Equilibrium Asset Pricing with Defaultable Debt

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Abstract

I study the asset pricing implications of default in an equilibrium model with incomplete markets. Defaultable debt is not redundant in my model since markets are incomplete and agents suffer a utility penalty when they default. I find that, compared to the standard incomplete markets model, the equity premium is larger in my model. I also consider the effects of a decrease in the default penalty and an increase in income inequality. Both increase default rates and thus, provide possible explanations for the rapid increase in personal bankruptcies during the 1990s. Because these two effects have different implications for asset prices, this type of model may shed further light on the causes of the increasing number of bankruptcies. Finally, I study the effects of an increase in stock dividend and the comparative static properties of credit spread.

Preliminary and Incomplete

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1 Introduction

Consumer debt is not default-free. The most recent data show that bankruptcy filings broke the record for filings in any 12-month period to reach 1.5 million cases (American Bankruptcy Institute). Yet in many equilibrium asset pricing models, agents never default on their borrowing and short selling. This makes us wonder how the asset pricing implications of current default-free models will change if we indeed allow for default. Specifically, will allowing agents to default help resolve puzzles such as the “equity premium puzzle” (Mehra and Prescott (1985))? Furthermore, what are the empirical implications for the relationships between the prices of riskless bonds, risky bonds and stocks?\footnote{Incomplete market models have been studied extensively in the asset pricing literature.}

Closely associated with the problem of consumer default is the puzzling phenomenon that default rates increased rapidly during the booming period of the 1990s. Two possible reasons among others have been given to explain this. One is that the default penalty (or social “stigma”) has decreased (White (1997), Gross and Souleles (2002)). The other is that income inequality has widened over time.\footnote{In this paper, I use default-free bond and riskless bond interchangeably, as well as defaultable bond and risky bond.} The question is then whether one source is mainly responsible for the observed fact. This becomes a hotly discussed issue since the policy implications of these two sources are dramatically different. Is it possible to differentiate the two stories by looking at their effects on the security prices?

This paper tries to provide a model to answer the above questions. I study an otherwise standard two-period pure exchange incomplete market model. There are two classes of agents facing undiversifiable individual labor income risk. Thus the markets are incomplete. The agents also receive income from investment in a stock, which provides a claim to a stochastic dividend and which is in positive net supply. Agents can borrow and lend among themselves using defaultable bonds, which promise to deliver one unit of good in the second period. The net supply of these bonds is zero. Unlike models without default, here I allow agents to default on their borrowing. Namely they may choose, as they like, how much to deliver on their promised amount as long as it is nonnegative.
(for instance, Mankiw (1986), Weil (1992)), since this provides a possible way to help resolve the equity premium puzzle.\(^3\) A sufficient condition to do so is that agents exhibit decreasing absolute risk aversion and decreasing absolute “prudence” (Weil (1992)). To focus on the effects of default, in this paper, I will let agents have quadratic utility functions so that the effects studied in the above models do not exist. Furthermore, I will impose a utility loss as a penalty for agents’ default.

As Mankiw (1986) shows, if agents’ utility functions are quadratic and they exchange bonds, individual heterogeneity has no effect on the equity premium. In my model, agents can hold riskless bonds, but they cannot issue riskless bonds. Riskless bonds are in net positive supply and I study the limiting case when this net positive supply goes to zero. The price of the riskless bond is potentially determined by the shadow price from only one of the two classes of agents.

Even with quadratic utility, my model is still very difficult to solve. As it is well known, even the optimal consumption and portfolio choice problems with stochastic labor income are already very difficult to analyze, prompting Zeldes (1989) to use numerical methods to attack these types of problems. Recent theoretical advancement makes it easier to analyze individual problems, yet general equilibrium models remain hard to solve.\(^4\) In my model, this task becomes ever much more difficult because of two reasons. First, the payoffs of risky bonds are like those of options. There have been few studies on solving the optimal consumption and portfolio choice problems when one of the securities is an option. Second, since borrowers can choose the default amount, the payoffs of the risky bonds are endogenously determined. This makes all the current methods useless when solving the model. To get sharp results, here I use an approximation method to study the case when default rates are small.

\(^3\)Later research in the dynamic case shows that this is indeed possible. As shown in Constantinides and Duffie (1996), one can get any equity premium if the idiosyncratic labor income shocks are permanent. (However, Telmer (1993), Heaton and Lucas (1996) show agents can achieve near perfect risk-sharing by saving a lot in good times if these shocks are transitory.) The question becomes how persistent these shocks are.

riskless bond while the stock price does not change. This latter effect is due to the quadratic utility functional form I use. Thus, the risk-free rate is lower and equity premium is larger. The reason is that, in my model, the risk-free rate depends only on the marginal utilities of the lenders instead of those of the whole population. Thus, heterogeneity plays an important role. In equilibrium, lenders hold less securities (risky bond and stock) than they do in the default-free case. Intuitively, there are two effects of default. Default increases risk-sharing over states, since borrowers condition their payoff on the states. Default also decreases consumption smoothing over time, since lenders can get less than what they are promised. In my model, the latter effect dominates.

I also study comparative static properties for portfolio choices and prices. Specifically, I study the effects from decreasing the default penalty and increasing income inequality. Decreasing the default penalty has similar effects as those described in the comparison between default and non-default cases, namely as the default penalty decreases, the prices of the riskless bonds increase. This is intuitive, since when the default penalty is high enough, there is no default.

When the expected endowment difference between lenders and borrowers increases, the expected default rate increases, but the above described increase in the price of the riskless bond does not necessary hold. When income inequality increases, borrowers pay less, as they have less endowment. But there is another effect, namely lenders' endowments increase, dominating the effects from the payoff of the defaultable bond, and consequently, changing portfolio choices dramatically. The price effect is still small yet with uncertain sign. If the effect of increasing risk-sharing over states is small so that the effect of decreasing consumption smoothing over time dominates, the price of the riskless bonds will decrease. This can also be compared with the nondefault case: although changing in inequality changes consumption and portfolio holdings, there is no price effect.

These results can help answer the above mentioned policy problem. My model gives a testable empirical prediction that can differentiate the two by looking at the trading of risky bonds and stocks. Of course, how large these effects are will be mainly an empirical issue.

In addition to these results, I also study credit spreads, defined as the difference between the inverse of price of risky bond and that of riskless bond. In general, when the default
penalty decreases, income inequality increases or expected payoff of stock increases, credit spreads increase. The only exception is when the income inequality is small. In this case, the credit spread decreases.

The structure of the paper is the following. In the next section, I present the basic setup of the model. Then, I study as a benchmark case a model without default. Next, I study in detail the case with default. Finally, I conclude. The proofs are in appendices A and B.

1.1 Literature Review

There are few works about default in general equilibrium models although the literature has been growing. In a complete market case, Kehoe and Levine (1993), Kocherlakota(1996), Alvarez and Jermann (2002a, 2002b) study an economy in which agents are constrained by their debts. The default penalty is so high that in equilibrium there are no defaults. Other closely related work is Lustig (2001), who applies the same idea to study the wealth distribution. This type of model cannot be used in my case since it is the existence of default I am studying.

The theoretical study of default in general equilibrium models with incomplete markets starts with the work by Dubey, Geanakoplos and Shubik (2000) and Zame (1993). They showed that in an incomplete market setting, allowing agents to default can be welfare improving since it gives agents an extra option to smooth consumption according to agents’ own states. Adopting similar ideas, Athreya (2001) studies the welfare implications of imposing tougher default penalty and found it is small. Santos (2000) studies the endogenous generated financial innovation. Zha (2001) studies capital accumulation. All these works regard the defaultable securities as a “pooling” device. Namely one defaultable security sold by different borrowers has the same price. In my model, I allow both agents to sell their own bonds and they cannot buy their own bonds.

In other quantitative works, Chatterjee, Corbae, Nakajima and Rios-Rull (2001) consider a dynamic model in which agents sell their own bonds and study the welfare effect from defaulting. The default penalty in their model is institutional, namely seizure of the financial asset and limiting access to the credit market. In my model, the penalty is utility penalty. Other works can be found in for example, Espinosa-Vega and Smith (2001), Lawrence (1995),

There are some other research works in the fields closely related to what I study here. For example, Davis, Kublier and Willen (2002) study the effects on equity premium when agents facing different borrowing and lending rates using a numerical method. While they show that indeed higher borrowing costs indeed imply higher equity premium, one has to note that one of main reason why borrowing rate is higher than lending rate is due to the borrowers’ default possibility. One can find further theoretical works in Karatzas and Shreve (1998) and the literature cited there. Another literature is in the continuous time framework with complete market, Basak and Shapiro (2002), Chang and Sundaresan (2000) etc. study the pricing implications from default. Their results show that defaulting indeed brings much richer structure to the known asset pricing literature.

2 The Economy

There are two periods, $t = 0, 1$ with uncertainties at $t = 1$. We denote the states at $t = 1$ as $\omega$. And there is one good for consumption in both periods.

There are two classes of agents in the economy, $i = 1, 2$.\footnote{One may assume there is a continuum of agents with total mass of one for each class. This is to insure that there is no strategic plan for each individual agent.} They are endowed with $e_i, t = 0, 1$ in the two periods. The agents are identical in terms of preferences and initial portfolio holdings. The difference between the two agents comes from their different stochastic endowments. More specifically, both agents have the same endowment at time $t = 0$, yet their time $t = 1$ endowments have different distributions with support on closed interval $[e_{\text{min}}, e_{\text{max}}]$ with $e_{\text{min}} > 0$. Within each class, agents have the same shocks to the endowments.

