The Nature of Market Growth, Risk, and Return

Michael J. Dempsey

In the model of asset appreciation advanced here, the market economy and the market of asset claims on the economy are modeled as organic (or exponential growth) processes, similar to those commonly seen in nature and the biological sciences. In this model, investors have a log-wealth utility function. Within the framework, the market risk premium is derived as the premium that balances supply and demand among risky and risk-free assets. The model indicates that the premium is less than is indicated by ex post returns observed on U.S. stock markets. The model is consistent, however, with empirical observations that idiosyncratic risk and small company size are rewarded by the markets. In terms of the model, investors choose to allocate their portfolios long in both the risky market and the risk-free asset. Furthermore, their portfolios are independent of the investment time horizon.

In the model presented here, “Dempsey’s organic growth model of appreciation” (DOGMA), I consider market growth, risk, and the manner in which market growth is generated consistent with risk. I represent the economy and the market of asset claims on the economy as organic processes of appreciation in which investors have a log-wealth utility function. In terms of the model, the market risk premium (the market’s expectation of return over and above the rate offered by risk-free government securities) is derived as the outcome of the desire of investors to maximize their (log-wealth) utility in such organic markets. When I allow for individual or idiosyncratic volatility in asset stocks, the model is predictive of historical trends in U.S. market returns.

Organic Growth Model of Appreciation

In this section, I consider the exponential growth or organic process itself and extend it to risky and risk-free assets in the U.S. markets.

Organic or Exponential Growth Process.

Exponential or organic growth occurs commonly in nature and in the biological sciences. Such growth may be represented as the outcome of continuously applied growth rates that are selected independently across time from a normal distribution. For such growth, the outcome valuation at the end of a time period is the starting valuation multiplied by \(\exp(y)\), where \(y\) is normally distributed.\(^1\) Stated alternatively, the end-of-period valuation is the starting valuation multiplied by \(\exp(\mu + x)\), where \(\mu\) represents the underlying mean (or drift) exponential growth rate for the period and \(x\) is normally distributed about zero with standard deviation (volatility) of \(\sigma\). The assumption that stock market growth can be modeled by such a process is justified by the evidence of past stock price performance (for example, Fama 1976 and, more recently, Jones and Wilson 1999). Fama observed that an a priori expectation for such a process is reinforced by the mathematics of selection as captured by the Central Limit Theorem.\(^2\)

Investors, however, assess stock prices not in terms of their potential exponential growth characteristics but in terms of their “expected periodic return,” by which I mean the expected percentage increase in wealth generated over some discrete period (a month, a year). Such return is familiarly expressed as:

\[
\text{Expected periodic return} = \frac{\text{Expected wealth outcome of investing } X \text{ for a single period} - X}{X}
\]

To generalize the relationships among the periodic return (the “surface” return that market participants seek to measure) modeled in Equation 1 and the “subsurface” continuously applied parameters \(\mu\) and \(\sigma\) that generate the return, the first step is to

Michael J. Dempsey is visiting academic at the University of Technology, Sydney, and senior lecturer at Griffith University, Australia.
define the “exponential growth rate for expected wealth” over a period $r$:

$$X_{\text{exp}}(r) = \text{Expected wealth outcome of investing} \ X \ \text{for a single period},$$

which provides

$$r = \ln(\text{Expected wealth outcome of investing} $1 \ \text{for a single period}),$$

where $\ln$ represents the natural log function. Because the wealth outcome of investing $1$ for a single period is assumed to be distributed as $\exp(y)$, where $y$ is normally distributed with mean $\mu$ and standard deviation $\sigma$, Equation 3 may be expressed as

$$r = \mu + \frac{1}{2} \sigma^2,$$

as used in the continuous-time framework of the Black–Scholes model. Combining Equations 1, 2, and 4 produces the relationship between an investment’s expected periodic return and the parameters $\mu$ and $\sigma$ as

$$\text{Expected periodic return} = \exp\left(\mu + \frac{1}{2} \sigma^2\right) - 1.$$

Because $1/2\sigma^2$ is necessarily positive, Equation 5 confirms that the volatility of returns, $\sigma$, in an exponential growth process necessarily acts to increase the expected periodic return.

The essential objective of the earlier sections of this article is the determination of likely values of $\mu$ and $\sigma$ for the risky market and, through this process, the likely value of the risky market’s expected return consistent with Equation 5. For this purpose, the aim is to determine the market’s mean exponential growth rate, $\mu_m$ (for a given volatility of market exponential growth rates $\sigma_m$), that will induce investors to arrange their portfolios between the risky market and risk-free assets (consistent with the availability of these assets). For that objective, a model of investors’ propensity to accept risk must be developed, and for both objectives, the organic growth model must be generalized so that it allows for a combination of risky and riskless assets. This generalization is the subject of the following subsection.

**Exponential Growth with Risky and Riskless Assets.** A normal distribution of exponential growth rates over an interval can be modeled as the outcome of a simple two-step or binomial distribution of exponential growth rates imposed over a sufficiently large set of subintervals. (Justification for the binomial modeling is demonstrated in Appendix A.) In the binomial framework, the exponential growth rate, $y$, over each subinterval is restricted to being either $\mu + \sigma$ or $\mu - \sigma$ with equal probability, where $\mu$ and $\sigma$ are equal to, respectively, the mean and standard deviation of the assumed underlying normally distributed exponential growth rates over the subinterval. The binomial process is demonstrated for three subperiods in Panel A of Figure 1, where the entries are the possible wealth outcomes multiplied by the probability of outcome. The structure of the mean and standard deviation of the outcome process is

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**Figure 1. Exponential Growth Characteristics: Probability-Weighted Wealth Outcomes**

<table>
<thead>
<tr>
<th>Starting Wealth</th>
<th>One Period</th>
<th>Two Periods</th>
<th>Three Periods</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A. Binomial Tree</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1.00$</td>
<td>$1/2 \exp(\mu + \sigma)$</td>
<td>$1/4 \exp(2\mu + 2\sigma)$</td>
<td>$1/8 \exp(3\mu + 3\sigma)$</td>
</tr>
<tr>
<td></td>
<td>$1/2 \exp(\mu - \sigma)$</td>
<td>$1/4 \exp(2\mu)$</td>
<td>$3/8 \exp(3\mu + \sigma)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1/4 \exp(2\mu - 2\sigma)$</td>
<td>$3/8 \exp(3\mu - \sigma)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$1/8 \exp(3\mu - 3\sigma)$</td>
</tr>
<tr>
<td><strong>B. Mean exponential growth rate</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu$</td>
<td>$2\mu$</td>
<td>$3\mu$</td>
<td></td>
</tr>
<tr>
<td><strong>C. Standard deviation of exponential growth rates</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$\sqrt{2} \sigma$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\sqrt{3} \sigma$</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>D. Log-wealth utility (starting = 0)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu$</td>
<td>$2\mu$</td>
<td>$3\mu$</td>
<td></td>
</tr>
</tbody>
</table>
presented in, respectively, Panels B and C (discussed in Appendix A). The distribution of potential wealth outcomes for a portfolio that combines risky assets with a riskless asset may be represented by a binomial process as that of Figure 1 simply by determining \( \mu \) and \( \sigma \) as

\[
\mu = \frac{1}{2} \ln(\exp(\mu_m + \sigma_m) + (1 - \omega)\exp(r_f)) + \ln(\exp(\mu_m - \sigma_m) + (1 - \omega)\exp(r_f)) \tag{6}
\]

\[
\sigma = \frac{1}{2} \ln(\exp(\mu_m + \sigma_m) + (1 - \omega)\exp(r_f)) - \ln(\exp(\mu_m - \sigma_m) + (1 - \omega)\exp(r_f)) \tag{7}
\]

where

- \( \mu_m \) = mean exponential growth rate for the risky market
- \( \sigma_m \) = volatility (standard deviation) for the risky market
- \( \omega \) = proportion of the portfolio allocated to the risky market (as opposed to the riskless asset)
- \( r_f \) = riskless exponential growth rate

The derivation of Equations 6 and 7 is demonstrated in Appendix A. The usefulness of these equations lies in the fact that for any portfolio combination of \( \mu_m, \sigma_m, r_f \) and \( \omega \), one simply substitutes the values into the equations, calculates \( \mu \) and \( \sigma \), and substitutes those values in Figure 1. Thus, Figure 1 (together with Equations 6 and 7) represents my model of portfolio price behavior for a general combination of risky and riskless assets. I will complement it in the next section with a model of investor propensity to undertake the risk implied by this price behavior.

