9328	5.8920	4.2219	3.6316	2.8121	1.5909	0.5137
Default risk	0	0	0	0	0.00014	0.0025	0.0558	0.3037

Table 2 λ-interval of fuzzy default probability with lognormal assumption

Membership level	default risk（lower bound）	default risk（upper bound）
1	0.0025	0.0025
0.9	0.0017	0.0035
0.8	0.0014	0.0041
0.7	0.0012	0.0046
0.6	0.0010	0.0050
0.5	0.0009	0.0054
0.4	0.0008	0.0057
0.3	0.0007	0.0059
0.2	0.0007	0.0062
0.1	0.0006	0.0064
0	0.0006	0.0066

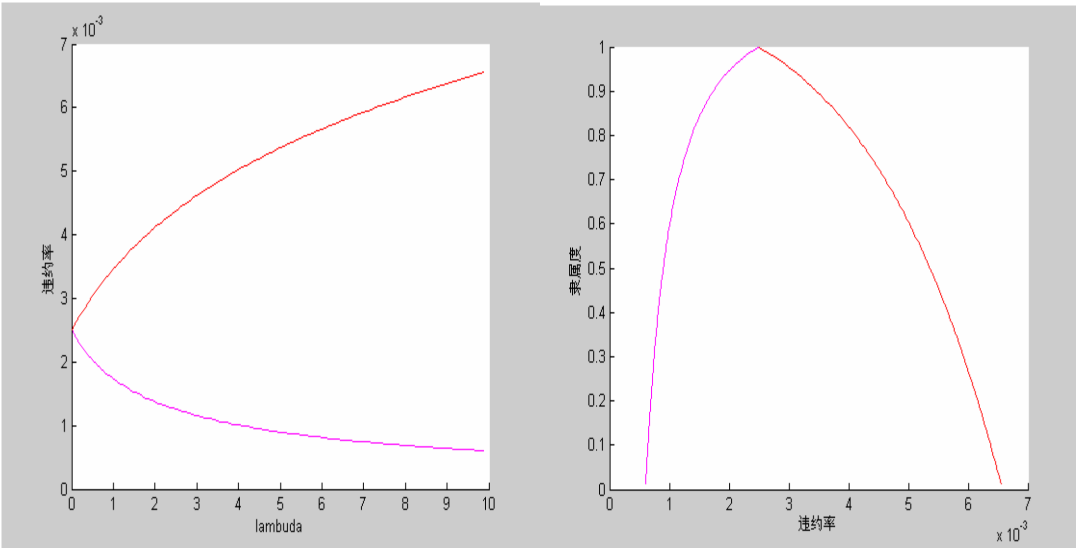

Figure 9 Fuzzy default ratio (liability size =17.5 bill., logarithm normal case)

From the tables above we can see, assuming that the fiscal revenue follows logarithm normal distribution, the default probability with the size being 17.5 billion is only 25 BP, as 17.5 billion is already 74% of the whole amount of hypothecation. In fact the historical data graph of fiscal revenue Beijing does not support the logarithm normal assumption, which density has a bigger tail. Therefore we introduce “the real distribution of fiscal revenue” statistically.

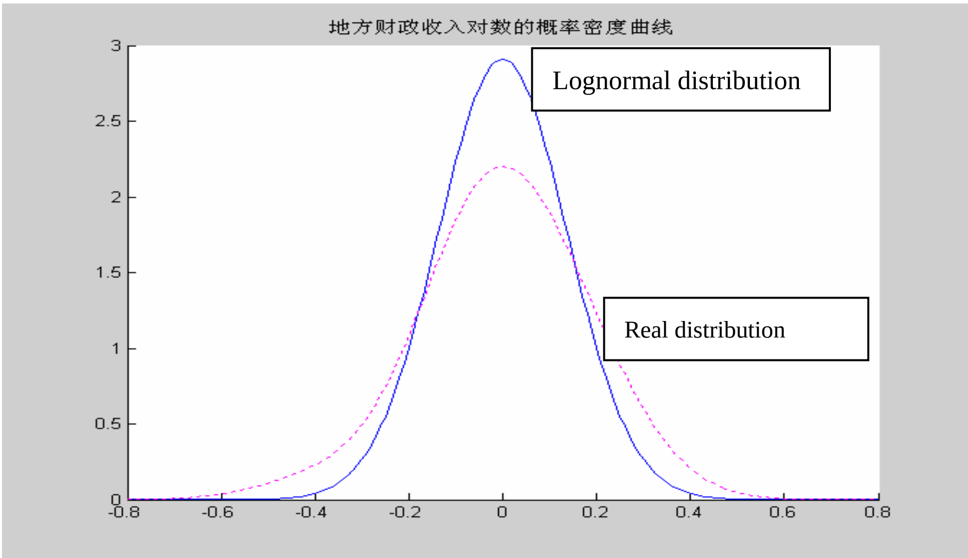

Figure 10 Comparison between the lognormal distribution and the real (implied) distribution

In fitting the real distribution for fiscal revenue, we specifically use the kernel estimation and numerical approaching methods, and get the result by operating the software Mathlab. The fitted real distribution is more flat and more scattered compared with that of the logarithm normal one. So under the real distribution, the default risk is much greater than that under the logarithm normal assumption corresponding to the same issue size, see table 3. This is consistently accorded with the conservative attitude to the default risk.

