Will the true marginal investor please stand up?:
Asset prices with immutable security trading by investors+ 

by

Peter L. Swan∗

School of Banking and Finance, Commerce Faculty, University of New South Wales
Version: February 11, 2002

ABSTRACT

The important and highly influential Amihud and Mendelson (1986) model of asset pricing incorporating immutable security trading by a continuum of investors/traders is unnecessarily opaque because of a flaw in the numerical simulation which I correct. Moreover, the exposition by Kane (1994) and Bodie, Kane and Marcus (2002) also fails to faithfully present the model. Putting aside these presentational issues, I show that the model is flawed because the marginal investor, on whom the analysis rests, cannot be identified. More fundamentally, the model fails to identify the benefits of trading whilst taking account of the costs. This one-sided treatment means that the model as it stands cannot be used to make valid predictions of the impact of transaction costs on asset prices.

Key words: asset pricing, liquidity, security trading, transactions cost, equity premium

JEL Classification: G120, G110, G200

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† I wish to thank the Australian Research Council, Grant Number A00104302, for financial support and Jay Muthuswamy for drawing to my attention the relevance of the Kane (1994) and Bodie, Kane and Marcus (2001) contributions. I also wish to thank Yakov Amihud for comments. Contact details: Peter L. Swan, School of Banking and Finance, University of New South Wales, Sydney NSW 2052 Australia. Email: peter.swan@unsw.edu.au.
ABSTRACT
The important and highly influential Amihud and Mendelson (1986) model of asset pricing incorporating immutable security trading by a continuum of investors/traders is unnecessarily opaque because of a flaw in the numerical simulation which I correct. Moreover, the exposition by Kane (1994) and Bodie, Kane and Marcus (2002) also fails to faithfully present the model. Putting aside these presentational issues, I show that the model is flawed because the marginal investor, on whom the analysis rests, cannot be identified. More fundamentally, the model fails to identify the benefits of trading whilst taking account of the costs. This one-sided treatment means that the model as it stands cannot be used to make valid predictions of the impact of transaction costs on asset prices.

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The pathbreaking study by Amihud and Mendelson (1986) of the impact of transaction costs on asset prices is not only famous in its own right but has motivated a large body of empirical work. It is justly renowned as the first attempt to derive asset prices from a specified set of preferences by investors for trading. Alternative approaches have tried to find a motivation for trading based on portfolio rebalancing or life-cycle consideration, for example, Constantinides (1986) and Vayanos (1998). However, despite the attempt by Kane (1992) and numerous editions of Bodie, Kane and Marcus (2002) to provide a simple closed-form solution, the characterization of the work by Kane (1994) as “quite opaque” is as true today as it was at the time Kane was contributing. One of the aims of this paper is to show that one of the reasons for this opaqueness is an incorrect characterization by Amihud and Mendelson of their own numerical solution that is critical to an understanding of what they have done. Their solution purports to include all possible values of the required return on the perfectly liquid asset with no transaction costs. In fact it only pertains to a liquid asset with a dividend stream of $1 per period and an asset price of $1. That is, to a peculiar asset with a required return of 100% per period. I hasten to add that this problem with their numerical simulation does not, of itself, affect their basic conclusions.

Kane’s contribution along with Bodie, Kane and Marcus was to place the Amihud and Mendelson solution explicitly within the context of a continuum of investor types, each with perfectly inelastic trading demands, so as to provide a closed form solution. Unfortunately, the curvilinear relationship they derive I show to be inconsistent with the solution of Amihud and Mendelson that is strictly linear. Kane (1994) failed to incorporate the optimizing solution of Amihud and Mendelson in his reformulation of the problem.

This linear solution is strictly defined by the trading characteristics of the marginal investor. If, as is the case, the identity of this marginal investor is unknown, the asset price of each partially illiquid asset is also unknown. Thus, not only is there no closed-form solution, there is no solution to the problem analyzed by Amihud and

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1 The Social Sciences Citation Index indicates that it has been cited 176 times up until September 2001. Some of the better know empirical contributions are Eleswarapu and Reinganum (1993), Brennan and Subrahmanyam (1996), Barclay, Kandel and Marx (1998), Chalmers and Kadlec (1998) and Datar, Naik and Radcliffe (1998).
Mendelson, Kane and Bodie, Kane and Marcus. Alternatively, and equivalently, there is a solution for each of the infinite number of potential marginal investors. However, the identity of the marginal investor remains unknown. As an illustration, I make one of the four investors in the Amihud and Mendelson simulation the marginal investor across all ten asset types. Now only the zero-transaction-cost asset and highest-transaction-cost asset are held in equilibrium by other than the marginal investor. The relationship between transaction costs and returns is no longer strictly concave as in the original simulation but, being linear, it remains concave.

What can be done about this unsatisfactory state of affairs? The existence of an infinite number of potential solutions but no actual solution is intimately related to the assumption by Amihud and Mendelson that trading demand by an investor-type is immutable. It is mandated by some unspecified perfectly inelastic preference for trading that makes trading costs irrelevant. Virtually every empirical study ever undertaken shows that security turnover is responsive to transaction costs. It would thus appear that an extension is required to, (a), make the model more realistic and (b), amenable to a unique solution. However, when I introduce well-behaved downward-sloping investor trading demand functions into the Amihud and Mendelson model all sorts of perverse outcomes appear including asset prices which increase as trading costs rise.