For an illustration of what is at stake when choosing a portfolio of risky and riskless assets, consider an investment opportunity that, with equal likelihood, will either increase wealth by one-third or decreases it by one-quarter. The following observations can be made about these three portfolios. With an initial investment of $12, the wealth outcome for Portfolio A remains at $12. The wealth outcome possibilities for Portfolios B and C over the first few periods are presented in Panel A of Figure 2. For example, commencing with an initial investment of $12, the upper outcome of $14 at the end of the first period for Portfolio B was calculated as the outcome of $6 invested at 0 percent in the riskless asset = $6 plus $6 that increased by one-third = $8, yielding $6 + $8 = $14. The entry $16.33 at the end of the second period was calculated as the outcome of $7 invested at zero percent in the riskless asset = $7 plus $7 that increased by one-third = $9.33, yielding $16.33.

The expected wealth outcomes, \( E(W_N) \)—that is, the probability-weighted sum of wealth outcomes—at the end of period \( N \) for Portfolios B and C are presented in Panel B of Figure 2. The expected periodic return over the period is then calculated as \( \frac{E(W_N) - E(W_{N-1})}{E(W_{N-1})} \), yielding the constant expected periodic returns given in Panel C. For comparison, note that for Portfolio A, the expected periodic return is trivially zero. In addition, observe in Panel A that the wealth outcomes for Portfolio C remain “centered” through time on the initial investment outlay, $12. The reason is that sequences of a gain of one-third followed by a loss of one-quarter and a loss of one-quarter followed by a gain of one-third bring the investment back to $12. In the case of Portfolio B, however, these sequences create an upward drift; that is, the centered or mean outcome increases through time, which produces $12.25 by the end of the second period.

Appendix B demonstrates the applicability of Equations 6 and 7 by showing how these equations reproduce this example in a particularly straightforward manner. The question remains, however, of how an investor might be expected to choose between such investment opportunities. For an answer, the model of stock behavior (Equations 6 and 7) must be complemented with a model of investor propensity for risk, which is done in the following section.

**Portfolio B**—50 percent risky. Half of investment wealth is in the riskless asset (with zero growth) as in Portfolio A, and half is in the risky investment opportunity as in Portfolio C.

**Portfolio C**—100 percent risky. Investment wealth is 100 percent in the risky investment opportunity that, with equal likelihood, will either increase wealth by one-third or decreases it by one-quarter.

Appendix B demonstrates the applicability of Equations 6 and 7 by showing how these equations reproduce this example in a particularly straightforward manner. The question remains, however, of how an investor might be expected to choose between such investment opportunities. For an answer, the model of stock behavior (Equations 6 and 7) must be complemented with a model of investor propensity for risk, which is done in the following section.
Utility Function and Portfolio Selection

In his introduction of Brownian motion as a model of share-price behavior, Osborne (1964) also introduced the Weber–Fecher law, which states that equal ratios of a physical stimulus—for example, of sound frequency, light intensity, or sound intensity—correspond to equal intervals of subjective sensation, such as pitch, brightness, or noise. Thus, equal intervals of sensation correspond to equal intervals of the log of the physical change. When equal ratios of wealth correspond to equal intervals of investor utility, the investor is said to be subject to a log-wealth utility function. The log-wealth utility function has the properties of both (1) decreasing absolute risk aversion (meaning that equal absolute losses decline in importance as wealth increases) and (2) constant relative risk aversion (meaning that one is equally averse to proportional losses in wealth). Interestingly, Copeland and Weston (1988) argued that a decreasing marginal utility of wealth is probably genetically coded because without it we would exhibit extreme impulsive behavior. We would engage in the activity with the highest marginal utility to the exclusion of all other choices. Addictive behavior would be the norm. (p. 88)

As the inverse of the exponential growth function, a natural log-outcome utility function has the effect of balancing the instinct to pursue risky growth opportunities with the need to safeguard current wealth. Although economists generally agree that investors have decreasing absolute risk aversion, less consensus exists that investors maintain strict constant relative risk aversion at the prospect of an increasingly severe loss at an increasingly low probability; “prospect theory” suggests that investors are typically less averse to such downside losses than is implied by log-wealth utility. Nevertheless, in the remainder of the article, I take it as a worthwhile hypothesis that investors can be described by a natural log-wealth utility.

For a natural log-wealth utility investor, the important outcome is that the per period utility, $U$, provided by investment in organic or exponential growth is equal to the per period mean exponential growth rate, $\mu$, of the growth process. That is, for such an investor,

$$ U = \mu, \quad (8) $$

which may be interpreted as either the utility gain over a single identified period with per period mean exponential growth rate $\mu$ or the utility gain over $N$ such consecutive periods divided by $N$ (see Panel D of Figure 1). The outcome is demonstrated in Appendix C. The important observation, therefore, is that an investor with a natural log-wealth utility function enjoys per period utility $U = \mu$ from an investment in exponential growth quite independently of the dispersion of possible outcomes at the termination of each stage of the investment process. Only mean exponential growth rate $\mu$ is important.

Thus, the per period utility offered by an investment’s expected periodic return as it derives from the per period continuously applied parameters $\mu$ and $\sigma$ as given in Equation 5 is equal to $\mu$ independent of $\sigma$. In other words, the contribution to the expected periodic return generated by the dispersion of outcomes, $1/2\sigma^2$, is precisely sufficient to compensate the investor for bearing such
dispersion. In a fundamental sense, risk has created its own reward.

Equations 6–8 constitute the joint model of stock market behavior and investor utility in the context of risk. Before discussing the implications for stock market performance generally, I first highlight the way in which the model resolves the issue of how an investor might choose between Portfolios A, B, and C—the 100 percent riskless, 50 percent risky, and 100 percent risky portfolios considered in the previous section. As in the previous section for Equations 6 and 7, the approach is to compute the utility of the portfolio from first principles and thereafter confirm that Equation 8 reproduces the identical outcome.

The log-wealth utility at the end of each period may be calculated from first principles as the sum of the utilities afforded by each outcome, with each utility multiplied by the probability of the outcome’s occurrence (the Von Neumann–Morgenstern theorem expressed as the first equation in Appendix C). That is, we have:

Portfolio A—100 percent riskless. At the end of each period, the utility remains \( \ln(12) = 2.4849 \). For this portfolio, the per period utility gain is, therefore, zero.

Portfolio B—50 percent risky (see Figure 2).

Original wealth utility = \( \ln(12) = 2.4849 \),
End-of-Period 1 utility = \( 0.5 \ln(14) + 0.5 \ln(10.5) = 2.4952 \),
End-of-Period 2 utility = \( 0.25 \ln(16.33) + 0.5 \ln(12.25) + 0.25 \ln(9.19) = 2.5055 \).