Remark 1 The fact that their distributions have the same support is without loss of generality. That the lower bounds of agents’ endowments are larger than zero is necessary to get the benchmark case in which agents exchange default-free bonds.
We denote preferences over consumption as \( u^i(c) \), where \( c \) must be nonnegative. Furthermore if an agent defaults amount \( d^i \) at \( t = 1 \), the utility will be denoted as \( w^i(d^i) \). So an agent \( i \) will maximize the utility function as follows:

\[
v^i = u_0^i(c_0^i) + E \left\{ u_1^i(c_1^i) + w^i(d^i) \right\}
\]  
(1)

**Assumption 2** I assume that \( u_0^i, u_1^i \) are strictly increasing, strictly concave, \( w^i \) is strictly decreasing, concave. And they are all smooth functions. They satisfy the following:

\[
\begin{align*}
    u_c &> 0, \quad u_{cc} < 0, \\
    w_d &< 0, \quad w_{dd} \leq 0.
\end{align*}
\]

**Remark 3** The assumptions about \( u(\cdot) \) are standard. While \( w_d < 0 \) simply says when an agent defaults more, the disutility is larger. The requirement \( w_{dd} \leq 0 \) has the similar purpose as the concavity of \( u(\cdot) \), namely the concavity over \( d \) will guarantee the existence of a solution to the individual optimization problem.

**Remark 4** I assume that \( u(\cdot) \) and \( w(\cdot) \) are additively separable. A general form would be \( u(c, d) \), which I will discuss briefly later on.

To make the argument as transparent as possible, I will further specify the utility functions to be quadratic, namely:

\[
\begin{align*}
    u_0^i(c) &= u_1^i(c) = -\frac{1}{2}(c - c^*)^2, \quad i = 1, 2
\end{align*}
\]  
(2)

I will assume that the “satiation point” \( c^* \) will never be reached.

**Remark 5** As Mankiw (1986) and Weil (1992) have shown, for incomplete markets to play a role in the equilibrium asset prices, the third order derivatives of the utility function must not be zero. In other words, heterogeneity across agents with quadratic utility functions does not affect equilibrium asset prices when the market is incomplete. Here I choose the quadratic utility form simply wanting to show that the results I obtain do not come from agents’ precautionary motives. I will give some discussions on power utility functions later on.
There is one stock in the economy, which is a claim to some dividend payoff at $t = 1$. The net supply of the stock is assumed to be one share. I further assume that the dividend is continuously distributed on the closed interval $[0, D_{\text{max}}]$ and the dividend is uncorrelated with the endowments $e_i$.

**Remark 6** The support of the dividend is upper bounded to make sure that the “satiation point” $c^*$ is never reached. The assumption that the dividend is uncorrelated with the endowments is for simplicity.

The agents can also trade bonds. When an agent sells one share of defaultable bond, he promises to pay one unit of good at $t = 1$. However at $t = 1$, he may not deliver that much, namely he may default. When an agent buys one share of bond from other agents, he will believe the delivery rate of other agents to be, say $\kappa(\omega)$. Here $\kappa(\omega)$ is state dependent. Rational expectations then require that the (believed) delivery rate $\kappa$ at each state $\omega$ will be the same as the actual one.

Note the difference between the defaultable bond and the default-free (riskless) bond. For a riskless bond, the payoff is predetermined and the same no matter which agent sells the bond, while for a defaultable bond, the bonds sold by different agents potentially have different payoffs. So the two bonds sold by the two agents are actually two different (risky) securities. I call the bond sold by agent $i$ to be $B_i$ with price $P^{B_i}$. Let me denote by $\alpha^i_j$ to be the absolute number of shares of bond held by agent $i$ and sold by agent $j$.

To compare the asset price implications with the case in which there are no risky bonds, I introduce riskless bond into the model. Agents can only hold nonnegative amounts of the riskless bond, which is in positive but small net supply. In other words, agents can save but cannot borrow using riskless bonds. This is consistent with the fact that agents cannot commit themselves to fulfill the promises they make. I denote the price of the riskless bond $B_0$ to be $P^{B_0}$ and the agents’ holdings of riskless bonds at the beginning of the second period to be $\alpha^0_i$. Similarly, I denote the stock as $S$ with price $P^S$ and the stock holdings for agent $i$ at the beginning of second period to be $\theta^i$. Again I impose the same restraints on agents’ stock holdings as those on riskless bonds, namely agents must also hold a nonnegative amount of stocks.
I assume initially agents do not hold any defaultable bonds, and each agent holds half of the stock and riskless bond supply.

**Remark 7** The positive net supply of riskless bond is not new. It does not mean that agents cannot exchange riskless bonds. As one will see, when the default penalty is high enough, the bonds agents trade with each other become default-free. In this case, the prices of all the bonds are the same. I introduce positive net supply of riskless bonds to get the riskless rate when borrowing is defaultable. Specifically I study the limiting case in which this net supply goes to zero. The sole purpose of this procedure is to endogenize the price of the riskless bonds, which, I will show, is determined by the lender’s marginal rate of substitution. A usual way to get the riskless rate is to use the shadow price from agents’ intertemporal rate of substitution directly. However in this model, there are two types of agents and it is not clear ex ante whose utility will imply the higher shadow price. Using the above approach this becomes transparent and indeed I can show that lenders, who will hold riskless bonds, have a higher intertemporal rate of substitution.

### 2.1 The Choice of Default Function

There are many different penalties agents may suffer when they choose to default. For example, there are institutional penalties such as lower credit score, limiting access to the credit market and seizure of non-exempt assets etc. Here I model the penalty as a loss of utility. I interpret the utility penalty as a form of “stigma” or “social norm”. Some previous exercises in choosing utility penalty include very simple examples such as fixed cost and sometimes proportional cost.

I first assume that the default penalty is a function of the default amount $d$. Often one imposes a fixed penalty on the borrowers upon default. The argument here is that defaulting is a zero-one event: either agents default or they do not. Although institutionally this can be considered as one event, it seems that a person who defaults $1$ million will get different social judgment than will a person who defaults $1$, if only in terms of public scrutiny. Furthermore as I will show below, by letting the default penalty to be a function of default amount I can obtain an optimal default decision rule such that agents can default partially.
This actually happens quite often in the real world.\footnote{Again I thank Satyajit Chatterjee for kindly pointing this out to me.}

I will adopt the following form for $w(d)$:

$$w(d) = -\frac{1}{2} \left( (d^* + d)^2 - d^*2 \right), \quad (3)$$

where $d^* \geq 0$ is a constant. $d^*$ is a measure for default penalty. This penalty form has the following properties:

- $w(0) = 0$: When there is no default, there is no penalty.
- $w'(d) < 0, w''(d) < 0$.
- The default amount $d$ is measured in the units of the consumption good. Indeed as I will show later on, default amount $d$ is a linear function of agents’ endowments. Using proportional cost for default penalty does not have this property and has the undesired the results that in the (partial) default region, consumption is a constant.
- To see that $d^*$ is indeed a measure for default penalty, I want that as $d^*$ becomes larger, $w(d)$ becomes lower. This can be seen immediately by rewriting $w(d)$ as $w(d) = -1/2d^2 - d^*d$. And as I will show later on, when $d^*$ goes to zero, agents will always want to (partially) default.

2.2 Budget Constraints and Equilibrium

I am now in a position to state the budget sets for the agents. An agent $i$ will take the prices of the bonds $B$ and stock $S$, and the delivery rate of the other agent’s bond $\kappa(\omega)$ as given. The budget set $B^1(P^B_0, P^B_1, P^B_2, P^S, \kappa_2(\omega))$ for agent 1 is then a set consisting of $(c_0, c_1(\omega), \alpha_0, \alpha_1, \alpha_2, \theta, d(\omega))$ (I omit the superscript 1 here, and agent 2’s budget set can be obtained similarly). Here all the variables are nonnegative with $(c_0, c_1(\omega))$ being the consumption in both periods, $(\alpha_0, \alpha_1, \alpha_2, \theta)$ being the agent 1’s portfolio holdings in riskless bond, bond sold by agent 1 himself, bond sold by agent 2 and stocks, and finally $d(\omega)$ being
the amount agent 1 decides to default. They have to satisfy:

\[
c_0 \leq e_0 - \left( (\theta - \frac{1}{2})P^S + (\alpha_0 - \frac{b}{2})P^B_0 - \alpha_1 P^B_1 + \alpha_2 P^B_2 \right),
\]

\[
c_1(\omega) \leq e_1(\omega) + \theta D_1(\omega) + \alpha_0 - \alpha_1 + d(\omega) + \alpha_2 \kappa_2(\omega),
\]

\[
d(\omega) \geq 0,
\]

\[
d(\omega) \leq \alpha_1.
\]

Note that \(\alpha_1\) is a nonnegative number. The first two equations are just the usual budget constraints. At \(t = 0\), agent 1 has endowment \(e_0\). The agent chooses how much stock \(\theta\), riskless bond \(\alpha_0\), agent 2’s bond \(\alpha_2\) to buy and how much bond \(\alpha_1\) to sell as well as how much goods to consume. Note that initial holdings for stock and riskless bond are \(1/2\) and \(b/2\) respectively. At \(t = 1\) and state \(\omega\), he gets \(e_1(\omega) + \theta D_1(\omega) + \alpha_0 - \alpha_1 + d(\omega) + \alpha_2 \kappa_2(\omega)\) from endowment, riskless bond, stock holding and the other agent’s delivery for bond. He also has to deliver \(\alpha_1\) for the bond he sold, yet he can default by amount \(d(\omega)\). So the net payout for the risky bond is just \(\alpha_1 - d \geq 0\). The next two constraints say that the default amount must be nonnegative and not larger than the total promised delivery amount.