Table 3 Two different default risk corresponding to the issue size when the membership being 1

Issue size (unit: billion RMB)	7.5	10	10.5	11	12.5	15	17.5	20	22.5

Theoretic default risk	0	0	0	0	0	0.0001	0.0025	0.0558	0.3037
Real default risk	0.0000	0.0019	0.0031	0.0049	0.0156	0.0611	0.1513	0.2795	0.4246

It is clear that under the real distribution, the default risk is more sensitive to the issue size, which explores the importance of controlling the default risk when issuing municipal bonds in Beijing. Table 4 gives membership of the fuzzy default probability.

Table 4 λ-interval of fuzzy default probability with real-distribution assumption

Membership level	default risk（lower bound）	default risk（upper bound）
1	0.0019	0.0019
0.9	0.0013	0.0026
0.8	0.0010	0.0031
0.7	0.0009	0.0035
0.6	0.0008	0.0038
0.5	0.0007	0.0041
0.4	0.0006	0.0043
0.3	0.0006	0.0045
0.2	0.0005	0.0047
0.1	0.0005	0.0049
0	0.0005	0.0050

Looking at table 4 one can get the default interval under the given membership. Similarly one can build this kind of tables for the fuzzy default risk corresponding to any given issue size of a municipal bond.

For the convenience of decision maker of issuing municipal bonds, we provide a three-dimension graph which connecting those three: the size of bonds (the liability size), the default risk and its membership to the fuzzy default risk, as in Figure 3. From the figure one can see the fuzzy default risk demonstrating by its membership curve, and watch clearly that for any given membership level, the default risk rises with the rising of the size of bonds, such as the “ridge line” stating the situation of the membership 1.

We can also construct the indifference curves for given default probabilities, showing in Figure 4. A dished line stands for the upper boundary of the equal default risk line, and the solid line stands for the lower boundary of the equal default risk line, which means for any given default risk there exist two the equal default risk lines that tell you the upper boundary and the lower one. The region lying on the left-above of the dished line is the absolutely safe one, the region lying on the right-below of the solid line is the absolutely unsafe region, and the one between that two lines is both safe and unsafe region.

KMV corporate formulas the rating table for corporate bonds. Upon that standard it is held the corporate bonds would be safe when their rating ranks above BBB (Standard poor) or Baa3 (Moody). As the issuer of municipal bonds are local governments or their agencies, the default risk should be smaller than that of corporate bonds. So the rating rank of a municipal bond should be qualified at least BBB+ or Baa1, with the value of the expected default probability ling within 0.4%, and in fact the average default risk in US for about 55 years is only 0.5%, usually as 0.3%, which gives the solid support to find the suitable risk level for the Chinese situation.

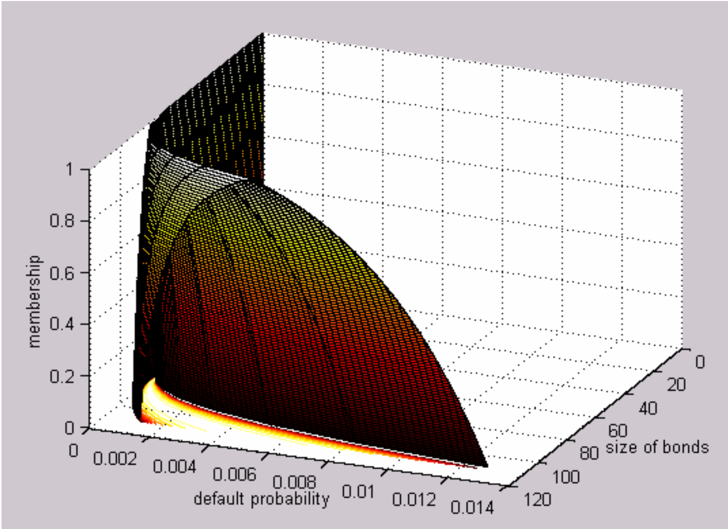

Figure 11 The fuzzy default risk parameterized with the issue size

Referring to the correspondence above, we can find that with the default risk as 0.4%, the issue size of municipal bonds in Beijing in 2001 should lye between 10 billion RMB and 12 billion RMB with the membership level as 0.7. In fact, it is quite reasonable if we take 11 billion RMB as the upper limit of issue size of municipal bonds in Beijing in 2001, for it is about 10%-20% of the expected fiscal revenue Beijing and 1%-3.5% of Beijing's expected GDP. The suggestion is supported by US experience, for the issue size of long-term municipal bonds takes averagely 12% of local government's fiscal revenue, and takes 2%-3% of GDP.