For this portfolio, the per period utility gain is, therefore, \( 2.5055 - 2.4952 = 2.4952 - 2.4849 = 0.0103 \), or 1.03 percent.\(^9\)

Portfolio C—100 percent risky (see Figure 2).

Original wealth utility = \( \ln(12) = 2.4849 \),
End-of-Period 1 utility = \( 0.5 \ln(16) + 0.5 \ln(9) = 2.4849 \),
End-of-Period 2 utility = \( 0.25 \ln(21.33) + 0.5 \ln(12) + 0.25 \ln(6.75) = 2.4849 \).

For this 100 percent risky portfolio, the per period utility gain is, therefore, zero.

Thus, the per period utility of 1.03 percent offered by the “hybrid” 50 percent risky portfolio actually exceeds the utility offered by either the 100 percent risky or 100 percent riskless portfolio. These outcomes for Portfolios B and C are summarized in Panel D of Figure 2. In Appendix B, the outcomes are revealed immediately in terms of Equations 6 and 8.

Does some other portfolio combination of the risky and riskless asset offer a log-wealth utility investor an even greater utility than that offered by a 50 percent risky/50 percent riskless split? For an answer, note that per period log-wealth utility \( U \) offered by a portfolio with proportion \( \omega \) in the risky asset may be expressed (with Equations 6 and 8) as

\[
U = \frac{1}{2} \{ \ln[\omega \exp(\mu + \sigma) + (1 - \omega)\exp(r_f)] + \ln[\omega \exp(\mu - \sigma) + (1 - \omega)\exp(r_f)] \},
\]

where \( \mu = r_f = 0 \) and \( \sigma = 0.28768 \). In this case, the value of \( U \), as derived here, is maximized with \( \omega = 0.5 \). The outcome is demonstrated algebraically in Appendix D. We thus conclude that a log-wealth utility investor chooses the portfolio comprising 50 percent risky and 50 percent riskless assets in preference to any other combination.

In the next section, I generalize the analysis to allow the mean exponential growth rate for the risky-asset component of a portfolio to not necessarily equal the exponential growth rate of the risk-free component. In this context, I consider the market risk premium required to induce investors to arrange their investment portfolios between assets that are risk free and the risky market in proportion to the actual market availability of these assets.

Allocation and the Risk Premium

This section extends the analysis of an investor’s portfolio choice to the choice between the risky market (with \( \mu = \mu_m, \sigma = \sigma_m \)) and risk-free assets, such as U.S. T-bills (with exponential return \( r_f \)). As developed in the previous section, a log-wealth utility investor seeks to optimize the utility function \( U \) that is the outcome of Equations 6 and 8. Combining these two equations produces directly

\[
U = \frac{1}{2} \{ \ln[\omega \exp(\mu_m + \sigma_m) + (1 - \omega)\exp(r_f)] + \ln[\omega \exp(\mu_m - \sigma_m) + (1 - \omega)\exp(r_f)] \},
\]

Consider as a base case that the underlying exponential growth rate for the risky market, \( \mu_m \), is equal to that of the riskless asset, \( r_f \). In this case, the value of \( U \) is maximized, as in the previous section, when the investment portfolio is divided equally between the risky market and a riskless asset (\( \omega = 0.5 \)). The outcome is presented numerically in the first row of Table 1. Table 1 provides the per period utility offered by a portfolio of risky and riskless assets as a function of the proportion, \( \omega \), of the portfolio assigned to the risky (as opposed to the riskless) asset. In the first row, the utility is...
simulated with $\mu_m = r_f = 2.0$ percent (annualized, reflecting a real return on government bonds) and standard deviation (volatility) of market returns of 15 percent (annualized, reflecting the monthly performance of U.S. stocks over the 1980–94 period). Consistent with the last panel of Figure 1, per period utility is determined independently of the number of periods over which the simulation was run.

The equality of $\mu_m$ and $r_f$ in the base case does not of itself imply a zero market risk premium, defined as the difference between the market’s expected periodic return (Equation 1) and the return of a risk-free asset. The reason is that, from Equation 5, the market risk premium, $MRP$, may be expressed as

$$MRP = \left[\exp\left(\mu_m + \frac{1}{2}\sigma_m^2\right) - 1\right] - \left[\exp(r_f) - 1\right]$$

$$= \exp\left(\mu_m + \frac{1}{2}\sigma_m^2\right) - \exp(r_f),$$

which confirms that with $\mu_m = r_f$, the market risk premium is greater than zero. Empirical measurements of the stock market’s periodic return are typically made on a monthly basis. On a monthly basis, the market exponential growth rates may be assumed to be sufficiently small that Equation 11 can be expressed as

$$MRP = \left(\mu_m + \frac{1}{2}\sigma_m^2\right) - r_f,$$

so that for the case of $\mu_m = r_f$,

$$MRP = \frac{1}{2}\sigma_m^2.$$  

With $\sigma_m = 15$ percent (annualized), the market risk premium (annualized) would thus be estimated to be $1/2(0.15)^2 = 1$ percent, whereas with $\sigma_m = 30$ percent, the market risk premium would be estimated to be $1/2(0.30)^2 = 4.5$ percent.

The question now is: Does the equality of $\mu_m$ and $r_f$—with the implication that log-wealth utility investors allocate their portfolios 50 percent in the risky market and 50 percent in riskless assets—represent actual market equilibrium? To begin to answer, note first that the market has historically been unable to offer riskless participation in equal measure with risky participation; the result is that equity assets have historically constituted the greater proportion of asset portfolios (for example, see the Mutual Fund Fact Book, Investment Company Institute, quoted by Madura 1995). The base case in Table 1 with $\mu_m = r_f$ cannot, therefore, be considered representative of historical equilibrium. Equilibrium requires that investors choose to allocate their portfolios between the market of risky assets and a riskless asset so as to be consistent with the relative availability of these asset classes. The

### Table 1. Per Period Utility Offered by a Hybrid Portfolio as a Function of the Proportion of Its Risky Assets

<table>
<thead>
<tr>
<th>Proportion in Risky Assets, $\omega$</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
<th>1.2</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Base case: $Utility$ with $\mu = 2.0%; r_f = 2.0%; \sigma = 15%$</td>
<td>2.0</td>
<td>2.1</td>
<td>2.2</td>
<td>2.2</td>
<td>2.3</td>
<td>2.3</td>
<td>2.3</td>
<td>2.2</td>
<td>2.2</td>
<td>2.1</td>
<td>2.0</td>
<td>1.7</td>
<td>1.1</td>
</tr>
<tr>
<td>B. Utility with $\mu = 2.5%; r_f = 2.0%; \sigma = 15%$</td>
<td>2.0</td>
<td>2.2</td>
<td>2.3</td>
<td>2.3</td>
<td>2.4</td>
<td>2.5</td>
<td>2.6</td>
<td>2.6</td>
<td>2.6</td>
<td>2.5</td>
<td>2.3</td>
<td>1.9</td>
<td></td>
</tr>
<tr>
<td>C. Utility with $\mu = 3.0%; r_f = 2.0%; \sigma = 15%$</td>
<td>2.0</td>
<td>2.2</td>
<td>2.3</td>
<td>2.4</td>
<td>2.5</td>
<td>2.7</td>
<td>2.8</td>
<td>2.9</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
<td>2.9</td>
<td>2.7</td>
</tr>
<tr>
<td>D. Utility with $\mu = 3.0%; r_f = 2.0%; \sigma = 20%$</td>
<td>2.0</td>
<td>2.3</td>
<td>2.6</td>
<td>2.8</td>
<td>3.0</td>
<td>3.1</td>
<td>3.2</td>
<td>3.2</td>
<td>3.2</td>
<td>3.1</td>
<td>3.0</td>
<td>2.7</td>
<td>1.8</td>
</tr>
<tr>
<td>E. Utility with $\mu = 3.5%; r_f = 2.0%; \sigma = 20%$</td>
<td>2.0</td>
<td>2.3</td>
<td>2.7</td>
<td>2.9</td>
<td>3.1</td>
<td>3.3</td>
<td>3.4</td>
<td>3.5</td>
<td>3.6</td>
<td>3.6</td>
<td>3.5</td>
<td>3.3</td>
<td>2.6</td>
</tr>
<tr>
<td>F. Utility with $\mu = 4.5%; r_f = 2.0%; \sigma = 30%$</td>
<td>2.0</td>
<td>2.7</td>
<td>3.2</td>
<td>3.7</td>
<td>4.1</td>
<td>4.4</td>
<td>4.6</td>
<td>4.7</td>
<td>4.7</td>
<td>4.5</td>
<td>3.9</td>
<td>2.4</td>
<td></td>
</tr>
</tbody>
</table>