This because there is a more fundamental problem with the Amihud and Mendelson approach. It is asymmetric in terms of its treatment of the costs and benefits of trading. Costs are incorporated but there is no consideration given to offsetting benefits of trading. This deficiency is hidden by the assumption that the desire for trading is so powerful that it is immutable, even in the face of trading costs that may approach infinity. In fact, a condition is missing from the Amihud and Mendelson approach: a requirement that the marginal investor trade until the marginal benefits of trade decline to equal the marginal cost of a trade. This means that the model cannot be generalized by simply adding in investors with elastic trading propensities.

Section 1 derives the Amihud and Mendelson model by incorporating the insights of Kane (1994). Section 2 considers the solution provided by Kane (1994) and by numerous editions of Bodie, Kane and Marcus (2002). Section 3 addresses the identity of the marginal investor. Section 4 relaxes the assumption of immutable trading propensities while some concluding remarks are contained in 5.
1. The model of Amihud and Mendelson

Following Kane (1994) and Bodie, Kane and Markus (2002) there is a continuum of investor types distributed uniformly over a range with an investment horizon ranging from an upper bound of $\bar{h}$ to a lower bound of $\underline{h}$. The corresponding immutable turnover or trading rates for this distribution of investors ranges from $\underline{\mu}$ to $\bar{\mu}$, where the turnover rate $\mu \equiv 1/h$. Thus, $\mu \in (\underline{\mu}, \bar{\mu})$. To keep it as elementary as possible, there are but two assets, one a bond or T-bill with zero transaction costs and rate of return $\rho$ and another, equity, with positive transaction costs $Sa \equiv pece$, given by the product of the asset ask price $p_e$ and the proportional transaction costs or bid-ask spread, $ce$. Amihud and Mendelson (1986) derive an expression for asset ask-price. The arrival of investors follows a Poisson process with interarrival times and holding periods being stochastically independent. The expected ask-price, $p_e$, for equity securities equals the expected discounted value of dividends at the constant dividend rate $d = $1 over the random exponentially-distributed horizon, $h \equiv 1/\mu$, plus the expected net receipts from disposal of the asset at the bid-price, $(1 - c_e)p_e$, once the horizon is reached:

$$p_e = E_h \left[ \int_{s=0}^{h} e^{-rs} d ds \right] + E_h \left[ e^{-rh} p_e (1 - c_e) \right] = [\mu + \rho]^{-1} [d + \mu p_e (1 - c_e)].$$

(1)

Solving (1) for the ask price results in an ask price for equity of

$$p_e = d / (\rho + \mu c_e),$$

(2)

where the turnover rate $\mu$ is yet to be specified by an appropriate investor class, and for zero transaction cost bonds with the same dividend rate $d$ the perpetuity is valued at

$$p_b = d / \rho.$$  

(3)

The marginal investor with a horizon of $h_m$ and a turnover rate of $\mu^m$ who is indifferent between one equity share or one bond will value the equity at precisely

$$p_e = d / (\rho + \mu^m c_e),$$

(4)
on substituting $\mu^m$ for $\mu$ in (2). If he devotes his entire wealth $w$ to one equity security at a cost of $p_e$ and turns it over at the rate, $\mu^m$, at a present value cost of $\mu^m c_e p_e / \rho$ the outlay of wealth in its entirety is constituted by

$$w = p_e \left(1 + \mu^m c_e \right)/\rho = d/\rho = p_b,$$

(5)
on substituting for $p_e$ using (4) and for $p_b$ using (3). Hence for the marginal investor the perpetuity cost of the equity and bond purchase are the same. It is necessary to establish this as a requirement for the marginal investor to be capable of purchasing and trading either the equity or bond security.

The gross return or yield on the equilibrium value of equity is

$$\text{Gross return} \equiv d/p_e = \rho + \mu^m c_e,$$

(6)found by rearranging (4). It depends only on the preferences of the marginal investor with respect to trading equity as well as on the general discount rate and equity transaction cost. The net yield on the equity security takes into account the trading costs per unit of the outlay. The net yield of intra-marginal investors whose preferences dictate less trade than the marginal investor, $\mu \leq \mu^m$, depends on individual trading preferences and is given by an amount in excess of the bond yield:

$$\text{Net yield} \equiv d/p_e - \mu c_e = \rho + c_e \left(\mu^m - \mu\right); \mu < \mu^m,$$

(7)and simply the bond yield, $d/p_e - \mu c_e = \rho; \mu = \mu^m$, for the marginal investor. Note for future reference that the net yield equation (7) is linear in trading preferences given by the investor class, $\mu$.

The marginal investor’s portfolio is either all equity or all bonds with equity selling at a discount relative to bonds. However, the wealth devoted to equity and bonds is the same and the net return after transaction costs of $\rho$ is also the same. Intra-marginal investors in equity lie to the left of the marginal investor and earn a premium net of transaction costs shown by the downward sloping linear net yield line shown in Figure 1 with a slope of $c_e$. Since the premium becomes negative for investors with a greater propensity to trade beyond that of the marginal investor, only bonds free of transaction costs are held to the right of this point. If the marginal investor lies further to the right, indicating a greater propensity to trade, the net yield line will shift
vertically to the right but will maintain the same slope with intra-marginal investors receiving a greater net return. However, the observed equity premium or gross return over bonds is the product of equity transaction costs and turnover rate of the marginal investor given by the amortized spread, \( c_e \mu^m \).