Before I go on to give the definition for the equilibrium, I want to point out a unique property of defaultable bonds in the model. There is a crucial difference between the defaultable bonds and the usual securities. The traditional non-defaultable securities such as stocks and riskless bonds have the property that their unit payoffs are exogenously given. Thus the only macro variables in the economy are the prices of these securities. The payoffs of defaultable bonds, however, are endogenously determined. As already pointed out in Dubey, Geanakoplos and Shubik (2000), there exists a somewhat “awkward” representation for the default amount in this type of models. One can see from the above budget constraints that for the same risky bond, say bond \(B_1\), the sellers (agent 1) and the buyers (agent 2) regard the payoff of the bond differently. For the sellers, the default amount (thus the total payoff) of the bond is a choice variable. For the buyers, the unit payoff of the bond \(\kappa(\omega)\) is taken as given, as well as the prices of the bonds. In other words, buyers of the defaultable bonds have to form beliefs on the payoffs of the defaultable bonds. To differentiate these two views for the defaultable bonds, I use default amount \(d(\omega)\) for the sellers’ defaultable amount and \(\kappa(\omega)\) for the unit payoffs to buyers. These two views have to be the mutually
consistent so that “rational expectation” holds. Specifically, the following hold:

\[ \alpha_1^2 \kappa_1(\omega) = \alpha_1^1 - d^1(\omega), \]
\[ \alpha_2^2 \kappa_2(\omega) = \alpha_2^2 - d^2(\omega). \]

Given the individual agent’s problem, I can now define the equilibrium. The equilibrium is a set \( \mathcal{E} = \{ P^{B0}, P^{B1}, P^{B2}, P^S, \kappa_1(\omega), \kappa_2(\omega), (c_0^i, c_1^i(\omega), \alpha_0^i, \alpha_1^i, \alpha_2^i, \theta^i, d^i(\omega))_{i=1,2} \} \) that satisfies the following:

1. Individual optimality:

\[ (c_0^i, c_1^i(\omega), \alpha_0^i, \alpha_1^i, \alpha_2^i, \theta^i, d^i(\omega)) \in \text{arg max} \, v^i \text{ over } \mathcal{B}^i(P^{B0}, P^{B1}, P^{B2}, P^S, \kappa_{j\neq i}(\omega)). \]

2. Markets Clear:

(a) Goods Markets:

\[ c_0^1 + c_0^2 = Y_0, \]
\[ c_1^1 + c_1^2 = Y_1, \]

where \( Y_0, Y_1 \) are the aggregate endowments for periods 0 and 1.

(b) Security Markets:

\[ \alpha_0^1 + \alpha_0^2 = b, \]
\[ \alpha_1^1 = \alpha_1^2, \quad j = 1, 2 \]
\[ \theta^1 + \theta^2 = 1, \]

where \( b > 0 \) is the total net supply of riskless bond, and again, \( \alpha_j^i \geq 0 \).

3. The expectation of delivery rate for the other agent is rational, namely the equations (4) hold.

\[ ^{\text{7}} \text{Dubey, Geanakoplos and Shubik (2000) discuss further the refinements of the equilibrium because of the fact that agents have to form beliefs on the defaultable bonds’ payoff. See also the appendix at the end of this paper. I do not pursue on this issue further. Since I am mainly interested in the asset pricing implications, I will take the existence of equilibrium as given and study the equilibrium in which payoffs of the defaultable bonds are what actual payoffs are.} \]
3 The Non-Default Case

As a benchmark case, I study here the standard incomplete market model without default. Now agents invest in stock and exchange riskless bonds. Total net supply of riskless bond is zero. Each agent maximizes the utility function of the form: (In this section I omit the superscripts of agents unless otherwise denoted)

\[ v(c_0,c_1) = -\frac{1}{2}(c_0 - c^*)^2 + E\left(-\frac{1}{2}(c_1 - c^*)^2\right), \]  

such that

\[ c_0 = e_0 - \left((\theta - \frac{1}{2})P^s + \alpha_0P^B_0\right), \]  

\[ c_1 = e_1 + \alpha_0 + \theta D. \]  

I obtain the following:

**Proposition 8** Suppose there is an interior solution, then the equilibrium prices of stock and bond are:

\[ P^B_0 = \frac{E(2c^* - Y_1)}{2c^* - Y_0}, \]  

\[ P^S = \frac{E((2c^* - Y_1)D)}{2c^* - Y_0}, \]  

and agent 1’s portfolio choice is:

\[ \alpha^* = -\frac{E(\Delta)}{A_0} A_\alpha, \]  

\[ \theta^* = \frac{1}{2} - \frac{E(\Delta)}{A_0} A_\theta, \]  

where

\[ \Delta \equiv \frac{e_1^2 - e_1^2}{2}, \]  

\[ A_0 = (1 + (P^B_0)^2)\text{var}D + (P^B_0E(D) - P^S)^2 > 0. \]  

\[ A_\alpha = \text{var}(D) - P^S(P^B_0E(D) - P^S) > 0 \]  

\[ A_\theta = P^B_0(P^B_0E(D) - P^S) > 0. \]
Note here I only specify the portfolio choices for agent 1. The results for agent 2 are similar. The following observations are immediate from above:

- The heterogeneity in income does not have any effect on equilibrium prices, while it does affect portfolio choice.

- For agent 1 to be a lender ($\alpha_0 \geq 0$), it must be the case $E(\Delta) \leq 0$. This is not surprising, since currently both agents have the same consumption. Those who lend must have lower growth in endowment than those who borrow. I have denoted this solution to be $\alpha^*$ as in equation (10).

- The above holds when there is no binding portfolio constraints. Otherwise the price will be determined by those agents who are not at the corner solution.

Thus a sufficient condition for the existence of an interior solution is that the following holds:

$$e_{\min} > |\alpha^*|.$$  

It is also a necessary condition if $\text{Prob}\{D < \varepsilon\} > 0$ for any $\varepsilon > 0$.

To make sure that I have interior solutions with nonnegative consumption, I will assume in the following that $e_{\min}$ is high enough to allow positive consumption even in the worst state, namely $e = e_{\min}, D = 0$. In other words, $e_{\min} \gg |\alpha^*|$. Furthermore, I will assume that agent 1 will be the lender, namely $\alpha^* > 0$ in the following.

4 The Default Case

4.1 Optimal Default Decision For Individual Agents

If agents can default, the payoff of the risky bond will be endogenously determined. Borrowers will decide how much to deliver conditional on the state. In fact I have the following optimal default decision of the borrowers:

**Proposition 9** Given the portfolio choice $(\alpha_0, \alpha_1, \alpha_2, \theta)$ and the delivery rate of the other agent $\kappa$, the optimal default amount for an agent (say agent 1) is the following: Denote the
“disposable income”, the income from endowment and investment, by \( y \):

\[
y \equiv c_1 + \theta D_1 + \alpha_0 + \alpha_2 \kappa_2.
\]

(16)

There are two boundaries

\[
\bar{y} = c^* - d^* + \alpha_1,
\]

(17)

\[
y = c^* - d^* - \alpha_1.
\]

(18)

which separate \( y \) into three regions:

- **In the region of nondefaulting:**

  \[
y \geq \bar{y},
  \]

  the optimal consumption and defaulting satisfies:

  \[
d = 0,
  \]

  \[
c_1 = y - \alpha_1 \geq c^* - d^*.
  \]

- **In the region of partial defaulting:**

  \[
y \leq y < \bar{y},
  \]

  the optimal consumption and defaulting satisfies:

  \[
d = \frac{1}{2}(\bar{y} - y)
  \]

  \[
  = \alpha_1 - \frac{1}{2}(y - \bar{y})
  \]

  \[
c_1 = y - \alpha_1 + d < c^* - d^*.
  \]

- **In the region of full defaulting:**

  \[
y < \underline{y},
  \]

  the optimal consumption and defaulting satisfy:

  \[
d = \alpha_1,
  \]

  \[
c_1 = y.
  \]
The intuition behind this is straightforward: when the disposable incomes are low, the marginal utilities for consumption are high. Agents default more to increase consumption. Similarly when disposable incomes are high, agents’ marginal utilities for consumption are low. They default less to lower consumption. The agents’ ability to default are limited by the fact that the total defaulting amounts cannot be more than the total borrowing amount and not less than zero. Thus one will have the three regions as specified above. This can be easily seen from Fig. 1.

As \( d^* \) increases, the two boundaries move to the left. The default regions (both full-defaulting and partial defaulting region) shrink. This simply represents the fact that as defaulting penalty increases, default decreases. As \( \alpha_1 \) increases, partial-defaulting region becomes larger and full-defaulting region becomes smaller (as does non-defaulting region).

The amount \( c^* - d^* \) is the consumption level below which borrowers will default. If the default penalty \( d^* \) is higher than \( c^* \), agents never default since consumption is nonnegative. When \( d^* \) is lower, agents begin to default. In the extreme case when \( d^* = 0 \), since consumptions \( c \) are always less than \( c^* \), it means agents will always want to default (at least partially). It is in this sense that I regard \( d^* \) as a measure of default penalty.

Now I define some specific \( d^* \), which I will use later on. Suppose agents do not lend to other agents \( (\alpha_2 = 0) \) and hold no riskless bonds \( (\alpha_0 = 0) \). The first of such \( d^* \) is \( d^*_P \equiv c^* - e_{\min} \) and I obtain the following property:
Lemma 10 If $d^* \geq d^*_p$, agents never fully default.

The second of such values is $d^*_N \equiv c^* - e_{\min} + \alpha^*$, where $\alpha^*$ is the equilibrium borrowing amount without default, and obtain the following property:

Lemma 11 If $d^* \geq d^*_N$, agents never default. And when $d^* < d^*_N$, agents begin to default.

For future reference, here I write down the first order equations for individual agents: (we consider only the case for agent 1)

\[
(c^* - c_0)P^{B_0} = E(c^* - c_1) + \lambda_{B_0},
\]
\[
(c^* - c_0)P^{B_1} = E((c^* - c_1)\kappa_1 + (d^* + d)(1 - \kappa_1)) - \lambda_{B_1},
\]
\[
(c^* - c_0)P^{B_2} = E(c^* - c_1)\kappa_2 + \lambda_{B_2},
\]
\[
(c^* - c_0)P^{S} = E(c^* - c_1)D + \lambda_{S},
\]

where $\lambda_{B_0}, \lambda_{B_1}, \lambda_{B_2}, \lambda_S$ are the Lagrangian multipliers associated with borrowing and lending constraints.

4.2 Equilibrium Price Analysis

Riskless Bond

The riskless bond is only a saving tool in my model and it is this feature that distinguishes my model from others. The price of the riskless bond is determined by the intertemporal marginal rate of substitution of the lenders who save through riskless bonds. In the benchmark case above, the price of riskless bonds is not determined by any heterogeneity among agents when it is traded by both agents. Now as I show later on, not all agents hold riskless bonds in the situation I study. As such, the price of riskless bond depends on the distribution of income.