Note: In the first panel, the mean exponential growth rate of the risky asset is equal to that of the riskless asset. In the other panels, $\mu$ is set greater than $r_f$ and $\sigma$ is the standard deviation assigned to the potential exponential growth rate for the risky asset. Data from Ibbotson Associates (see Brealey and Myers 1996).
implication is that the risky market over the period has been priced so that \( \mu_m \) is greater than \( \mu_f \).

In modeling an investment portfolio with just the two asset classes, I assume that investors, both institutional and individual, have tended historically to have portfolios dominated by risky equity investments (tending toward, say, two-thirds of their portfolio assets) with high-quality bonds typically representing a perceived secure component.\(^\text{11}\) Interestingly, it is possible, without being much more specific, to specify in terms of the model fairly close constraints on the market risk premium. The reason, as I will explain, is that quite modest shifts in the premium predict fairly large shifts in the proportions of risky and risk-free assets held in an investment portfolio.

To determine the relationship between the market risk premium and investors’ portfolio compositions, the first step is to work through the implications of the difference between the market’s mean exponential growth rate (for a given market volatility) and that of the risk-free rate for portfolio compositions. The next step will be to transform the implications to a determination of the market risk premium in accordance with Equation 12.

The base outcome is that with \( \mu_m = \mu_f \), a log-wealth utility investor chooses to allocate investment wealth equally between the market of risky and riskless assets. The outcome utilities for \( \mu_m = \mu_f = 2.0 \) percent (real) and \( \sigma_m = 15 \) percent with Equation 10 are presented in Panel A of Table 1. Setting \( \mu_m \) equal to 2.5 percent while maintaining \( \mu_f = 2.0 \) percent leads to the outcomes in Panel B of Table 1. It shows that a portfolio invested 100 percent in a riskless asset (\( \omega = 0 \)) has utility equal to \( \mu_f \) (2.0 percent), as in the base case, whereas a portfolio invested 100 percent in the market (\( \omega = 1.0 \)) has utility equal to 2.5 percent. In this case, the utilities offered in the range of \( \omega \) equal to 55–90 percent remain essentially flat at 2.6 percent. That is, investors with log-wealth utilities now choose to hold between 10 percent and 45 percent of their portfolio in the riskless asset. Thus, an important observation is that the “flatness” of the resulting utilities allows for a fair degree of investment flexibility consistent with log-wealth utility. Setting \( \mu_m \) equal to 3.0 percent while maintaining \( \mu_f \) at 2.0 percent leads to the outcomes in Panel C of Table 1. In this case, utilities offered in the range of \( \omega \) equal to 65–100 percent remain essentially flat at 3.0 percent, implying that investors with log-wealth utility choose to hold between 0 and 35 percent of their portfolio in the riskless asset. The conclusion is that the underlying mean exponential growth rate of investments in risky markets need be only as little as 1 percent above that offered by riskless assets, such as government bonds, before log-wealth utility investors are induced to allocate greater portions of their portfolios to the risky markets than to riskless assets.

Although the difference between \( \mu_m \) and \( \mu_f \) is determined ultimately by market supply and demand, the relationship depends on the market’s volatility, \( \sigma_m \). The reason is that with increasing \( \sigma_m \), investors choose to allocate the riskless asset to their portfolios at increasingly lower returns relative to the expectation of return on the risky market. Consider market volatility at the 20 percent measured over the long 1926–94 period. To achieve the same profiles for investor allocations between risky and riskless assets as in Panels B and C of Table 1, the difference between \( \mu_m \) and \( \mu_f \) must be increased.

Thus, setting \( \mu_m = 3.5 \) percent generates the utility patterns in Panel E, where the utilities remain essentially flat in the range of \( \omega \) equal to 65–100 percent (as for \( \sigma = 15 \) percent in Panel C), implying that investors choose to invest 0–35 percent in the riskless asset. Finally, the last panel of Table 1 shows that for a market volatility as high as 30 percent (a value closer to the 33.6 percent actually measured for U.S. stock returns over the 1926–39 Depression period) and the return on government bonds at 2.0 percent (it was closer to 1.85 percent during the Depression), the estimate is that the risky market must be priced to offer a mean exponential growth rate of 4.5 percent before investors will allocate their portfolios to the risky market in the 65–90 percent range. Again, consistent with Panel D of Figure 1, the period utilities are independent of the number of periods over which the simulation was run.

In terms of Equation 12, the implications for the market risk premium may be generalized as follows. With an annualized volatility of market returns equal to 15.0 percent (as measured for the 1980–94 period) and a real rate on government bonds of 2.0 percent, the exponential growth rate for the risky market, \( \mu_m \), must be 3.0 percent to induce investors to put 0–35 percent of their portfolio in such risk-free assets as government bonds. That is, I estimate that to induce investors to allocate their portfolios between the risky market and risk-free assets in a way that reflects the actual proportions of these assets, \( \mu_m - \mu_f \), is no more than 1.0 percent. Moreover, the market investor enjoys
the additional expectation of return $1/2 \sigma^2_m$ that is generated by the riskiness of the market itself, which with $\sigma_m = 0.15$, is also approximately 1.0 percent. I thus estimate the required annualized market risk premium over and above the rate offered by government bonds (Equation 12) to be no more than $(\mu_m - r_f) + 1/2 \sigma^2_m = 1.0$ percent + 1.0 percent = 2.0 percent. By setting market volatility $\sigma_m$ equal to the 1926–94 volatility of 20.0 percent while maintaining $r_f$ at 2.0 percent, however, I estimate a $\mu_m$ of 3.5 percent to be necessary to induce investors to allocate 0–35 percent of their portfolio to risk-free assets. In this case, I estimate $\mu_m - r_f$ to be no more than 1.5 percent whereas the expectation of return generated by volatility $1/2 \sigma^2_m$ is approximately 2.0 percent, so the annualized market risk premium is estimated to be no more than $(\mu_m - r_f) + 1/2 \sigma^2_m = 1.5$ percent + 2.0 percent = 3.5 percent.

Such market risk premiums are clearly much less than have been supposed on the basis of the historical performance of U.S. stocks. For example, average premiums for U.S. stocks over government bonds in the 1926–94 period were closer to 7.0 percent (Brealey and Myers 1996). By setting market volatility $\sigma_m$ as high as 0.30 percent while maintaining $r_f$ at 2.0 percent, I calculate a mean exponential growth rate of 4.5 percent to be necessary to induce investors to allocate 10–35 percent of their portfolio to risk-free assets. The expectation of return generated by the riskiness of the market itself is then approximately 4.5 percent ($\sigma_m = 0.30$). Under such conditions, the annualized market risk premium is estimated to be close to 7.0 percent.