Table 1 shows the solution values for the numerical example provided by Amihud and Mendelson (1986, p.229). It includes the set of prices for each security relative to the bond with zero transaction costs assuming a bond yield \( \rho = 0.05 \). I have computed this series which is not in the original Table. Transaction costs \( c \) range from 0 to 0.045 for the 10 different asset types. The shaded rectangles represent the highest net yield or return for each of the four investor types with investment horizons ranging from 5 years for Investor #1 to one month for Investor #4 and with corresponding mandated and immutable turnover propensities ranging from 0.2 to 12. Within the shaded regions each investor-type is indifferent between the respective asset-types. Hence for these assets the investor is the marginal investor indifferent between adjacent assets with different transaction costs and prices. Every column except column 3 is taken directly from Table 1 of Amihud and Mendelson (1986, p.229). Thus in this solution each of the four investor-types is a marginal investor for at least two assets. The Amihud and Mendelson solution is depicted graphically in Figure 2 for the investors with the highest trading propensity and in Figure 3 for all investors and the 10 asset types. Note that the rays for each of the ten asset types, depicting the net rate of return according to the investor-type located along the horizontal axis, are linear and radiate out from the point of indifference for each marginal investor. The negative return segments will clearly never be chosen by any investor-type.

Amihud and Mendelson appear to obtain one set of market clearing prices for the 10 assets irrespective of the bond discount rate \( \rho \) so long as all prices for assets with positive transaction costs are expressed relative to the price of the asset with zero transaction costs. It is apparent from the ratio of the equity price to the bond price (eqs. (2) and (3) above) that, contrary to the statements of Amihud and Mendelson, the ratio is dependent on the bond discount rate, \( \rho \). Moreover, for a 5% bond return, that is \( \rho = 0.05 \), the relative prices are as shown in column 3, not as given by Amihud and Mendelson in column 4. I find that the price of the bond with zero transaction costs is $20, the next asset $9.09 and the last asset $4.83 with relativities for the 2\textsuperscript{nd}
and last of 0.4545 and 0.157 respectively. The Amihud and Mendelson relativities specified at 0.943 and 0.864, respectively, are much higher and in fact implicitly assume a value for $\rho = 1$ or a required return of 100% per period for transaction cost free bonds.

To see this we need to examine how Amihud and Mendelson derive their Table. The relationship between the gross yield and net yield, as derived by Amihud and Mendelson, is given by (6). They use the formula

$$ \frac{d}{p_e} = 1/p_e = 1 + \mu e $$

where the dividend $d = $1 and the asset price is defined relative to the price of the asset with zero transaction costs. Hence an asset price of $1 implies a net return of zero corresponding to the zero transaction cost, an asset price of 0.943 implies a net return of 0.06, and so on. Since (8) differs from (6) effectively only by the substitution of 1 for the unknown bond return, $\rho$, their relationship is only true if the required return on transaction cost free bonds is $\rho = 1$ or 100% per period with a bond paying $1 in perpetuity with no transaction costs worth only $1. This explains why an asset with a huge return of 0.157 in excess of the return on transaction-cost-free bonds sells for a relatively high discounted price of 0.846 rather that the more realistic value of 0.24 with $\rho = 0.05$. This slip in the main simulation table of Amihud and Mendelson has doubtless contributed to the opaqueness which is the main subject of Kane’s (1994) comment. Fortunately, it does not affect the validity or otherwise of their findings. More substantive issues are addressed below.
2. The model of Kane and Bodie, Kane and Marcus

Kane (1994) and Bodie, Kane and Marcus (2002, 5th ed., pp.279-284, and a number of earlier editions) set out to build a model that is a less complex and more intuitive version of Amihud and Mendelson (1986). It is indicated that they have discovered the closed form solution to the Amihud and Mendelson problem. They think it is implausible that assets with different transaction costs would yield lower and lower net of transaction cost returns as the investment horizon shortens and the propensity to trade increases. This is because investors would prefer to invest in the zero transaction cost asset. The framework and notation is identical to that described above with a continuum of investor types with increasing propensities to trade. They suppose that the gross return on equity and bonds/T-bills is higher by unknown constant amounts, $x_e$, and $x_b$, that are proportional to the transaction costs on each class of asset, $c_e$ and $c_b$. Thus on our equity asset with transaction cost $c_e$ the gross return for investor class $\mu$ is

\[
\text{Gross return} = \frac{d}{p_e} = \rho + x_e c_e, \tag{9}
\]

The proportionate amount must be less than unity, $x_e < 1$, according to the authors (Kane, 1994, and Bodie, Kane and Marcus, Fourth ed., 1999, p. 269).\(^2\) This is because diversified stock portfolios would dominate the asset with zero transaction costs in terms of net returns.