4. Conclusions

On the research of behavioral finance by many literatures, it is necessitated to break through the identical rationality in option pricing. Having reference to Cherubini's work (1997，2001), instead of the unified fuzzy measure, we use a family of λ-additive fuzzy measures, which induced from risk-neutral probability measure. Furthermore with the Choquet expectations, a set-valued correspondence from λ to option prices is formed, which naturally induces a fuzzy set, i.e. the fuzzy option. Some basic properties and applications to financial analysis are developed in our succeeding study, and here a case of default risk analysis for municipal bonds in China is demonstrated, in which the mapping between the issue size and fuzzy default probabilities is established, and the safe issue size is implied. By the comparison study with normal KMV model, the default risk estimate based on fuzzy options is more tolerated with varying of states of nature, and more approaching to human being's decision.

There are many interesting topics related with fuzzy options could be considered, such as the choice of the family of fuzzy measures, multiple- equilibrium's existence, hedge ratio with fuzzy options, the distribution of parameter indicating the family of fuzzy measures, etc.. Besides those topics we are trying to develop the applications of fuzzy options on portfolio insurance, convertible bonds pricing and risk analysis for corporate bonds. In theoretical field, set-valued integrals and set-valued martingale measures are other alternatives in framing the fuzzy options.

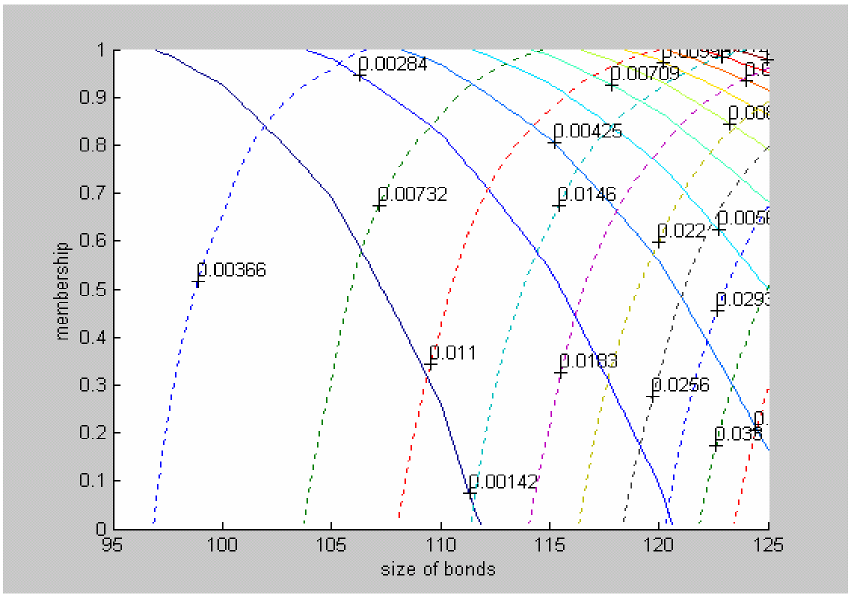

Figure 12 The indifference curves for the given default probability

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Appendix

Basic Properties of Choquet integrals:

Let A be a subset of X , and f and g be two functions on X , then we have:

(1) $(c)\int 1_A d\mu = \mu(A)$

(2) If $f \le g$, then

$$(c)\int f d\mu \le (c)\int g d\mu$$

(3) If a is a non-negative number and b any real number, then

$$(c)\int (af + b) d\mu = a \cdot (c)\int f d\mu + b \cdot \mu(X)$$

(4) $(c)\int (-f) d\mu = -(c)\int f d\overline{\mu}$

(5) $(c)\int f d\mu = (c)\int f^+ d\mu - (c)\int f^- d\overline{\mu}$

where $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$

(6) If a is any real number, then

$$(c)\int f d(a \cdot \mu) = a \cdot (c)\int f d\mu$$

(7) If μ and ν are two fuzzy measures on X , and $\mu \le \nu$, $\mu(X) = \nu(X)$, then for any function defined on X , we have

$$(c)\int f d\mu \le (c)\int f d\nu$$

(8) If N is a null measurable set, and for all $x \notin N$, $f = g$, then

$$(c)\int f d\mu = (c)\int g d\mu$$

(9) If f and g are homologic, then

$$\int (f + g) d\mu = \int f d\mu + \int g d\mu$$