Thus, in my model, quite large variations in the market risk premium (2–7 percent) are implied by the historical variation of stock market volatility (15–30 percent). Nevertheless, the kind of returns actually enjoyed in later years by U.S. stock investors appear more consistent with a risk premium calculated with stock market volatility more typical of the 33.6 percent level of the Depression years of 1926–1939.

Schwert (1991) observed that, although the data might now indicate that by the late 1980s and early 1990s the volatility of U.S. stock market returns had come closer to 15 percent, the widespread impression at this time remained that of especially volatile stock prices (see also Brealey and Myers). In this case, historical U.S. market returns may be interpreted as partly the result of investor perceptions of volatility. Or it may be that, as a consequence of the market’s potential to vary the volatility of its returns as I have discussed, investors conceptualize market volatility in terms of a somewhat higher volatility than they know to be currently operative. Alternatively, in line with Jorion and Goetzmann (1999), the high level of historical U.S. stock market returns may be interpreted as the phenomenon that U.S. stocks have continued to surpass investor expectations. That is to say, historical U.S. market returns are not actually representative of investor expectations.

An interesting conclusion to this section is to apply the utility model of Equation 10 to the case of a “depression” economy—one that offers only market investment opportunities with zero or uniformly low mean exponential growth rates (for example, $\mu_m = r_f = 0$). In such a case, the model implies that log-wealth utility investors persist in committing half of their investment wealth (either enhanced or depleted) to the upside potential of risk. Furthermore, the greater market volatility $\sigma_m$, the greater the propensity of investors to undertake risky investment. Thus, even with economic prospects so reduced, risky investment persists.

The Organic Growth Model versus the CAPM

How do the implications of the organic growth model of appreciation in conjunction with log-wealth utility investors compare with the elements of the capital asset pricing model? In his introduction to the foundations of the CAPM, Sharpe (1964) identified the availability of portfolios to the investor over a single period as the “broken egg” combinations of expected return, $r$, and level of risk (volatility of return), $\sigma$, depicted in Figure 3. The introduction of a risk-free asset with return $r_f$ allows the risk-return portfolio combinations on the dotted line in Figure 3. The portfolio combinations are formed by combining the risk-free asset at point $r_f$ with the portfolio that is at point $M$ (points above $M$ are obtained by borrowing the risk-free asset at the same rate). Investors A, B, and C have rates of trade-off between risk and reward as represented by, respectively, the indifference curves A, B, and C in Figure 3 (see also Kritzman 1992). Higher combinations provide greater utility than lower ones because for a given level of risk, $\sigma$, all investors are assumed to prefer a higher expectation of return, $r$, to a lesser one. The risk-averse investor is deduced, therefore, to maximize utility by finding the point of tangency between the efficient set represented by the capital market line and his/her highest risk–return indifference curve. For example, Investor A is the most risk averse of the three pictured in Figure 3 and will choose to invest all of his portfolio in the risk-free asset. Investor C,
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who is the least risk averse, will borrow at the risk-free rate to invest more than 100 percent of her portfolio in the market portfolio. Investor B has chosen to invest 100 percent in the market portfolio. Nevertheless, in the context of the model I have advanced, investors rationally choose to be long in both the risky and nonrisky components of the market. Thus, the model actually precludes all three of these investors considered by Sharpe.

Samuelson has argued for many years (Samuelson 1963, 1969, 1989, 1994; Merton and Samuelson 1974) that the investment horizon can have no effect on portfolio proportions (see, for example, Samuelson 1994). Kritzman (1994), quoting Samuelson, observed that a rising mean does not overcome the increase in dispersion. My model is consistent in this respect with Samuelson. Nevertheless, this outcome is not necessarily consistent with the foundations of the CAPM. To understand, consider a possible portfolio over a single period that is characterized in Figure 3 by expected return \( r \) with level of risk \( \sigma \). Over \( N \) periods, the portfolio is characterized by a compounded expected return, \( N r \), with level of risk \( \sqrt{N} \sigma \).\(^{13}\) The risk–return relationship offered by the single-period portfolios in Figure 2, when taken over \( N \) periods, is thus represented on a per period basis as in Figure 4, which is Figure 3 with the \( x \)-axis compressed by dividing each standard deviation by \( \sqrt{N} \) (thus, the gradient of the capital market line is \( \sqrt{N} \) times the gradient for a single investment period).\(^{14}\)

Now, consider Investor A in Figure 3 (who chooses rationally to abstain completely from the market over a single period because this investor’s highest attainable position of utility is achieved with 100 percent investment in the risk-free portfolio). Following Sharpe, Investor A construes risk as the variability of returns, for which he requires an enhanced expectation of return. On this basis, he constructs his utility curves as an indifference between such risk and return. He thus imposes these indifference curves, as presented in Figure 3, on the capital market line offered by an \( N \)-period investment as represented by the solid line in Figure 4. In this case, Investor A clearly achieves a higher utility indifference curve by investing a proportion of investment wealth in the market portfolio (at point \( A_N \) in Figure 4). For such an investor, the portfolio allocation between risky and risk-free assets is dependent on the time horizon. Thus, the model presented here (1) restricts investor portfolio allocation choices, as compared with the CAPM, and (2) unlike the CAPM, denies the possibility that longer investment horizons might induce investors to greater risk taking.

The Organic Growth Model and Idiosyncratic Risk

Portfolio theory states that outcome deviations \( \pm x \) percent and \( \pm x \) percent from an asset’s expected periodic growth rate are equally likely. Hence, when a sufficient number of assets are allowed, nonmarket or idiosyncratic deviations from the expected growth rates of individual assets effectively cancel each other (that is, the principle of

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**Figure 3.** Market Capital Line with One Risk-Free Asset and Many Risky Assets

![Figure 3](image3.png)

**Source:** After Sharpe.

**Figure 4.** Capital Market Line with an \( N \)-Period Investment Horizon

![Figure 4](image4.png)
Markowitz portfolio diversification). For this reason, the theory assumes that nonmarket or idiosyncratic volatility of individual assets need not be rewarded by the markets (for example, see Malkiel and Xu 1997). The organic growth model, however, implies that it is the deviations $\pm x$ percent from an asset’s expected exponential growth rate that are equally likely. Because $[\exp(x) + \exp(-x)]/2$ always exceeds 1, deviations $+x$ percent and $-x$ percent from the expected exponential growth rate do not cancel. With Equation 5, exponential growth rates normally distributed with volatility $\sigma$ about a mean growth rate $\mu$ equal to zero would generate an expected periodic return of $\exp(1/2\sigma^2) - 1$ (which is always greater than zero). Furthermore, given a sufficiently large number of assets with idiosyncratic volatility $\sigma$ about such a zero mean exponential growth rate, all idiosyncratic deviations occur in proportion to their probability of occurrence and the expected return $\exp(1/2\sigma^2) - 1$ generated by idiosyncratic volatility approaches a certain outcome.\textsuperscript{15}

The empirically observed relationship between the idiosyncratic volatility of the component assets in a portfolio and the average return on the portfolio by Malkiel and Xu offers an opportunity to test my model. Malkiel and Xu reported that the idiosyncratic volatility of small-company stocks increased over the 1963–94 period (a phenomenon that appears to have been unrecognized at the time). For this period, the authors constructed portfolios as a function of the idiosyncratic volatility of the portfolio’s assets as measured individually. They then plotted the average portfolio returns as a function of the underlying idiosyncratic volatility (the circles shown in Figure 5). Their findings appear consistent with the prediction, as shown by the curved line in Figure 5, that the returns on the portfolios should be a quadratic function of the idiosyncratic volatility of their component assets. The average annual returns they found ranged from a low of close to 12 percent for the portfolio with the lowest idiosyncratic volatility (5 percent a month) to a high close to 19 percent for the portfolio with the highest idiosyncratic volatility (13 percent a month).