By contrast, Amihud and Mendelson (1986) specify the gross return as

\[
\text{Gross return} = \frac{d}{p_e} = \rho + \mu^m c_e, 
\]

from (6) above. Since the assumptions of the two sets of authors is for all practical purposes identical, equation (9) can only be logically correct if and only if the unknown proportionate amount in (9) is identically equal to the trading propensity of the marginal investor, $x_e = \mu^m$. Since this marginal trading propensity can take on any value including numbers greater than unity, it is true in this framework that for certain investors with low trading propensities the net return on positive transaction cost

\(^2\) This inappropriate inequality restriction is maintained in Kane (1994) and Bodie, Kane and Marcus up to the Forth edition, but is relaxed without comment in the latest (Fifth) edition, 2002.
assets can dominate the zero transaction cost asset. Such an outcome is not surprising in what is after all a model of specialization, and hence should not be ruled out a priori. The net return on taking into account transaction costs for an investor with trading propensity $\mu$ now becomes

\[
\text{Net return} = d/p_e - \mu c_e = \rho + c_e(x_e - \mu),
\]

which should be equal to the Amihud and Mendelson expression (7) given the identity $x_e \equiv \mu^m$.

For the investor indifferent between the equity asset and bonds or T-bills with zero transaction costs the authors correctly identify that the net return in (10) must equal the bond yield $\rho$ and thus $x_e \equiv \mu^m$, even though this may violate the restriction imposed by the authors that $x_e < 1$ (that is, until the most recent 2001 edition). However, things go awry when the authors equate using (10) the net return for the equity asset and a bond asset with a low but positive transaction cost $c_b$:

\[
\rho + c_e(x_e - \mu^m_b) = \rho + c_b(x_b - \mu^m_b),
\]

where $\mu^m_b$ is the trading propensity of an investor indifferent between the higher and lower transaction cost asset. The authors then solve for the apparent unknown $x_e$ in terms of the second unknown $x_b$ as:

\[
x_e = \mu^m_b + \frac{c_e}{c_b}(x_b - \mu^m_b),
\]

which is then substituted into the expression for the gross return on the equity asset given by their equation (Bodie, Kane and Marcus, 2002, 5th ed., eq. (9.10), p. 283).

Of course, since we know that $x_e \equiv x_b \equiv \mu^m_b$, (11) is satisfied as an identity it is unnecessary to solve for $x_e$ in terms of $x_b$ and other variables. The Amihud and Mendelson difference on gross returns comes directly from (6) above for equity and its counterpart for bonds with transaction costs yielding:

\[
d/p_e - d/p_b = \mu^m_b(c_e - c_b),
\]

which is the difference in transaction cost times the propensity to trade of the marginal investor. Given that this reflects the period over which the transactions are
amortized, the return difference can be thought as the difference in amortized transaction costs.

Figure 4a depicts a graphical representation of the ‘closed-form’ solution of Kane (1994, p. 1181) and Bodie, Kane and Marcus (2002, p. 282) for the net yield transformed such that turnover rather trader horizon is displayed on the horizontal axis. Note that the simulated relationship is curvilinear rather than the strictly linear relationship implied by the Amihud and Mendelson (1986) solution. The traders endowed with the highest demand to trade are indifferent between bonds/T-bills and the Liquid asset at $\mu_{pl}^m$, while investors endowed with a lesser propensity to trade are indifferent at $\mu_{Lj}^m$. Investors with a greater propensity to trade than $\mu_{pl}^m$ trade only T-bills. Intermediate ones trade only the Liquid asset between $\mu_{Lj}^m$ and $\mu_{pl}^m$, whilst those with the least propensity to trade, only invest in and trade the Illiquid asset.

In Figure 4b, I show the correct linear relationship of Amihud and Mendelson (1986) consistent with the simple linear relationships shown in Figures 1, 2 and 3, rather than the curvilinear one of Kane and Bodie, Kane and Marcus. Everything else remains the same. The inability of Kane and Bodie, Kane and Marcus to solve for the two unknowns, $x_e$ and $x_b$, as well as the incorrect characterization of the solution as curvilinear rather than linear must be of concern. However, as will become apparent, they also fail to recognize that their proposed solution also involves a third unknown, $\mu_{eb}^m$. 
3. Who is the marginal investor?

So far it has been taken for granted that the marginal investor exists and can be identified out of the continuum of investor types with different trading propensities along the interval, $\mu$ to $\overline{\mu}$. If the marginal investor cannot be identified then the analysis is rather like the Cheshire cat in Lewis Carroll’s, *Alice in Wonderland*, which disappears on closer examination. The simplest case would be one in which there exists a fixed supply of (say) the asset with zero and positive trading costs respectively. If these assets were to be discreet, such as different types of widgets or products such as motor vehicles, with a discreet number of investors, then supply restrictions could be used to identify the marginal investor. Claims such as financial assets and securities are typically not limited in supply in this way and are generally divisible. Hence it would appear that supply restrictions cannot be used in general to identify the marginal investor.

In Amuihud and Mendelson’s (1986) numerical example the marginal trader is identified by a restriction on the analysis to just four investors, each of whom is assumed to be a marginal investor for two or more asset classes. The zero transaction cost asset is in unlimited supply while all other assets are restricted to just one unit. However, there is no need to restrict the analysis in this way.