Furthermore, to compare with the benchmark case analyzed before, I consider the situation in which the total supply of riskless bond goes to zero in the main text and discuss the extension to positive net supply case at the end of the paper.
Close To Non-Default Assumption And Approximation

Even though in reality the default rate has increased significantly, the absolute value of default rate is still small. It is reported that the rate of consumer bankruptcies per 100,000 adults increased from 201 to 650 between 1980 and 1999. It is a three time increase but small in absolute value. I concentrate on the case in which the default penalty is such that agents just go into the default region. Using the value I denote before, I assume throughout the paper the following:

Assumption 12 (SMALL) I assume that $d^* < d^*_N$ and $|d^*_N - d^*| \ll \alpha^*$. 

As I mentioned before, the difficulties associated with solving incomplete market models are what prompted Zeldes (1989) to use numerical methods, when he claimed that “No one has derived closed-form solutions for consumption with stochastic labor income and constant relative risk aversion utility.” Since then, people have made great advancements in solving optimal consumption and portfolio choice problems when agents face stochastic labor income. Yet there has not been much work on solving equilibrium models with incomplete markets. Here there are additional difficulties beyond those presented in the traditional general equilibrium models with incomplete markets (GEI). One is the fact that the payoff of the risky bond is like that of an option. In this case, what distributions one should use is not clear. In fact, there have been few works studying the portfolio choice problems with options. The most serious difficulty comes from the fact that the payoffs of defaultable bonds are endogenously determined. Recall that a standard way to solve the equilibrium model with heterogeneous agents is to first solve a representative agent’s model. This way one does not need to solve portfolio choice problems to determine the equilibrium security prices. Then one goes on decentralizing to obtain individual agents’ optimal consumptions and portfolio choices. This procedure is still valid even in GEI models. But because of the endogeneity of payoffs, all these methods cannot be used.

So to obtain sharp results, I use quadratic preferences over consumption and defaulting. This way I can obtain expressions for the equilibrium prices of defaultable bonds and stocks first. Then I can solve the portfolio choice problem of agents. But because of the endogeneity of payoffs of risky bonds, closed form solutions for reasonable distributions can not be
obtained. Instead, I will use an approximation method to obtain the results while avoiding
distribution of state variables.

With assumption (SMALL), I immediately obtain the following:

**Proposition 13** In equilibrium, agents who borrow using defaultable bonds do not hold
riskless bonds.

This simplifies the result since now there is only one defaultable bond traded. I denote
the bond as $B_1$ and call those agents who sell it (agent 1) as the borrowers. Agent 2 will be
lenders, who also invest in riskless bonds. The price of riskless bonds is then determined by
lenders’ marginal utility. As in the case of the non-default model, I denote $\Delta = e^l - e^b$ and
assume that $E(\Delta) < 0$ in the following.8

**The Price of Riskless Bond And Equity Premium**

The following is the first main result of the paper:

**Proposition 14** The equilibrium price of riskless bond is higher in the default case than that
in the benchmark non-default case, while the stock price has the same form in both default
case and non-default case:

$$p^s = \frac{E((2e^s - Y_1)D)}{2e^s - Y_0}.$$  

The stock price is the same as that in the non-defaulting case because agents have
quadratic utility function. As the marginal utility of agents are linear, the prices of any
securities traded by all the agents are determined by the aggregate endowment (Mankiw
(1986)).

---

8It is easy to see that there exist cases in which both bonds are traded. One example is the following: suppose there are two states with equal probabilities to happen at the second period. Two agents are endowd with $(1, 0)$ and $(0, 1)$ respectively at second period. Agents have no preferences over first period consumption so they only maximize second period sum of expected utility for consumption and utility for defaulting. Because of the symmetry, agents have the same holdings of defaultable bonds and stocks. Now consider the case when default penalty goes to infinite. Then no defaultable bond will be traded since otherwise agents cannot pay in the state with zero income. And when the default penalty goes to zero, again no defaultable bond will be traded since sellers will always default fully. Somewhere in between, one can imagine that there are nonzero amounts of defaultable bonds traded.
The basic intuition for this proposition can be seen from the lenders’ marginal utility function. Indeed, the price of riskless bond is determined as:

\[ P_{B0} = \frac{E(c^* - c_1^*)}{c^* - c_0^*} \]  \hspace{1cm} (19)

\[ = \frac{E(c^* - (c_1^* + \alpha_1 \kappa + \theta D))}{c^* - (c_0^* - \alpha_1 P^{B1} - (\theta - 1/2) P^S)} \]  \hspace{1cm} (20)

where \( \alpha_1, \theta \) are lenders’ demand for risky bond and stock. So for example, if \( \alpha_1 P^{B1} + \theta P^S \) and \( E(\alpha_1 \kappa + \theta D) \) decrease comparing to the non-default case, \( P_{B0} \) increases. More generally, if comparing to the non-defaulting case, the change of \(-E(\alpha_1 \kappa + \theta D)\) is more than the change of \( B_0(\alpha_1 P^{B1} + \theta P^S) \), \( P_{B0} \) is then larger than \( B_0 \). Here \( B_0 \) is the equilibrium price of riskless bond in the non-defaulting case.

To see what happens in the proposition, let me write out the price of risky bond:

\[ P_{B1} = \frac{E((2c^* - Y_1) \kappa) + E((d^* + d)(1 - \kappa))}{2c^* - Y_0} \]  \hspace{1cm} (21)

The first term represents the fact that a defaultable bond is just a contingent claim paying \( \kappa(\omega) \) at state \( \omega \). Put the other way, if instead of defaultable bonds, agents can trade a contingent claim with payoff \( \kappa(\omega) \), then the price of this contingent claim is \( \frac{E((2c^* - Y_1) \kappa)}{2c^* - Y_0} \) (c.f., the equilibrium stock price). The second term represents the fact that borrowers suffer additional utility penalty, they require higher prices to compensate for this loss. It is this second term that plays a major role.

Specifically, demand for the risky bonds as a result of lenders’ first order equation are :

\[ \alpha_1 \approx \alpha^* + \delta_\alpha, \]  \hspace{1cm} (22)

where \( \alpha^* \) is the equilibrium demand for riskless bond in non-defaulting case. One can show that if not for the default penalty term in the price of the risky bonds, \( \delta_\alpha > 0 \), namely the shares of the bond hold by the lender will increase unambiguously. But with this extra penalty term in the risky bond prices, this increase on the holdings of bond is mitigated. As to \( \alpha_1 P^{B1} + \theta P^S \), one can show that if the second term in the price of risky bond does not exist, the change is ambiguous and depends on the distributions of endowment and dividends. But with the penalty term, the result is unambiguous: the price of riskless bond increases.
Putting together, the intuition is the following. From the above expression in the prices of the risky bond, there are two possible effects of default from lenders’ point of view. First, default increases risk-sharing (over states $\omega$) since instead of a riskless bond with unit payoff, now agents trade a security with payoff $\kappa(\omega)$ and $\kappa$ is state contingent. Second, because of borrowers’ default penalty, the prices of risky bonds are higher than the case with simple contingent claims (the second term). In other words, the ability of the risky bonds to provide intertemporal consumption smoothing is smaller than that of a contingent claims with the payoff $\kappa(\omega)$. The net effects are such that financial wealth from lenders’ portfolio choice decreases compared with the situation with only riskless bond. Thus the first period consumption increases. Similarly, next period consumption decreases. The result is that price of riskless bond increases.

**Comparative Static Analysis**

In this subsection I want to do some comparative static analysis in the region of default. Specifically, I want to study the effects of changing in default penalty and income inequality. Namely I want to study when $d^*$ changes and when $E(\Delta) = E(e^d - e^b)/2$ changes (while fixing $E(e^d + e^b)$) what the effects on the portfolio choices and prices are.

First of all, I want to make sure that both a decrease in default penalty $d^*$ and an increase in $E(\Delta)$ indeed increase the (expected) default rate $E(1 - \kappa)$:

**Proposition 15** $\partial E(1 - \kappa)/\partial d^* < 0, \partial E(1 - \kappa)/\partial E(\Delta) > 0.$

It is this fact that stimulates policy discussions about the true source of the increasing default rate during the booming period of the 1990’s. It is important to determine whic effect, if either, is the quantitatively most important source of increased bankruptcies because these two stories have totally different policy implications. To be able to shed further light on this issue becomes quite important. Here I want to show that indeed these two stories have different effects on the security prices and portfolio choices of agents. Indeed, I obtain the following:
Proposition 16 When $d^*$ decreases, the price of riskless bond increases, $\partial P^B_0/\partial d^* > 0$ while $\partial \alpha_1/\partial d^* = O(\alpha^*)$, $\partial \theta/\partial d^* = o(\alpha^*)$.\footnote{A variable $y$ is of order $o(x)$ if $\lim_{x \to 0} (y/x) = 0$. A variable $y$ is of order $O(x)$ if $\lim_{x \to 0} (y/x) \in (-\infty, \infty)$ and $\lim_{x \to 0} (y/x) \neq 0$.}

Since the non-default case is just a default case when the default penalty is very high (so there is no default), the previous comparison between default and non-default cases is just a special example of low penalty and high penalty. There should not be a qualitative difference between the two in terms of riskless bond price. A more intuitive explanation is that when the default penalty decreases, borrowers pay less. Lenders forsee this and hold less risky bonds. Again, the consumption of the lenders in the first period increases so that the price of riskless bonds increase. The reason that the change of portfolio holdings is small is simply because the change of defaultable bond payoffs is small.

Proposition 17 When income inequality $E(\Delta)$ increases, $\partial P^B_0/\partial E(\Delta) < \partial P^B_0/\partial d^*$, while $\partial \alpha_1/\partial E(\Delta) = O(\alpha^*)$, $\partial \theta/\partial E(\Delta) = O(\alpha^*)$. Specifically, if $|E(\Delta)|$ and $A_0$ are small enough, then the price of riskless bond decreases, namely $\partial P^B_0/\partial E(\Delta) < 0$.