Malkiel and Xu considered such outcomes to be “surprising,” but the outcomes appear to be consistent with my model. In terms of the model, an idiosyncratic volatility of asset returns of 5 percent a month contributes approximately $(1/2\sigma^2) = 1/2 [0.05^2] = 0.125$ percent to the expected monthly periodic return of a portfolio of such assets (annualized by multiplying by 12 to equal 1.50 percent) whereas an idiosyncratic volatility of asset returns of 13 percent a month contributes approximately $1/2(0.13^2) = 0.84$ percent to the expected monthly periodic return (annualized to 10.10 percent).\textsuperscript{16} The difference of 8.6 percentage points (10.10 percent – 1.50 percent) is actually somewhat more than is required to account for Malkiel and Xu’s measured difference of 7 percentage points (19 percent – 12 percent).

Extrapolating back to zero idiosyncratic volatility in Figure 5, one might judge that a portfolio with zero idiosyncratic volatility would have experienced an average annual return of no more than

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**Figure 5. Portfolio Returns versus Idiosyncratic Volatility: Empirical Results of Malkiel and Xu versus Predictions of the Model**

![Image of Figure 5](image-url)
approximately 10 percent (as opposed to an average return of closer to 14 percent for the market over the same period), in which case, we arrive at the rather remarkable conclusion that idiosyncratic volatility may have contributed as much as 4 percentage points annually to the market return.

Malkiel and Xu observed further that the volatility of their stocks was correlated closely with the inverse of company size. They noted that their results are thus consistent with the finding of Fama and French (1992, 1996) that the portfolios of smaller companies’ stocks provide rates of return that are greater than the returns from portfolios of larger companies’ stocks. The Fama–French results remain controversial, but nevertheless, my model may be interpreted as predicting their finding.

**Conclusion**

I have advanced Dempsey’s organic growth model of appreciation for the market with log-wealth utility investors. 17 DOGMA is necessarily a simplification of reality in the following ways. A key feature of the model is that growth occurs continuously with rates that are normally distributed and independent of each other across time. Thus, for example, the model does not allow for consideration of non-random-walk changes in stock prices that result in serial correlation—mean reversion, for example—of stock returns (see Poterba and Summers 1988). Furthermore, in developing the model, I characterized the distributions of possible growth rates over each investment period as identical with mean \( \mu_m \) and standard deviation \( \sigma_m \). Thus, the model does not allow for the possibility that the volatility of stock returns might depend on price—for example, that high volatilities might be associated with lower prices (see Chriss 1997).

In the model, investors in the market have log-wealth utility. I attempted to justify log-wealth utility as representative of investor preferences, but I noted that, although a decreasing marginal utility of wealth appears reasonable, investors may actually be less averse to downside losses than log-utility suggests. In that case, investors are, on balance, actually less risk averse than is implied by my log-wealth utility function.

Notwithstanding the limitations, the model incorporates a fundamental understanding of stock markets and investors’ propensity to invest in such risky markets. As such, it offers at least a first insight into how markets might be expected to perform. The model predicts that the growth generated by volatility is sufficient to compensate log-wealth utility investors for bearing such volatility to the extent that they are prepared to allocate their investment portfolios in equal proportion between volatile growth and riskless growth. In this case, “risk creates its own reward.” Ultimately, however, the market risk premium is determined to be the premium that balances supply and demand among both risky and risk-free assets in the market. Because the markets have historically been unable to supply risk-free assets in the same proportion as risky assets, investors have required a sufficient risk premium to induce them to allocate the greater proportion of their portfolios to the risky assets.

On this basis, I estimated the market risk premium as a function of stock volatility. Stock volatility characterized by a standard deviation of returns of 15 percent (such as measured over the 1980–94 period) predicts a risk premium of about 2.0 percent to be sufficient to induce investors to allocate less than 35 percent of their portfolios to risk-free assets; a standard deviation of 30 percent (it was close to 33.6 percent over the 1926–39 period) would be consistent with a risk premium closer to 7 percent (close to the historical average offered by U.S. stock returns) as sufficient to induce investors to allocate less than 35 percent of their portfolios to risk-free assets. Thus, on balance, the model determines a current market risk premium that is less than the premium implied by actual U.S. stock price performance.

The model is consistent with the empirical findings of Malkiel and Xu that the idiosyncratic risk of individual stocks appears to be rewarded by the markets. The model implies that deviations of \( +x \) percent and \( -x \) percent from an asset’s expected exponential growth rate are equally likely for all \( x \) and, in combination, generate the growth factor \[ \frac{\exp(x) + \exp(-x)}{2}, \] which is always greater than unity. The resulting quadratic relationship between portfolio returns and the idiosyncratic volatility of the portfolio’s assets, as predicted by the model, appears to be consistent with the Malkiel–Xu findings. When the inverse correlation between a stock’s idiosyncratic volatility and underlying company size, as observed by Malkiel and Xu, is allowed, the model further suggests a satisfying explanation for the controversial observation of Fama and French that portfolios of small companies have provided superior returns. Remarkably, the model indicates that idiosyncratic volatility in stock returns may have contributed more significantly than market or nonidiosyncratic volatility to observed U.S. stock performance.

The model further indicates that investors in the market choose to be long in both the risky market and risk-free assets. This outcome pertains even when the underlying mean exponential growth rate is reduced to zero. The implication is that, even with...
reduced investment opportunities (in a depressed economy), risky investment in the market economy endures. Finally, within the constraints of the model and consistent with Samuelson’s famous proofs, a rational investor is indifferent to the investment time horizon when assessing the risk profile of an investment portfolio.

I would like to express my appreciation for helpful conversations with many people along the way, notably, Darren Duxbury, Bruce Grundy, Robert Hudson, Kevin Keasey, and Graham Partington, and for the comments of participants at presentations of earlier versions of the article at the universities of Griffith, Melbourne, and New South Wales.

Appendix A. Binomial Frameworks

This appendix shows how a normal distribution of exponential growth rates over an interval can be modeled as the outcome of a simple binomial distribution and expands the binomial framework to a portfolio of risky and riskless assets.

Framework as a Substitute for Normally Distributed Growth Rates. When a binomial process over a single interval with equally likely exponential growth rates \( \mu + \sigma \) and \( \mu - \sigma \) (as in Figure 1) is extended over \( N \) intervals, the resulting exponential growth rates over such an extended period achieve a distribution with mean rate \( N\mu \) and standard deviation \( \sqrt{N}\sigma \) (as indicated in Panels B and C of Figure 1). For example, over two intervals, the mean exponential growth rate in Figure 1 is determined as

\[
\frac{1}{4}(2\mu + 2\sigma) + \frac{1}{2}(2\mu) + \frac{1}{4}(2\mu - 2\sigma) = 2\mu,
\]

with standard deviation (the square root of the probability-weighted average of the squares of the differences between each possible exponential growth rate and the mean exponential rate, \( 2\mu \)) about such rate determined as

\[
\sqrt{\frac{1}{4}[(2\mu + 2\sigma) - 2\mu]^2 + \frac{1}{2}(2\mu) + \frac{1}{4}[(2\mu - 2\sigma) - 2\mu]^2} = \sqrt{\frac{1}{2}\sigma}.
\]

(A2)

In the case of normally distributed potential growth rates over each interval (with mean \( \mu \) and standard deviation \( \sigma \)) with independent outcomes, the mean growth rate and standard deviation about such rate over \( N \) intervals are also \( N\mu \) and \( \sqrt{N}\sigma \) (see, for example, Fabozzi and Modigliani 1996). The outcome is that when a sufficient number of intervals is allocated to the investment period under consideration, the distribution of binomially generated exponential growth rates converges to the normal distribution that would be generated by assuming normally distributed rates over each individual interval (see, for example, Cox and Rubinstein 1985).