Table 2 provides an equally valid set of results to those shown in Table 1. This indicates that the results using this model are not unique, and in fact there are an infinite number of potential solutions with a continuum of investor types. Once again the bond yield for zero transaction cost asset is $\rho = 0.05$. Transaction costs $c$ range from 0 to 0.045 for the 10 different asset types. The investment horizons range from 5 years for Investor #1 to one month for Investor #4 and with corresponding mandated turnover propensities ranging from 0.2 to 12. There is by assumption a single shaded region for investor of Type #3 with a turnover rate of 2 (see column 6). This investor-type is indifferent between all the respective asset-types. Equally, investor types #1, #2 or #4 could have become the marginal investor. It can be seen that the results are quite different from those of Amihud and Mendelson in Table 1 above.

All investors along the continuum from $\mu = 12$ to $\mu < 2$ hold and trade only the zero transaction cost asset. Investors with turnover rates $\mu < 2$ hold and trade only the highest transaction cost asset since it yields the highest return premium in excess of
the bond return. The market return (column 2) now increases linearly with transaction costs so that the strictly concave diminishing marginal returns no longer pertains. The bond security with zero transaction costs yields a return of $\rho = 0.05$. The prices of all assets normalized relative to this asset are shown in column 3. Relative asset prices decline uniformly with higher transaction costs with the rate of decline far more gradual than in Table 1.

The solution provided in Table 2 is depicted graphically in Figure 5. There are nine equity securities with positive transaction costs and the most costly to trade with a transaction cost $c_e=0.045$. There is also a bond with zero transaction costs. The marginal investor is assumed to have a horizon of 0.5 and a turnover rate of 2. He is assumed to be indifferent between all ten asset classes. His portfolio is either all equity or all bonds. The net return after transaction costs of $\rho = 0.05$ is also the same.

Intra-marginal equity investors in equity lie to the left of the marginal investor, $\mu^e=2$, and earn a premium net of transaction costs shown by the downward sloping net yield line with a slope of $c_e = 0.045$. These investors hold and trade only the highest transaction cost asset that yields the highest return net of transaction costs. No investor with the possible exception of the marginal investor holds or trades equity assets with transaction costs in the range, $> 0$ or $< 0.045$. Since the premium becomes negative for investors with a greater propensity to trade beyond that of the marginal investor, only bonds free of transaction costs are held to the right of this point. If the marginal investor lies further to the right indicating a greater propensity to trade the net yield line will shift vertically to the right but will maintain the same slope with intra-marginal investors receiving a greater net return. However, the observed equity premium or gross return over bonds is the product of equity transaction costs and turnover rate of the marginal investor given by the amortized spread, $c_e\mu^e = 0.045*2$.

The relationship between the returns on the ten assets with transaction costs ranging from zero to 0.045 is exactly linear as is therefore concave but not strictly so. The strictly concave result obtained by Amihud and Mendelson is a product of their assumption of a succession of marginal investors. However, as succession of marginal investors allocated in the opposite way to Amihud and Mendelson such that investors with higher propensities to trade prefer higher transaction cost securities is not
possible. Hence a strictly convex relationship can be ruled out. This finding is supportive of Amihud and Mendelson. There are an infinite number of such concave or quasi-concave solutions depending on the assumptions made about the marginal investor.

The key equation describing the Amihud and Mendelson equilibrium is the gross rate of return relationship, eq. (6). The asset price of the equity security is known once the return on bonds is known, together with the dividend rate, the cost of transacting on the equity asset and, most importantly, the turnover rate for the marginal investor type. Equivalently, knowing the asset price and gross yield, together with bond yield, dividend and transaction cost, the marginal investor trading propensity and investor-type can be identified. However, it its present form the model is not capable of yielding either the asset price or the marginal investor. Hence it cannot be used for carrying out comparative-static or similar analyses. Nor can it be used to compute the equity premium in terms of the amortized spread of the marginal investor or to describe how the premium varies as transaction costs increase. Not only is there no ‘closed form’ solution, there is no solution at all.

4. **Dispensing with immutable trading propensities**

A possible reason for this inability to solve the model for a unique marginal investor and asset price is the stark and counter-intuitive assumption of a fixed propensity to trade irrespective of the cost of trading. An investor’s propensity to trade must surely depend on the cost of trading such that as trading costs fall as a proportion of the security price, the propensity to trade intensifies. A number of authors have followed this path to generalize Amihud and Mendelson including Hubbard (1993).

A simple linear (in logarithms) specification that has gained considerable empirical support is the simple constant elasticity of turnover demand formulation:

$$\mu = \alpha c^{-\beta}, \quad \mu \in \left[0, \bar{\mu}\right], \quad \alpha \in \left[\underline{\alpha}, \bar{\alpha}\right], \quad \text{and} \quad \beta \geq 0,$$

where turnover demand depends on a parameter, $\alpha$, and a constant demand elasticity, $\beta$, for each investor type along the continuum $\mu \in \left[0, \bar{\mu}\right]$. The demand elasticity remains the same, but the intrinsic liquidity parameter, $\alpha$, varies along the continuum so that for any given transaction cost parameter, $c$, only the intrinsic liquidity parameter will vary to alter turnover. The Amihud and Mendelson and Bodie, Kane
and Marcus analyses are now special limiting cases as $\beta \to 0$ and demand becomes perfectly inelastic.