Let us first look at the portfolio choices. Recall in the benchmark non-default case, the portfolio holdings are linear functions of $E(\Delta)$ and the coefficients are expectations over the whole distribution of endowments and dividends, while in the case of change in default penalty, changes in portfolio choices only come from expectations over the default region. So the change in portfolio holdings is of a different order (by the assumption (SMALL)) for the decrease in default penalty and increase in income inequality.

Then let us look at the prices. The difference between changing income inequality and changing the default penalty is that changing the default penalty only changes borrowers’ intentions to pay, while changing income inequality also changes lenders’ endowments (keeping the total endowment fixed). The fact that we need a small $E(\Delta)$ and $A_0$ (to get the decreasing price of the riskless bonds) is because of the two effects we have mentioned before. Small $E(\Delta)$ and $A_0$ make the demand for risky assets from intertemporal consumption smoothing motive be sensitive to the change in $E(\Delta)$. So when $E(\Delta)$ increases, on the one hand, borrowers become poorer and they pay less (default more). On the other hand,
lenders demand less because of two factors, higher default rate and higher (expected) future endowment. If only the first factor in demand dominates, the results are similar as the case of decreasing in default penalty. However, when the amount \( \alpha_1 \) and \( \theta \) are very sensitive to change of \( E(\Delta) \), lenders decrease a lot more demand in financial assets (risky bond and stocks) because of the second factor. Thus the price of riskless bond decreases.\(^{10}\)

These two results make it possible to distinguish the two stories underlying the increasing in default rate by looking at riskfree rate and portfolio holdings. Specifically when \(|E(\Delta)|\) and \( A_0 \) are small, there is qualitative difference between the riskless rate. As to how large this effect will be mainly an empirical issue.

In the study I have done so far the stock price does not change. In the following I want to see the effect on the equity premium if, say the stocks pay a higher dividend. Here equity premium, \( EP \), is defined as:

\[
EP = \frac{E(D)}{Ps} - \frac{1}{P^{B_0}}.
\]

In other words, suppose there is a boom in stock valuation, what will happen to equity premium?

Looking back at the benchmark case, one can easily see that as \( E(D) \) increases while keeping \( \text{var}(D) \) fixed, \( P^{B_0} \) decreases and \( P^S \) increases. I will show that the net result is that (expected) equity premium decreases. Yet there is one variable that does not change, namely \( P^{B_0} E(D) - P^S = \text{var}(D)/(2c^*-Y_0) \). In the case of defaulting, this value will increase. Indeed we have:

**Proposition 18** With an increase in \( E(D) \) while keeping \( \text{var}(D) \) fixed, equity premium \( EP \) decreases. Yet \( \partial EP^{def}/\partial E(D) > \partial EP^{non}/\partial E(D) \). Furthermore, if we look at \( P^{B_0} E(D) - P^S \), it does not change in the non-defaulting case, and yet it increases in the case of defaulting.

In other words, the equity premium will decrease less than that in the case of non-defaulting.

**Credit Spread**

\(^{10}\)Of course one has to be careful so that the resulting \( |\alpha_1 - \alpha^*| \) is still much smaller than \( |\alpha^*| \) (see appendix for details).
Since I have riskless bond and risky bond in one equilibrium setting, I can also study the comparative static properties of credit spread $CR$, which is defined as the following:

$$CR = \frac{1}{PB_1} - \frac{1}{PB_0}.$$  \hspace{1cm} (24)

Indeed I obtain the following:

**Proposition 19** When $d^*$ decreases, $CR$ increases. When $E(\Delta)$ increases, $CR$ also increases unless $E(\Delta)$ is small (the cutoff level is smaller than that in the previous case for riskless bond). Lastly, when $E(D)$ increases (while holding $\text{var}(D)$ fixed), $CR$ increases.

The first two are not surprising since in both cases the payoff of the risky bonds becomes less, thus the credit spread increases. The last effect comes mainly from the fact that an increase in $E(D)$ has a first order effect to increase $P^{B_0}$.

5 Discussions

In this section I give some extensions of the model.

5.1 Positive Net Supply of Riskless Bond

It is straightforward to extend the model to the case with positive net supply riskless bond. Now the riskless bond is another security (Lucas trees) which pays a unit amount of goods at time 1. This does not change the basic point of the paper. Denote the new default-free solution for the bond holdings to be $\overline{\alpha}$. Then the separation $d^*$ value between defaulting and partial delivery region is $d^\text{pu}_p = c^* - e_{\text{min}} + \overline{\alpha}$. Everything else goes through. The main results in the text still hold.

5.2 Other Utility Function

The setup in the main text is actually quite general. Indeed, for any utility function $u(c)$, a similar default penalty function can be expressed as:

$$w(d) = u(d^* - d) - u(d^*).$$  \hspace{1cm} (25)
The last \( u(d^*) \) in the above expression is to make sure that \( w(0) = 0 \). Using this setup, one can easily show that the linear relationship for the default decision and the three region default decision rule still holds. Indeed, looking at the first order condition for the optimal default decision, the following holds:

\[
u'(c) - u'(d^* - d) + \lambda_{d0} - \lambda_{d1} = 0,
\]

where \( \lambda_{d0}, \lambda_{d1} \) are the Lagrangian multipliers associated with the constraints that default amount must be nonnegative and less than the total promised amount. It is straightforward to see that a linear relationship still holds between default amount and consumption, thus endowments and portfolio holdings. The two Lagrangian multipliers again imply there are three regions.

As I have pointed out, adopting quadratic functional form enables me to study the model for quite general distribution, while using other forms basically it is hopeless to do so. So numerical analysis is warranted in this case. In the model studied in the previous sections, the price of the stock does not change because of the fact that I use quadratic utility functions, which implies that the marginal utility is of linear form. If I use some other utility functions such as power utility, the stock price changes since the marginal utility functions of agents will be determined by the heterogeneity of the agents. And this makes the model much richer.

A more general form of preferences including consumption and default would be just \( u(c, d) \). Note that the above additive separable form may not be the most natural form for default penalty. Some other non-separable form can be adopted. But again, linear default decision rule seems to hold even in this general case. I will pursue this further when I extend this to the real data.

6 Conclusion

In this paper I study an equilibrium model with incomplete market and defaultable bond. I show that allowing agents’ default has different effects on the equilibrium riskless bond price and thus the equity premium than non-defaulting models. Not only changing default penalty will change the equilibrium asset prices and portfolio choices, changing the heterogeneity of
agents’ endowment will also have effects. This is not shown in the model without default with agents having quadratic utility function. The model also provides a testable prediction which can differentiate the causes of the increasing in default rate by looking at the equilibrium asset prices, portfolio holdings and credit spread.
Appendix

A Proofs

Proof for Proposition (8): The first order conditions for individual agents are:

\[(c^* - c_0)P^B_0 = E(c^* - c_1)\]  \hspace{1cm} (1)
\[(c^* - c_0)P^S = E(c^* - c_1)D.\]  \hspace{1cm} (2)

Because they are linear functions of consumption, one can add them up to obtain the equilibrium prices. Substitute back these prices into the above equations to obtain the portfolio choices in the main text. Q.E.D

Proof for proposition (9): Denote the Lagrangian multipliers \(\lambda_{d_1}, \lambda_{d_2}\) to be associated with the constraints \(d \geq 0, d \leq |\alpha_1|\) respectively. The first order condition for the default decision is:

\[(c^* - c) - (d^* + d) + \lambda_{d_1} - \lambda_{d_2} = 0.\]  \hspace{1cm} (3)

Substitute the representation for \(c\) into the equation and consider the constraints, I obtain the results in the main text. Q.E.D.

Proof for Lemma (10): Since stock dividend is lower bounded at zero and agents do not have other income, the disposable income \(y\) is

\[y = e_1 + \theta D.\]

It then follows that \(y \geq e_1 \geq e_{\text{min}}\). So I obtain \(y + \alpha_1 \geq c^* - d^*_p\). For any \(d \geq d^*_p\), the results follows from proposition (9). Q.E.D.

Proof for Lemma (11): To prove the first part, I just need to show that no defaulting is indeed an equilibrium. This is true from the non-defaulting case and the optimal default decision shown in the proposition (9).

To show the second part, note that when \(d < d^*_N\), the solution in the benchmark case is not applicable anymore since it does not satisfy the optimal default decision. If the resulting
\( \alpha_1 \) increases, agents default at the state \((e = e_{\text{min}}, D = 0)\). If the resulting \(|\alpha_1|\) decreases, it is not a non-defaulting optimal solution. Otherwise, the price of the bond is the same as that in the case without default and lenders will want to buy more. This is not an equilibrium. Q.E.D.

Proof for Proposition (13): For any defaultable bond traded, say \(B_1\), from lenders’ (agent 2’s) point of view, its price must be lower than the riskless bond price if borrowers default on the bond.

Borrowers (agent 1), under the assumption, indeed default on the bond. Also from the assumption, I have \(d^* > c^* - e_{\text{min}}\). This means that borrowers will only partially default (there is no full default region). As such if I rewrite borrowers’ first order equation for their own bonds, I obtain:

\[
(c^* - c_0)P^{B_1} = E ((c^* - c_1)\kappa_1 + (d^* + d)(1 - \kappa_1)) .
\]  

(4)

Remember from optimal default decision, in the partial default region, I have:

\[
(c^* - c_1) = (d^* + d) .
\]  

(5)

Put these together, I obtain:

\[
(c^* - c_0)P^{B_1} = E (c^* - c_1) .
\]  

(6)

It is the same equation for riskless bond if the riskless bond is also traded by borrowers. This is a contradiction from lenders’ point of view. So in equilibrium, lenders will save through both defaultable bond and riskless bond while borrowers will only borrow. The price of the riskless bond is determined by lenders’ intertemporal rate of substitution. Q.E.D.

Proof for Proposition (14): In this proof, I denote the lenders’ holdings of defaultable bond and stock to be \((\alpha_1, \theta + \frac{1}{2})\) and use the fact that the net supply of riskless bond is close to zero (so I omit the riskless bond position) and \(\alpha_1 \geq 0, \theta \in [-\frac{1}{2}, \frac{1}{2}]\).