Framework for a Portfolio Comprising the Risky Market with a Riskless Asset. In a binomial framework, an investor who is long in both the market of risky assets, as depicted in Figure 1, and a riskless asset, such as U.S. Treasury securities, has equally likely wealth outcomes at the end of the first period (\( S_u \) and \( S_d \)) per dollar of investment, which may be represented as

\[
S_u = \omega \exp(\mu_m + \sigma_m) + (1 - \omega)\exp(r_f)
\]

and

\[
S_d = \omega \exp(\mu_m - \sigma_m) + (1 - \omega)\exp(r_f),
\]

where \( \omega \) is the proportion of the portfolio invested in the market of risky assets (with mean exponential growth rate \( \mu_m \) and standard deviation about such rate \( \sigma_m \)) and where \( 1 - \omega \) is the proportion invested in the riskless asset (with exponential growth rate \( r_f \)). Equations A3 and A4 can be expressed as

\[
S_u = \exp(\mu + \sigma)
\]

and

\[
S_d = \exp(\mu - \sigma).
\]

(A5)

(A6)

Combining Equations A3–A6 produces

\[
\begin{align*}
\mu &= \frac{1}{2} \left[ \ln[\omega \exp(\mu_m + \sigma_m) + (1 - \omega)\exp(r_f)] + \ln[\omega \exp(\mu_m - \sigma_m) + (1 - \omega)\exp(r_f)] \right] \\
\sigma &= \frac{1}{2} \left[ \ln[\omega \exp(\mu_m + \sigma_m) + (1 - \omega)\exp(r_f)] - \ln[\omega \exp(\mu_m - \sigma_m) + (1 - \omega)\exp(r_f)] \right].
\end{align*}
\]

(A7)

(A8)

If the investor’s portfolio is rebalanced at the end of each period to maintain the proportion of the portfolio in risky assets as \( \omega \), outcomes \( S_u \) and \( S_d \) per dollar of investment are repeated over successive intervals. Furthermore, if \( \mu_m, \sigma_m, r_f, \) and \( \omega \) are assumed to be fixed across time, then \( \mu \) and \( \sigma \) are also fixed across time. The significance of these observations is that the growth behavior of a portfolio of the market of risky assets and the riskless asset continues to be characterized across time by the binomial framework of Figure 1 with \( \mu \) and \( \sigma \) determined by Equations A7 and A8.
Appendix B. Figure 2 Constructed Algebraically

Figure 2 states that Portfolio C (100 percent risky) was characterized by $\mu = 0$ and $\sigma = 0.28768$. Substituting these values for, respectively, $\mu_m$ and $\sigma_m$ in Equations 6 and 7 produces (with $\omega = 0.5$ and $r_f = 0$) the characteristics of Portfolio B (50 percent risky), namely, $\mu = 0.01031$ and $\sigma = 0.14384$. We are now in a position to reconstruct algebraically the entries in Figure 2.

For example, the upper outcome of $16.33 at the end of the second period for Portfolio B in Panel A is calculated as

$$12 \times \exp(\mu + \sigma) \times \exp(\mu - \sigma) = 12 \times \exp(2\mu) = 12 \times \exp(2 \times 0.01031) = 12 \times \exp(0.02062) = 12 \times 1.02072 = 12.25$$

because $\mu = 0$. That is, the condition for such centered growth is that the mean exponential growth rate equals zero. Portfolio B, however, is characterized by upward drift. Its mean outcome at the end of two periods is calculated as

$$12 \times \exp(\mu + \sigma) \times \exp(\mu - \sigma) = 12 \times \exp(2\mu) = 12 \times \exp(2 \times 0.01031) = 12 \times \exp(0.02062) = 12 \times 1.02072 = 12.25$$

The mean outcome for Portfolio C at the end of two periods in Panel A is calculated as

$$12 \times \exp(\mu + \sigma) \times \exp(\mu - \sigma) = 12 \times \exp(2\mu) = 12 \times \exp(2 \times 0.01031) = 12 \times \exp(0.02062) = 12 \times 1.02072 = 12.25$$

because $\mu = 0.01031$.

For Portfolio C, $\mu = 0$, so from Equation 8, the per period utility, $U$, is zero (whereas for Portfolio B, $\mu = 0.01031$). Therefore, from Equation 8, the per period utility for Portfolio B is 0.01031, or 1.031 percent, as shown in Panel D of Figure 2.

Appendix C. Log-Wealth Utility Identified as the Mean of the Distribution of Exponential Growth Rates

The Von Neumann and Morgenstern (1947) theorem is that utility $U$ derived from an investment’s possible outcomes is determined as

$$U = \sum_{i=1}^{N} \Pr(x_i)u(x_i)$$

where $\Pr(x_i)$ is the probability of each possible wealth outcome $x_i$ and $u(x_i)$ is the corresponding utility. Thus, for example, the natural log-wealth utility offered by an investment of $1 over two periods in the exponential growth process modeled by Panel A in Figure 1 is determined as

$$\frac{1}{4} \ln[\exp(2\mu + 2\sigma)] + \frac{1}{2} \ln[\exp(2\mu)] + \frac{1}{4} \ln[\exp(2\mu - 2\sigma)] = 2\mu;$$

that is, the natural log-wealth utility offered by an investment of $1 over two periods is the mean exponential growth rate over two periods, $2\mu$. Generally, the utility offered by an investment of $1 over $N$ periods remains the mean exponential growth rate over $N$ periods, $N\mu$. Stated alternatively, per period utility $U$ is determined as per period mean exponential growth rate $\mu$.

Appendix D. Proposition Demonstration

In this appendix, I demonstrate the following proposition: When the mean exponential growth rate for risky assets, $\mu_m$, is equal to the riskless rate, $r_f$, portfolio utility is maximized with equal investments in risky and riskless assets.

First, Equation 9 gives per period utility $U$ as

$$U = \frac{1}{2} \left[ \ln[\omega \exp(\mu_m + \sigma_m)] + (1 - \omega)\exp(r_f) \right] + \ln[\omega \exp(\mu_m - \sigma_m)] + (1 - \omega)\exp(r_f).$$

We also have the general rule that when $y = \ln[F(x)]$, where $F(x)$ is a continuous function of $x$, the partial differentiation of $y$ with respect to $x$ is

$$\frac{\delta y}{\delta x} = \frac{\delta[F(x)]/dx}{F(x)}.$$  

(D1)

Using the rule to differentiate $U$ with respect to $\omega$ produces

$$\frac{\delta U}{\delta \omega} = \frac{1}{2} \left[ \frac{\exp(\mu_m + \sigma_m) - \exp(r_f)}{\omega \exp(\mu_m + \sigma_m) + (1 - \omega)\exp(r_f)} + \frac{\exp(\mu_m - \sigma_m) - \exp(r_f)}{\omega \exp(\mu_m - \sigma_m) + (1 - \omega)\exp(r_f)} \right].$$  

(D2)

The value of $U$ is maximized with respect to $\omega$ at the point that $\delta U/\delta \omega = 0$. With $\mu_m = r_f$, we can see that the right-hand side of Equation D2 = 0 when $\omega = 1/2$. 