Substituting for the marginal equity investor in (14), we have

$$\mu^m = \alpha^m c_e^{-\beta},$$

and incorporating (15) into (6) we have last a unique closed-form solution to the Amihud and Mendelson (1986) and Kane (1994) problem in terms of transaction costs for the equity investor:

$$\beta^\alpha r = +\frac{1}{\text{yield}}G\text{ Gross}e\text{ m}e\text{ cpd}.$$  (16)

This result preserves the basic result of Amihud and Mendelson (1986) in that the equity premium given by $\mu^m c_e \equiv \alpha^m c_e^{-\beta}$ from (16) is still the amortized spread of the marginal investor.

It is also possible to carry out comparative-statics so long as we treat the intrinsic liquidity parameter, $\alpha^m$, of the marginal investor as approximately constant or replace it by some average propensity in empirical work. Thus the gross equity asset yield, $d/p_e$, is increasing in the bond yield parameter, $\rho$, the intrinsic liquidity parameter, $\alpha^m$, and at the rate, $1-\beta$, in transaction costs, $c_e$, on differentiating (16) with respect to the parameters.

These last results with respect to transaction costs seem sensible at first blush so long as the demand elasticity is bounded from below by zero and from above by unity: $0 \leq \beta < 1$. Thus higher transaction costs result in higher yields. However, for the case in which demand becomes elastic, $\beta \to 1$ or $\beta > 1$, either there is no effect or the yield falls and the asset price increases with further increases in transaction costs. This is because the principle underlying the model is that amortized transaction costs act as a discount to the asset price. The higher the amortized spread, $\mu^m c_e = c_e^{1-\beta}$, the lower the asset price. The asset price reflects the present value of an infinite sequence of asset trades, with expected transaction costs being incurred on a continuous basis.

This modified Amihud and Mendelson model solution is illustrated in Table 3 and Figure 6. Simulations using eq. (16) are made of the turnover rate $\mu$, gross yield and asset price for two different values of the intrinsic liquidity parameter $\alpha$ and trading.
responsiveness parameter $\beta$, with the dividend set at $1$ and the return on assets with zero transaction costs, $\rho = 0.05$. The simulated outcomes with $\alpha = 0.2; \beta = 0.5$ at least appear sensible with the yield increasing in transaction costs while the inverse, the asset price, diminishes in transaction costs. The required yield increases in transaction costs and the asset price declines so long as the elasticity $\beta < 1$. The simulated outcomes with $\alpha = 0.002; \beta = 1.5$ are peculiar and violate the law that consumption opportunities are finite with the yield reducing in transaction costs while the inverse, the asset price, increasing in transaction costs. The required yield reduces in transaction costs and the asset price increases so long as the elasticity $\beta > 1$. These simulation results indicate serious flaws in the model.

With a demand elasticity greater than unity, this amortized spread falls as the transaction costs are raised, resulting in a lower present value of transaction costs and hence a higher asset price. Surprisingly, investors become wealthier as transaction costs increase and in the limit as transaction costs approach infinity and trading ceases the amortized spread approaches zero. Thus the price of an asset with infinite trading costs converges to the price of a bond with zero transaction costs as trading costs approach infinity, so long as $\beta > 1$. Such an outcome clearly cannot exist. In generalizing the model to enable an internal closed-form solution to be obtained and in the process making it considerably more realistic, a more fundamental flaw has been exposed.

In fact the Amihud and Mendelson model does not generalize for any value of the transaction cost elasticity $\beta$, not just the values of $\beta > 1$ which are obviously seriously flawed. This is because the model does not set out to express the benefits from trading along with the costs of trading. By assuming that trading propensities are immutable, and hence the desire to trade up to a specified amount essentially infinite, the opportunity to obtain more powerful results has been missed.
5. Concluding remarks

While there are a number of flaws in the exposition of the Amihud and Mendelson model, both in the original article and in the attempt by Kane (1994) and Bodie, Kane and Marcus (2002) to improve on the original exposition, two insurmountable problems remain even when expositional problems are overcome:

(a), the marginal investor on whom the analysis is based cannot be identified and
(b), explicit attention is paid to the cost of trading but no consideration what so ever is paid to the benefits of trading once it ceases to be immutable.

This means that we cannot rely on the main conclusions from the Amihud and Mendelson model. In general the equity premium for assets is not given by the difference between the amortized spread of the equity security relative to the bond or T-bill security for the marginal investor, even if the investor could be identified. Moreover, there is no valid proof that the relationship between transaction costs such as the bid-ask spread and the equity premium is concave. However, it cannot be strictly convex under the Amihud and Mendelson assumptions. In a companion piece, Swan (2002), I show how the impact of transaction costs on asset prices can be computed, without the serious problems encountered by Amihud and Mendelson model, by explicitly modeling the benefits of trading as well as the costs. This paper, together with Swan and Westerholm (2002), contains a number of empirical tests of the endogenous trading model incorporating the benefits of trading.
References