I prove the proposition in the following steps. During the process, I only keep the terms with the form \(E(f(\alpha_1, \theta)(1 - \kappa))\). And I will justify this approximation at the end of proof.

Step One: From lenders’ and borrowers’ first order equation, I get the equilibrium prices for stock and bond:
\[ p^s = \frac{E(2c^* - Y_1)D}{2c^* - Y_0} \]
\[ p^{B_1} = \frac{E(2c^* - Y_1)\kappa + E(d^* + d)(1 - \kappa)}{2c^* - Y_0} , \]

where \(d, \kappa\) are the default amount and delivery rate for the borrowers.

From this, I can rewrite the defaultable bond price as:
\[ P^{B_1} = \overline{B}_0 - E m_l (1 - \kappa) , \]

where
\[ \overline{B}_0 = \frac{E(2c^* - Y_1)}{2c^* - Y_0} , \]

is the riskless bond price in non-default case and
\[ m_l = \frac{c^* - c_b}{2c^* - Y_0} . \]

Step Two: Taking these two prices as given, I solve the portfolio decision problem of lenders. Rewrite the first order conditions of lenders using their budget constraints, I obtain:
\[ \alpha_1 ((P^{B_1})^2 + E \kappa^2) + \theta (P^{B_1} p^s + E \kappa D) = -E \Delta \kappa - \frac{1}{2} E((c^* - c_b)(1 - \kappa)) \]
\[ \alpha_1 (P^{B_1} p^s + E \kappa D) + \theta ((p^s)^2 + ED^2) = -E \Delta D , \]

where \( \Delta = e^l - e^b \). Now I will use the assumption (SMALL), the portfolio choices of lenders are
\[ \alpha_1 = \frac{1}{A} (\alpha_{10} + \alpha_{11}) \]
\[ \theta = \frac{1}{A} (\theta_0 + \theta_1) , \]

where
\[ A = A_0 + A_1, \]
\[ A_0 = (B_0ED - P^S)^2 + (1 + B_0^2)\text{var}D, \]
\[ \alpha_{10} = -E\Delta(\text{var}D - P^S(B_0ED - P^S)) \]
\[ \theta_0 = -E\Delta B_0(B_0ED - S), \]
\[ A_1 = Em_l(1 - \kappa)(-2B_0ED^2 + 2P^SED) - \]
\[ E(1 - \kappa)(2(P^S)^2 + 2ED^2) + ED(1 - \kappa)(2B_0P^S + 2ED) \]
\[ \alpha_{11} = -E\left(\frac{c^* - c_b}{2} - \Delta\right)(1 - \kappa)((P^S)^2 + ED^2) - \]
\[ Em_l(1 - \kappa)P^SE\Delta D - ED(1 - \kappa)E\Delta ED. \]
\[ \theta_1 = E\left(\frac{c^* - c_b}{2} - \Delta\right)(1 - \kappa)(B_0P^S + ED) + \]
\[ E(m_l(1 - \kappa))(2B_0E\Delta D - SE\Delta) - ED(1 - \kappa)E\Delta + E(1 - \kappa)(2E\Delta D). \]

Step Three: With the portfolio choices, I get the financial wealth of lenders at time \( t = 0 \) and \( t = 1 \) (note that I define \( \theta \) to be the excess demand of stock in addition to the initial holdings).

\[ X_0 = \alpha_1P^{B_1} + (\theta + \frac{1}{2})P^S, \quad (16) \]
\[ X_1 = \alpha_1\kappa + (\theta + \frac{1}{2})D. \quad (17) \]

From the above results, I obtain the following:

\[ X_0 = \frac{1}{2}P^S + \frac{1}{A}(X_{00} + X_{01}), \quad (18) \]
\[ EX_1 = \frac{1}{2}ED + \frac{1}{A}(X_{10} + X_{11}), \quad (19) \]
where

\[
X_{00} = -E\Delta (\overline{B}_0 varD)
\]
\[
X_{10} = -E\Delta (varD + (\overline{B}_0 ED - P^S)^2)
\]
\[
X_{01} = E m_l (1 - \kappa) (E \Delta varD) + \\
E (c^s - c^i_0 - \Delta) (1 - \kappa) (-\overline{B}_0 ED^2 + SED) + \\
ED (1 - \kappa) (-\overline{B}_0 E \Delta ED - P^S E \Delta) + \\
E (1 - \kappa) (2 P^S E \Delta D),
\]
\[
X_{11} = E (1 - \kappa) (E \Delta (varD - P^S (\overline{B}_0 ED - P^S)) + 2 E \Delta (ED)^2) + \\
E (c^s - c^i_0 - \Delta) (1 - \kappa) (P^S (\overline{B}_0 ED - P^S) - varD) + \\
E m_l (1 - \kappa) (-P^S E \Delta D + 2 \overline{B}_0 P E \Delta (ED)^2 - P^S E \Delta D) + \\
ED (1 - \kappa) (-2 E \Delta D).
\]

Step Four: The equilibrium price of riskless bond is determined by the lenders’ marginal utility. Namely

\[
P^{B_0} = \frac{E(c^s - c^i_0)}{c^s - c^i_0}.
\]

Use the budget constraints for lenders and the above results, I obtain the following:

\[
P^{B_0} = \frac{\overline{B}_0 (A_0 (c^s - e_0) - E \Delta \overline{B}_0 varD) + \overline{B}_0 A_1 (c^s - e_0) - E \Delta A_1 - X_{11}}{(A_0 (c^s - e_0) - E \Delta \overline{B}_0 varD) + A_1 (c^s - e_0) + X_{01}}
\]

\[
= \overline{B}_0 + \frac{-E \Delta A_1 - X_{11} - \overline{B}_0 X_{01}}{(A_0 (c^s - e_0) - E \Delta \overline{B}_0 varD) + A_1 (c^s - e_0) + X_{01}}
\]

To compare this value with $\overline{B}_0$, it is sufficient to compare $-E \Delta A_1 - X_{11}$ and $\overline{B}_0 X_{01}$. Indeed

I obtain from the above results:

\[
\overline{B}_0 X_{01} < 0,
\]

\[
-E \Delta A_1 - X_{11} > 0.
\]

And this gives me the desired results:

\[
P^{B_0} > \overline{B}_0.
\]
Now I turn to the issue that the approximation I make is valid, namely the terms with the form \( E(f(\alpha_1, \theta)(1 - \kappa))E(g(\alpha_1, \theta)(1 - \kappa)) \ll E(f(\alpha_1, \theta)(1 - \kappa)) \). Note that for the term \( E(f(\alpha_1, \theta)(1 - \kappa)) \) the only nonzero part comes from the partial delivery region. Because of the assumption (SMALL) I make in the main text, two facts contribute to the result that this expectation is small. (In the following, by “small” I mean the term is of the order \( o(|d_N^* - d^*|) \).) Namely the region itself is small and also \((1 - \kappa)\) is small in this region. To show that this is the case, I want to show that \(|\alpha_1 - \alpha^*| \ll \alpha^* \) and \(|\theta - \theta^*| \ll \theta^* \). In other words, if I define:

\[
\alpha_1 = \alpha^* + \delta_\alpha, \\
\theta = \theta^* + \delta_\theta,
\]

I want to find a solution \((\delta_\alpha, \delta_\theta)\) such that \(|\delta_\alpha| \ll \alpha^*, |\delta_\theta| \ll \theta^* \). To solve for this, I want first to assume that such a solution exists, then I substitute them back into the first order equations and show that indeed the results satisfy the requirement.

First I want to show that the region is small. Recall that the boundary of the partial delivery region is a function of \((\alpha_1, \theta)\), which is endogenously determined. To express it in terms of exogenous variables such as \(\alpha^*, \theta^*, d^*\), I will use Taylor expansion to approximate the expectation around point \((\alpha^*, \theta^*)\). So instead of taking expectation over region upper bounded by \(e^b + (1/2 - \theta)D = c^* - d^* + \alpha_1\), it is taken over \(e^b + (1/2 - \theta^*)D = c^* - d^* + \alpha_1^*\). Namely we have:

\[
E(f(\alpha_1, \theta)(1 - \kappa)) = E^*(f(\alpha^*, \theta^*)(1 - \kappa^*)) + o(\delta_\alpha) + o(\delta_\theta),
\]

where \(E^*(\cdot)\) is taken over the region bounded by \((\alpha^*, \theta^*)\). Now for the term \(E^*(f(\alpha^*, \theta^*)(1 - \kappa^*))\). Note that \(f(\alpha^*, \theta^*)(1 - \kappa^*)\) is bounded, while the region is of order \((d_N^* - d^*)^2/\theta^*\), which by assumption is small: \(E^*(f(\alpha^*, \theta^*)(1 - \kappa^*)) \ll \min(\alpha^*, \theta^*)\).

Furthermore, note that the maximum value for \((1 - \kappa^*)\) is achieved at \((\hat{\theta}, D) = (\epsilon_{\min}, 0)\). Again by assumption the maximum default amount is of order \(|d_N^* - d^*|\). So I have \(1 - \kappa \ll 1\).

These two facts let me only consider the expectation in which both the boundary and integrand are valued at \((\alpha^*, \theta^*)\). Then from first order conditions obtained above I get the
following:

\[ \alpha_1 = \alpha^* + \frac{E(f_\alpha(\alpha_1, \theta)(1 - \kappa))}{A_0 + E(f_\alpha(\alpha_1, \theta)(1 - \kappa))}, \]  

\[ \theta = \theta^* + \frac{E(f_\theta(\alpha_1, \theta)(1 - \kappa))}{A_0 + E(f_\alpha(\alpha_1, \theta)(1 - \kappa))}. \]  

(29)  

(30)

So I have \(|\delta_\alpha| \ll \alpha^*, |\delta_\theta| \ll \theta^*\), which is exactly what I am looking for.

Q.E.D.

Proof for Proposition (15): Using the approximation arguments in the previous proof, I only need to calculate the derivatives with respect to \(d^*\) and \(E(\Delta)\) for \(E^*(1 - \kappa^*)\). The only concern is that boundary might contribute to the derivatives. But this contribution is small because of the fact that the remaining integral over either \(e^b\) or \(D\) is small and that \((1 - \kappa^*) \ll 1\). And this obtains the results in the main text. Q.E.D.