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Notes

1. By \(\exp(y)\), I mean “\(e\) to the power of \(y\),” where \(e\) is the exponential number, approximately equal to 2.7183. With \(\exp(y)\), the limit of \((1 + y/N)^N\) is reached as \(N\) reaches infinity—that is, the limit as the growth rate \(y\) is applied continuously, as opposed to discretely, over the time interval.

2. Recent studies indicate that stock returns exhibit leptokurtosis; that is, the values of exponential growth rate \(x\) both near the mean and highly divergent from the mean appear more likely than predicted by a normal distribution for \(x\) (Turner and Weigel 1992; Jackwerth and Rubinstein 1996). Such leptokurtosis is consistent with a nonconstant volatility for stock returns. In developing the arguments in this article, I make the assumption that the standard deviation (volatility) of stock returns is fixed in time. Such an assumption is, therefore, consistent with the assumption of zero leptokurtosis (see, for example, Chriss 1997).

3. I define the natural logarithm of \(x\), \(\ln(x)\), as that number \(y\) such that \(\exp(y) = x\). Hence, \(y = \ln(x)\) and \(\exp(y) = x\) are equivalent statements and \(\ln[\exp(y)] = y\) for all \(y\). Thus, Equation 3 follows from Equation 2.

4. In these cases, \(r\) is the \(\ln[\text{Expected value of } \exp(y)]\), where \(y\) is normally distributed with mean \(\mu\) and standard deviation \(\sigma\), which allows the general relationship between the logarithm of an expectation and the expectation of a logarithm, where \(z\) is a random variable: \(\ln[\text{Expected value of } \exp(z)] = \text{Expected value of } (\ln(z)) + 1/2 \text{ variance of } (\ln(z))\). On substituting \(z = \exp(y)\) in this relationship, the left-hand side is \(r\), the first term on the right-hand side is \(\mu\), and the final term is \(1/2\sigma^2\). Hence, Equation 4.

5. A U.S. dollar subject to exponential growth rates \(x_1\) and \(x_2\) over successive intervals has the outcome \(\exp(x_1) \times \exp(x_2)\). In Figure 1, I use the relationship \(\exp(x_1) \times \exp(x_2) = \exp(x_1 + x_2)\).

6. This particular example was considered by Kritzman (1994) and by Kritzman and Rich (1998), but these authors chose not to draw attention to the example as one of binomial exponential growth (with \(\mu = 0, \sigma = 0.28768\)). The result was that both Olsen and Khaki (1998) and Van Eaton and Conover (1998) interpreted the numbers 1/3 and 1/4 to be merely opportunistic in Kritzman’s (1994) argument that a log-wealth utility function implies nonincreasing utility with extended investment horizon (as I confirm). Also, Bierman (1998) used the example of a $100 investment with equally likely outcomes of $130 and $76.9 (to illustrate investor portfolio allocations) but did not draw attention to the example as one of binomial exponential growth [with \(\mu = 0, \sigma = 0.262364\), so that \(\exp(0.262364) = 1.3\) and \(\exp(-0.262364) = 0.769\)].

7. At this stage, before I reveal the choice of my model investor, readers might like to consider for themselves which of the three portfolios they would choose.

8. For an introduction to utility, see Kritzman (1992).

9. The designation “percent” for utility \(100\) is arbitrary. The effect of using the designation here is that the per period utility from investing a $W amount in a riskless investment with per period exponential growth rate \(r\) is normalized as \(r\%)\) percent. To see this effect, note that if we have an initial investment of $W subject to exponential growth rate \(r\), the outcome at the end of the period is $Wexp(r). Hence, the gain in log-wealth utility is \(\ln(W \exp(r)) - \ln(W) = \ln(W) + \ln[\exp(r)] - \ln(W) = r\). Representing utilities as “percentages” thus allows utilities to be compared directly with the utility derived from a riskless asset.

10. Historical data are from Ibbotson Associates (see Brealey and Myers 1996).

11. To determine the appropriate value of the historical market risk premium, bear in mind that investors choose to allocate their portfolios not only to risky equity markets and risk-free securities but also to municipal and corporate bonds and other types of assets. The risk in holding the assets depends on how they are integrated strategically in the investor’s portfolio: A long-term bond may be held short term as a risky capital exposure to interest rate movements or, alternatively, may be held until maturity as a secure source of income. Or the bonds held may be risky in their own right but their inclusion is regarded as reducing a portfolio’s overall risk by providing a hedge against market risk. Thus, the distinction between “risky equity markets” and “riskless assets” is not clear cut.

12. Because empirical observations are made on a monthly basis, we are justified in identifying the expected periodic return (the \(y\)-axis of Figure 3) with the “exponential growth rate for expected wealth,” \(r\). To understand, observe from Equations 5 and 4 that Expected periodic return = \(\exp(\mu + 1/2\sigma^2) - 1 = \exp(r) - 1\), confirming that the expected periodic return applied continuously is equal to \(r\). Thus, for reasonably small \(r\), the expected periodic return is approximately equal to \(r\). Similarly, we are justified in identifying the volatility of periodic stock returns (the \(x\)-axis of Figure 3) with the volatility, \(\sigma\), of exponential growth rates (consistent with, for example, Hull 1999, p. 241). For the more exact relationships between exponential and periodic-based returns and variances, see De La Grandville (1998).

13. For a portfolio with per period mean exponential growth rate \(\mu\) and standard deviation about such rate \(\sigma\), the compounded mean exponential growth rate and standard deviation about such rate over \(N\) periods are, respectively, \(N\mu\) and \(\sqrt{N}\sigma\) (see Figure 1, Panels B and C). The “exponential growth rate for expected wealth” for the portfolio compounded over \(N\) periods is determined, therefore, from Equation 4, with \(N\mu\) substituted for \(\mu\) and \(\sqrt{N}\sigma\) substituted for \(\sigma\), as: Exponential growth rate for expected wealth = \(N\mu + 1/2(\sqrt{N}\sigma)^2 = N\mu + 1/2\sigma^2 = N\mu + 1/2\sigma^2 = N\mu\).

14. Alternatively, I could have multiplied the standard deviation on the \(x\)-axis by \(\sqrt{N}\) and multiplied the expected periodic return on the \(y\)-axis by \(N\).

15. For example, a portfolio with a sufficiently large number of assets subject individually to normally distributed exponential growth rates about a mean rate of 0 with per period idiosyncratic volatility of \(\sigma = 10\) percent generates the period return \(\exp(1/2\sigma^2) - 1 = \exp(1/2 \times 0.10^2) - 1 = 0.005\), or 0.5 percent. To see the reasonableness of this outcome, consider that the portfolio comprises only two such assets. When one asset grows over a period at a continuously applied rate of 10 percent while the other declines at the rate of 10 percent, the assets in combination grow by a factor of \(\exp(0.10) + \exp(-0.10))/2\) is approximately 1.005, that is, they generate a periodic return of approximately 0.5 percent.

16. For ease of computation here, I have allowed the return generated by idiosyncratic volatility, \(\sigma\) (that is, the return with idiosyncratic volatility minus the return with no idiosyncratic volatility = \(\exp(\mu + 1/2\sigma^2) - \exp(\mu)\), using Equation 5), to be approximated as \(1/2\sigma^2\) [allowing that \(\exp(x) = 1 + x\) for small \(x\)].

17. In my 2001 working paper, I developed the implications of the DOGMA model in the context of idiosyncratic risk and market returns.
References