**Figure 1:** The equilibrium solution to the Amihud and Mendelson and Brodie, Kane and Marcus problem with a single equity security and a bond with zero transaction costs. The marginal investor’s portfolio is either all equity or all bonds with the market value of the equity portfolio less than that of the bond portfolio. However, the wealth required to buy and maintain each portfolio is the same. The net return after transaction costs of \( \rho \) is also the same. Intra-marginal investors in equity lie to the left of the marginal investor and earn a premium net of transaction costs shown by the downward sloping net yield line with a slope of \( c_e \). Since the premium becomes negative for investors with a greater propensity to trade beyond that of the marginal investor, only bonds free of transaction costs are held to the right of this point. If the marginal investor lies further to the right indicating a greater propensity to trade the net yield line will shift vertically to the right but will maintain the same slope with intra-marginal investors receiving a greater net return. However, the observed equity premium or gross return over bonds is the product of equity transaction costs and turnover rate of the marginal investor given by the amortized spread, \( c_e \mu^m \).
Table 1: The derivation of the numerical solution provided by Amihud and Mendelson (1986, p.229) including the corrected set of prices for each security relative to the bond with zero transaction costs assuming a bond yield $\rho = 0.05$. Transaction costs $c$ range from 0 to 0.045 for the 10 different asset types. The shaded rectangles represent the highest net yield or return for each of the four investor types with investment horizons ranging from 5 years for Investor #1 to one month for Investor #4 and with corresponding mandated turnover propensities ranging from 0.2 to 12. Within the shaded regions each investor-type is indifferent between the respective asset-types so that for these assets the investor is the marginal investor indifferent between adjacent assets with different transaction costs and prices. Every column except column 3 is taken directly from Table 1 of Amihud and Mendelson (1986, p.229). Hence in this solution each of the four investor-types is a marginal investor for at least two assets. Amihud and Mendelson appear to obtain one set of market clearing prices for the 10 assets irrespective of the bond discount rate $\rho$ so long as all prices for assets with positive transaction costs are expressed relative to the price of the asset with zero transaction costs. It is apparent from the ratio of the equity price to the bond price (equations 2 and 3 in the text) that the ratio is dependent on $\rho$. Moreover, for a $\rho = 0.05$ the relative prices are as shown in column 3, not as given by Amihud and Mendelson in column 4. I find that the price of the bond with zero transaction costs is $20, the next asset $9.09 and the last asset $4.83 with relativities for the 2\textsuperscript{nd} and last of 0.4545 and 0.157 respectively. The Amihud and Mendelson relativities specified at 0.943 and 0.864, respectively, are much higher and in fact implicitly assume a value for $\rho = 1$ or a return of 100% per period.
Figure 2: The Amihud and Mendelson numerical solution to the asset pricing problem for investors with trading/turnover demands ranging from 0.1 to 12 for three asset classes with transaction costs ranging from 0 to 0.01
Figure 3: Net rates of return for the 10 asset classes and four investor-types in the Amihud and Mendelson numerical solution. Where the lines cross indicates an investor who is indifferent between the intersecting net returns.
Figure 4a: A graphical representation of the ‘closed form’ solution of Kane (1994, p. 1181) and Bodie, Kane and Marcus (2002, p. 282) for the net yield is transformed such that turnover rather trader horizon is displayed on the horizontal axis. Note that the simulated relationship is curvilinear rather than the strictly linear relationship implied by the Amihud and Mendelson (1986) solution. The traders endowed with the highest demand to trade are indifferent between bonds/T-bills and the Liquid asset at $\mu_{pl}^m$ while investors endowed with a lesser propensity to trade are indifferent at $\mu_{LI}^m$. Investors with a greater propensity to trade than $\mu_{pl}^m$ trade only T-bills. Intermediate ones trade only the Liquid asset between $\mu_{LI}^m$ and $\mu_{pl}^m$ whilst those with the least propensity to trade, trade only the Illiquid asset.
**Figure 4b:** A graphical representation of the ‘closed form’ solution of Kane (1994, p. 1181) and Bodie, Kane and Marcus (2002, p. 282) for the net yield transformed such that turnover rather trader horizon is displayed on the horizontal axis. I show the correct linear relationship of Amihud and Mendelson (1986) rather than the curvilinear one of Kane and Bodie, Kane and Marcus. The traders endowed with the highest demand to trade are indifferent between bonds/T-bills and the Liquid asset at $\mu_{pl}^m$ while investors endowed with a lesser propensity to trade are indifferent at $\mu_{LI}^m$. Investors with a greater propensity to trade than $\mu_{pl}^m$ trade only T-bills. Intermediate ones trade only the Liquid asset between $\mu_{LI}^m$ and $\mu_{pl}^m$ whilst those with the least propensity to trade, only invest in the Illiquid asset.
Table 2: This Table provides an equally valid set of results to those shown in Table 1. This indicates that the results using this model are not unique. The derivation of the numerical solution provided by Amihud and Mendelson (1986, p.229) including the corrected set of prices for each security relative to the bond with zero transaction costs assuming a bond yield $\rho = 0.05$. Transaction costs $c$ range from 0 to 0.045 for the 10 different asset types. The investment horizons range from 5 years for Investor #1 to one month for Investor #4 and with corresponding mandated turnover propensities ranging from 0.2 to 12. There is by assumption a single shaded region for investor of Type #3 with a turnover rate of 2 (see column 6). This investor-type is indifferent between all the respective asset-types. Equally, investor types #1, #2 or #4 could have become the marginal investor. It can be seen that the results are quite different from those of Amihud and Mendelson in Table 1 above. All investors along the continuum from $\mu = 12$ to $\mu < 2$ hold and trade only the zero transaction cost asset. Investors with turnover rates $\mu < 2$ hold and trade only the highest transaction cost asset since it yields the highest return premium in excess of the bond return. The market return (column 2) now increases linearly with transaction costs so that the strictly concave diminishing marginal returns no longer pertains. The bond security with zero transaction costs yields a return of $\rho = 0.05$. The prices of all assets normalized relative to this asset are shown in column 3. Relative asset prices decline uniformly with higher transaction costs with the rate of decline far more gradual than in Table 1.