Proof for Proposition (16) and (17): From previous proof, I rewrite the form of as follows:

\[ P^{B_0} = \frac{-E\Delta A_1 - X_{11} - \overline{B_0}Y_{01}}{A_0(c^* - e_0) - E\Delta \overline{B_0} \text{var}(D)}. \]  

(31)

To calculate \(\frac{\partial}{\partial \Delta}\), I only need to calculate the numerator. This will have same sign as in the previous proof. For example, in calculating \(\frac{\partial X_{01}}{\partial \Delta}\), the only exception will be those additional \(d^*\) terms. These additional terms will be dominated by the other \((1 - \kappa)\) terms. And the portfolio choices are of order \(o(\alpha^*)\) and \(o(\theta^*)\) are straightforward. There are also extra terms associated with the boundary condition. At the boundary, the term of \(d^*\) also has negative sign. So when one take derivatives, it is again negative, which means \(P^{B_0}\) will only get lower when \(d^*\) increases.

As to changing in inequality, it is different since now the denominator will have some effects, plus the results from \(-E\Delta A_1\). The total effects will not be clear cut. Indeed if I calculate the following:

\[ \frac{\partial(1 - \kappa^*)}{\partial E(\Delta)} = \frac{1}{2\alpha^*}(1 - \frac{\text{var}(D)}{A_0}(1 + \frac{\overline{B_0}D - P^S}{2c^* - Y_0})). \]

So I obtain (ignoring higher order term):

\[ \frac{\partial P^{B_0}}{\partial E(\Delta)} = \frac{1}{(A_0(c^* - e_0) - E(\Delta)\overline{B_0} \text{var}(D))^2}[(A_0(c^* - e_0) - E(\Delta)\overline{B_0} \text{var}(D))(-p_1 + p_2) + \overline{B_0} \text{var}(D)p_3], \]
where

\[ p_1 = \text{var}(D) + P^S(\overline{B_0}E(D) + P^S) + A_0/2)E(1 - \kappa) \]
\[ + \overline{B_0}\text{var}(D)E(1 - \kappa) \frac{c^* - c' - D}{2c^* - Y_0}, \]
\[ p_2 = -E(\Delta)E\left(\frac{\partial(1 - \kappa)}{\partial E(\Delta)}\right) + A_0E\left(\frac{\partial(1 - \kappa)}{\partial E(\Delta)}\left(\frac{c^* - c^b - 2\Delta}{2}\right)\right), \]
\[ p_3 = -E(\Delta)p_1 + A_0E((1 - \kappa)\left(\frac{c^* - c^b - 2\Delta}{2}\right)) \]

which are positive. One can verify that the boundary terms disappear when taking derivatives with respect to \( E(\Delta) \).

If \( E(\Delta) \) and \( A_0 \) are of order \( O(|d^*_N - d^*|) \), then \( p_2 \) dominates. In this case \( \frac{\partial P^{B_0}}{\partial E(\Delta)} > 0 \). If \( E(\Delta) \) and \( A_0 \) are of order \( o(|d^*_N - d^*|) \) and their ratio is such that assumption SMALL still holds, then it is possible \( p_1 \) dominates. One can indeed find small enough \( E(\Delta) \) and \( A_0 \) so that \( p_1 \) dominates. In this case \( \frac{\partial P^{B_0}}{\partial E(\Delta)} < 0 \).

Q.E.D.

Proof for Proposition (18): In the non-defaulting case, one can calculate directly the value \( \overline{B_0}E(D) - P^S \). Indeed I get the following:

\[ \overline{B_0}E(D) - P^S = \frac{\text{var}(D)}{2c^* - Y_0}. \]  \hspace{1cm} (32)

From this expression, I immediately obtain that when \( E(D) \) increases while keeping \( \text{var}(D) \) fixed does not change the value of \( \overline{B_0}E(D) - P^S \). But in the case of defaulting, \( P^{B_0} > \overline{B_0} \).

It means that this term is larger than that in the nondefaulting case. And when taking derivatives in the region of partial default, the sign does not change. Q.E.D.

Proof for Proposition (19): The credit spread \( CR \) is defined as:

\[ CR = \frac{1}{P^{B_1}} - \frac{1}{P^{B_0}} \]
\[ = \frac{1}{B_0}E\left(\frac{1}{2}m_i(1 - \kappa)\right) + \frac{1}{A_0(c^* - c_0)} \frac{-E\Delta A_1 - X_{11} - \overline{B_0}X_{01}}{1/2 - \theta^*}. \]  \hspace{1cm} (34)

Taking derivative with respect to \( d^*, E(\Delta), E(D) \). From the discussion before, I immediately
obtain that
\[
\frac{\partial CR}{\partial d^*} > 0 \\
\frac{\partial CR}{\partial E(D)} > 0.
\]
(35)  
(36)

As to \(E(\Delta)\), the discussions from previous proof again gives similar results, namely when 
\(E(\Delta)\) and \(A_0\) are of order \(O(|d^*_N - d^*|)\), \(\frac{\partial CR}{\partial E(D)} > 0\). And when \(E(\Delta)\) and \(A_0\) are of order 
\(o(|d^*_N - d^*|)\) and small enough, \(\frac{\partial CR}{\partial E(D)} < 0\). The criterion is smaller than that in the previous 
proof. Q. E. D.

**B  The Existence of Equilibrium in Finite State Space**

In this part of the appendix, I prove the existence of equilibrium when the number of states 
at \(t = 1\) is finite. It not only serves as a theoretical interesting part, but also serves as a 
reassurance that I will indeed get results in the numerical calculation. In fact I will prove a 
more general results in which there are more than one agents and multi-goods case.

There are two period with \(t = 0, 1\). There is uncertainty about states in \(t = 1\). The 
uncertainty is characterized by \(S\) states. I sometimes write \(S^* \equiv \{0\} \cup S\).

There are \(I\) agents. And there are \(L\) goods for consumption in each state \(s \in S^*\). The 
initial endowment of the agent \(i\) is denoted by \(e^i \in \mathbb{R}^{S^* \times L}_{++}\).

There are one security for trading. Selling one unit of this security promises to pay 
\(A_s \in \mathbb{R}^L_{++}\) of goods at state \(s \in S\) at time \(t = 1\). However, the security is defaultable. 
Namely the seller of the security can default on the promise he makes at \(t = 0\). Of course, 
some penalties must be present to prevent the agents to default freely. I will penalize the 
default on agents’ preferences, which I now turn to.

For any agent \(i \in I\), the preference over the consumption is denoted by a utility function 
\(u^i : \mathbb{R}^{S^* \times L}_{++} \rightarrow \mathbb{R}\). I make the following assumption:

**Condition 20** I assume that \(u^i, i \in I\) is continuous, concave and strictly increasing in each 
of its arguments.

The utility for the default amount is denoted by a function \(w^i : \mathbb{R}^{S \times L}_{++} \rightarrow \mathbb{R}\), which I 
make the following assumption:
**Condition 21** I assume that $w^i, i \in I$ is continuous, convex and strictly increasing in each of its arguments.

The total preference for an agent $i$ is denoted by $v^i = u^i - w^i$.

**Example 22** If $I$ denote the consumption of the agent $i$ be $(c^i_0, c^i_1) \in \mathbb{R}^L_+ \times \mathbb{R}^{S\times L}_+$, and the amount of defaulting at $t = 1$ be $d^i \in \mathbb{R}^{S\times L}_+$. The preference of the agent is then $v^i = u^i(c^i_0, c^i_1) - w^i(d)$.

**Remark 23** I assume there exists one security for purely simplification purpose. Indeed, I can extend the following results to multiple securities without any change of arguments.

I now discuss the portfolio holdings of the agents. Agents can buy the defaultable security from other agents and also sell the security to other agents. Unlike the case with no default (so one can use one single variable to denote the portfolio holding for each individual agent), different agents will potentially deliver different amount of goods on their promises. So buying from each individual agent is actually buying different securities since the payoff are potentially different. Let us denote by $\theta_{ij} \geq 0, i \neq j \in I$ as the shares agent $i$ buys from agent $j$ and $\theta_{ii} \geq 0, i \in I$ as the shares agent $i$ sells. I also make the following assumption:

**Condition 24** I assume that there exists $Q_i \in \mathbb{R}_+, i \in I$ such that $\theta_{ii} \leq Q_i$.

This upper bound is necessary for the existence of the equilibrium in multi-good case. As we will see, in one-good case, this upper bound is not needed.

An economy is defined as a vector:

$$\mathcal{E} = \left((u^i, w^i, e^i)_{i \in I}, \left(A, (Q^j)_{i \in I}\right)\right).$$ \hspace{1cm} (1)

To get the competitive equilibrium, I will now state the “macro variables” such as the prices for the goods $p \in \mathbb{R}^{S\times L}_+^{S\times L}$ and prices for the securities sold by each agents $\pi \in \mathbb{R}_+^L$. Furthermore, each agent has to forecast the payoff of the other agents from whom he buys. Since the information is public and I assume the beliefs of the agents are the same, the forecasting will be the same across the agents. Denote the forecasting delivery rate for agent $i$’s promises to be $K_i \in \mathbb{R}^{S\times L}_+$ with each element between 0 and 1. I impose the following rational expectation requirement:
Condition 25 I assume that the forecasting of the agents satisfies rational expectation. Namely the forecasting delivery rate is actually what happens.

With these notations, I can write the budget set $B^i(p, \pi, K)$ of agent $i$ as follows:

$$B^i(p, \pi, K) = \left\{ (c, \theta, D) \in \mathbb{R}_+^{S \times L} \times \mathbb{R}_+^I \times \mathbb{R}_+^{S \times L} : 
\begin{align*}
p_0 \cdot (c_0 - e_0^i) + \sum_{j \neq i} \pi_j \theta_j^i - \pi_i \theta_i^i & \leq 0; \\
p_s \cdot (c_s - e_s^i) + p_s \cdot D_s & \leq \sum_{j \neq i} \theta_j^i K_{sj} p_s \cdot A_s.
\end{align*}
\right\}$$  \hspace{1cm} (2)

And the preference of agent $i$ is then:

$$v^i(c, \theta, D) = u^i(c) - \sum_{s \in S} w^i \left( \frac{(\theta_i p_s \cdot A_s - p_s \cdot D_s)^+}{p_s \cdot A_s} \right),$$  \hspace{1cm} (3)

where $\overline{A}_s \in \mathbb{R}_+^L, \overline{A}_s \neq 0$ are constants, and $(x)^+ \equiv \max(x, 0)$. The reason I scale the default by some fixed basket of goods $\overline{A}_s$ is that I want to impose the default on some “real” defaulting instead of nominal defaulting. Of course, in the case of one good economy, this is not needed.

B.1 Definition of The Equilibrium

Using the above setup, I define the equilibrium as a list of $(p, \pi, K, (c^i, \theta^i, D^i)_{i \in I})$ such that the following hold:

1. For each $i \in I$, $(c^i, \theta^i, D^i) \in \arg \max v^i(c, \theta, D)$ over $B^i(p, \pi, K)$.

2. $\sum_{i \in I} (c^i - e^i) = 0$.

3. $\sum_{i \neq j} \theta_j^i = \theta_j^i, j \in I$.

4. $K_{si} = \left\{ \begin{array}{ll} p_s \cdot D_s / p_s \cdot A_s \theta_i^i, & \text{if } p_s \cdot A_s \theta_i^i > 0 \\ \text{arbitrary} & \text{if } p_s \cdot A_s \theta_i^i = 0 \end{array} \right.$

for all $s \in S$ and $i \in I$.

Here condition (1) is the individual optimality condition. Conditions (2) and (3) are that the goods and security market clear. Condition (4) is the rational expectation requirement.
Remark 26 The equilibrium I have defined so far is very similar as that in Dubey, Geanakoplos and Shubik (2000) (Hereafter DGS). However unlike DGS, in which they emphasize the pooling structure of securities, here I emphasize the separating structure of the securities. Studying the symmetric information case will provide us a benchmark point in studying the cases for private information as I will do later on.

B.2 A Refinement of the Equilibrium

I adopt a refinement of the above equilibrium because of the following reason. Note that even though each agent can sell defaultable bond, not everyone will do so. For example, suppose an agent has very high time $t = 0$ endowment. Selling bonds to increase current period consumption is not attractive since the marginal utility of consumption for $t = 0$ is low. And for those non-traded securities there is no conditions for the expectations for the delivery. One potential problem is that this will always allow the trivial equilibrium of no trading. To avoid this, I have to have some refinements on the above equilibrium, as in the cases for most equilibriums with expectations on off-equilibrium paths in the game theory. Many possible refinements can be imposed, for example, those in DGS and Zame (1993). One may also impose stability refinement, as in Cho and Kreps (??). For our purpose, any one will do. Here I will adopt the refinement in DGS and use the stability refinement in the private information case.

Let $|| \cdot ||$ denote the supremum norm, and let $E$ be a candidate equilibrium satisfying conditions (1)-(4). Let $J(s) = \{ j \in I : p_s \cdot A_s \theta_{jj} = 0 \}$. Namely it is the set of defaultable bonds not traded in the market.

(5) For any $\varepsilon > 0$, there exists $E(\varepsilon) = (p(\varepsilon), \pi(\varepsilon), K(\varepsilon), (c^i(\varepsilon), \theta^i(\varepsilon), D^i(\varepsilon))_{i \in I})$ such that

(5a) $||E - E(\varepsilon)|| < \varepsilon$.

(5b) $(c^i(\varepsilon), \theta^i(\varepsilon), D^i(\varepsilon)) \in \arg \max v^i(c, \theta, D)$ over $B^i(p(\varepsilon), \pi(\varepsilon), K(\varepsilon))$

(5c) $K_{si}(\varepsilon) \geq \begin{cases} p_s(\varepsilon) \cdot D_s^i(\varepsilon)/p_s(\varepsilon) \cdot A_s \theta_{ii}(\varepsilon), & \text{if } p_s(\varepsilon) \cdot A_s \theta_{ii}(\varepsilon) > 0 \\ 1 & \text{if } p_s(\varepsilon) \cdot A_s \theta_{ii}(\varepsilon) = 0 \end{cases}$

for all $s \in S$ and $i \in I$.  

37
Simply saying, this implies that if one deviate from the equilibrium \( E \) a little bit, the expected delivery rate will be optimistic.

### B.3 Existence of Equilibrium

Here I state two results for the existence of equilibrium. One is the case for the above multi-goods model. The other is the case when there is one good (actually one only needs the condition that the securities pay only one fixed good).

**Theorem 27** For any \( Q \in \mathbb{R}_+^I \), there exists an equilibrium satisfying (1)-(5).

Proof: This is a special case for the theorem 1 in DGS. I just let the there are \( I \) securities with the exact promised payoff structure. And for \( i \in I \), let \( Q^j_i = 0 \), for all \( j \neq i \) and \( Q^i_i = Q^i \). Then I have from theorem 1 in DGS that there exists an equilibrium \( GE(A, w, Q) \) for any finite \( Q \). QED.

Although I will study the model based on the following theorem for one good financial market, the above theorem is interesting for its implications in the case of multi-period model, which I will explore in another paper. Furthermore, the proof of the following theorem is much easier when I have the above theorem.

In the one-good financial market, one can without loss of generality assume that the price of the good to be one. The real default is then just the default part, which one need not discount any more. However to prove the existence of the equilibrium, I need the following condition:

**Condition 28** For any \( c \in \mathbb{R}_+^S \), \( v^i(c, \theta, D) < u^i(c^i), i \in I \) if the default is large enough in any state.

**Theorem 29** Suppose the above condition is satisfied and there is only one good in the economy. Then there exists an equilibrium with \( Q^i = \infty, \forall i \in I \).

Proof: This is not implied by the theorem 2 of DGS since their theorem 2 says an equilibrium exists when all \( Q = \infty \). Yet what I really need is that those who sell the bonds will have \( Q = \infty \), while for those who buy others’ bond, the limit is still zero. The proof, however, is adapted from their proof.
From theorem 27, I have an equilibrium for all finite \( Q \). Consider a sequence of equilibria, 
\[
E(Q) = (\rho(Q), \pi(Q), K(Q), (\phi^i(Q), \theta^i(Q), D^i_s(Q)))_{i \in I},
\]
where \( Q^i = Q \in \mathbb{N} \), for all \( i \in I \).

If there is a \( Q \) such that \( \theta^i_1 \leq Q \) for all \( i \in I \), then by the concavity of each \( v^i \), this is the equilibrium I am looking for. Suppose this is not the case, I want to develop a contradiction.

Suppose for all \( i,j \in I \), (if necessary using a convergent subsequence),
\[
\frac{\theta^i_j(Q)}{Q} \to \bar{\theta}^i_j, \quad \frac{\theta^i_j(Q)}{Q} \to \tilde{\theta}^i_j, \quad j \neq i.
\]

Assume that for at least one \( j \) and some \( i, \bar{\theta}^i_j \neq 0 \) and \( \tilde{\theta}^i_j = 1 \).

Observe that by condition 28 the default amount for any agents at any state is upper bounded. For otherwise agent would just eat his endowment. So if \( \theta^i_j \to \infty \), the delivery ratio \( K_{s_i} \to 1 \) for all \( s \in S \).

Note also that \( \sum_{j \neq i} \theta^i_j(Q)K_{s_j}(Q)A_s - D^i_s(Q) \) is bounded since otherwise agent \( i \) would either consume negative amount or an amount exceed the aggregate endowment.

So for both finite and infinite convergent \( \theta^i_j \), I have
\[
\lim_{Q \to \infty} \sum_{j \neq i} \theta^i_j(Q)K_{s_j}(Q)A_s - D^i_s(Q) = A_s \left( \sum_{j \neq i} \bar{\theta}^i_j - \bar{\theta}^i_i \right) = 0
\]

for all \( i \in I, s \in S \).

Consider any \( i \) with \( \bar{\theta}^i_i \neq 0 \), and thus \( \sum_{j \neq i} \tilde{\theta}^i_j \neq 0 \). For any \( Q \geq 1 \), I can define new portfolio holdings as follows:
\[
\phi^i = \theta^i(Q) - \bar{\theta}^i \geq 0.
\]

And for large enough \( Q \), the agent can choose the new delivery decision rule:
\[
\Delta^i_s = D^i_s(Q) - A_s\bar{\theta}^i \geq 0, \quad \forall s \in S.
\]

These new portfolio choice and delivery decision will pay his the same penalty as in the equilibrium \( E(Q) \). He would receive same consumption if the delivery rate \( K^j(Q) = 1 \) for \( \bar{\theta}^i_j > 0 \) and strictly more otherwise. Since \( E(Q) \) is optimal, I must have time \( t = 0 \)
consumption to be less:
\[
\sum_{j \neq i} \pi_j \tilde{\theta}_j - \pi_i \tilde{\theta}_i \leq 0. \tag{8}
\]

Since \( \tilde{\theta}^i \) are limits of equilibrium portfolios, \( \sum_{j \neq i} \tilde{\theta}_j^i = \tilde{\theta}_i^i \). So I must have the above inequality to equality for every \( i \in I \). So time \( t = 0 \) consumption will not change. Again from optimality condition, if \( \tilde{\theta}_j^i > 0 \) for any \( i \neq j \), I will have \( K_{s_j}(Q) = 1 \). And if \( \tilde{\theta}_i^i > 0 \), there must be some \( \tilde{\theta}_j^i > 0 \), hence \( K_{s_i}(Q) = 1 \).

So if I replace the original \( E(Q) \) with the new portfolio choice and delivery decision rule, I have a new \( E(Q) \) with \( \phi_j^i < Q, i,j \in I \). (This will not affect \( K \) since I only change the portfolio choice for those securities with \( K_{s_i} = 1 \)). This is the contradiction I am looking for. QED.

References