<table>
<thead>
<tr>
<th>Transaction cost</th>
<th>Market return in excess of bond yield</th>
<th>Price of equity relative to bond value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.005</td>
<td>0.01</td>
<td>0.833333</td>
</tr>
<tr>
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<td>0.02</td>
<td>0.714286</td>
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<tr>
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</tr>
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<td>0.02</td>
<td>0.04</td>
<td>0.555556</td>
</tr>
<tr>
<td>0.025</td>
<td>0.05</td>
<td>0.5</td>
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<td>0.08</td>
<td>0.384615</td>
</tr>
<tr>
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<td>0.09</td>
<td>0.357143</td>
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</table>
Figure 5: This Figure illustrates another apparent equilibrium solution to the Amihud and Mendelson and Brodie, Kane and Marcus problem describing Table 2. There are nine equity securities with positive transaction costs and the most costly to trade with a transaction cost $c_e=0.045$. There is also a bond with zero transaction costs. The marginal investor is assumed to have a horizon of 0.5 and a turnover rate of 2. He is assumed to be indifferent between all ten asset classes. His portfolio is either all equity or all bonds. The net return after transaction costs of $\rho = 0.05$ is also the same.

Intra-marginal equity investors in equity lie to the left of the marginal investor, $\mu^m = 2$, and earn a premium net of transaction costs shown by the downward sloping net yield line with a slope of $c_e = 0.045$. These investors hold and trade only the highest transaction cost asset that yields the highest return net of transaction costs. No investor with the possible exception of the marginal investor holds or trades equity assets with transaction costs in the range, $> 0$ or $< 0.045$. Since the premium becomes negative for investors with a greater propensity to trade beyond that of the marginal investor, only bonds free of transaction costs are held to the right of this point. If the marginal investor lies further to the right indicating a greater propensity to trade the net yield line will shift vertically to the right but will maintain the same slope with intra-marginal investors receiving a greater net return. However, the observed equity premium or gross return over bonds is the product of equity transaction costs and turnover rate of the marginal investor given by the amortized spread, $c_e\mu^m = 0.045 \times 2$. There are an infinite number of such solutions depending on the assumptions made about the marginal investor.
Table 3: Modeling investors whose trading is responsive to transaction costs. Simulation of the turnover rate $\mu$, yield and asset price for two different values of the intrinsic liquidity parameter $\alpha$ and trading responsiveness parameter $\beta$ with the dividend set at $1$ and the return on assets with zero transaction costs, $\rho = 0.05$. The simulated outcomes with $\alpha = 0.2; \beta = 0.5$ at least appear sensible with the yield increasing in transaction costs while the inverse, the asset price, diminishes in transaction costs. The required yield increases in transaction costs and the asset price declines so long as the elasticity $\beta < 1$. The simulated outcomes with $\alpha = 0.002; \beta = 1.5$ are peculiar and violate the law that consumption opportunities are finite with the yield reducing in transaction costs and the inverse, the asset price, increasing in transaction costs. In fact, the asset price approaches zero and the yield infinity as the transaction costs approach zero for the peculiar case in which $\beta > 1$. The required yield reduces in transaction costs and the asset price increases so long as the elasticity $\beta > 1$. These results indicate serious flaws in the model.

<table>
<thead>
<tr>
<th>Transaction Cost, $c$</th>
<th>Turnover Rate $\mu$</th>
<th>Yield</th>
<th>Price</th>
<th>Turnover Rate $\mu$</th>
<th>Yield</th>
<th>Price $\rho_e$</th>
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<tbody>
<tr>
<td>0</td>
<td>Very large</td>
<td>0.0500</td>
<td>20.0000</td>
<td>Very large</td>
<td>5.6569</td>
<td>12.7740</td>
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<tr>
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<td>0.0700</td>
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<tr>
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<td>0.0745</td>
<td>13.4237</td>
<td>1.0871</td>
<td>0.0663</td>
<td>15.9004</td>
</tr>
<tr>
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<td>0.0783</td>
<td>12.7740</td>
<td>0.7071</td>
<td>0.0641</td>
<td>15.9004</td>
</tr>
<tr>
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<td>12.2515</td>
<td>0.5060</td>
<td>0.0626</td>
<td>15.9619</td>
</tr>
<tr>
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<td>0.0846</td>
<td>11.8146</td>
<td>0.3849</td>
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<tr>
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<td>11.1111</td>
<td>0.2500</td>
<td>0.0600</td>
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<tr>
<td>0.045</td>
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<td>0.0924</td>
<td>10.8194</td>
<td>0.2095</td>
<td>0.0594</td>
<td>16.8271</td>
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</table>
Figure 6: Simulated asset yields with transaction cost responsiveness of 0.5 and 1.5 based on Table 3.