

**‘Estimating the Expected Equity Market Risk Premium:
An Analytical Approach’**

Abstract

The article considers the nature of the market risk premium and seeks to estimate its value. To these ends, the market economy is modelled as an organic or exponential growth process in which investors have a log-wealth utility function. It is observed that such investors potentially accept the risk premium that derives from the additional expectation of return that is generated by the riskiness of the process itself. In other words, “risk creates its own reward”. Ultimately, we model the market risk premium as the outcome of the *supply and demand* of assets. The model implies that the market risk premium is actually much less than is indicated by ex-post returns observed on US stock markets, and is perhaps as little as two percent over and above the rate offered by Government bonds. In terms of the model, investors choose to allocate their portfolios long in both the risky market and the riskless asset. Further, their portfolios are *independent* of the investment time horizon.

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1. Introduction.

Recent studies of global equity markets have highlighted the superior performance of the US stock markets. Jorion and Goetzmann (1999) and Siegel (1997) conclude that the US stock market offers a clear case of survival bias. In this view, the US economy and its stock markets have consistently surpassed expectations in their ability to overcome challenges to their advancement: nuclear threat from the Soviet Union, the costs of Vietnam, inflation fears, OPEC strategy to create an economic fulcrum on oil, the economic competition threats from the Asian Tiger economies, and so on. In short, it was not always evident that the US stock market would become dominant.

The present paper advances a model that is consistent with a smaller *ex-ante* market risk premium than has previously been supposed on the basis of US market data. In terms of the model, the market risk premium, over and above that offered by Government bonds, may rationally be as little as two percent per annum. Our proposed model represents the economy - and the market of asset claims on the economy – as an organic or exponential growth process of appreciation. We consider that investors can be characterized as having log-wealth utility functions, and that the market risk premium may be modelled as the outcome of such investors’ desire to maximise utility in the context of the relative *supply and demand* of market and riskless assets. In which case, investors potentially accept the risk premium that derives from the additional expectation of return that is generated by the riskiness of the process itself; in other words, “risk creates its own reward”. The model further implies that log-wealth utility investors make investment choices that are long in both the risky market and the riskless asset, and, consistent with Samuelson’s arguments (1963, 1989, 1994), that such portfolio choices are *indifferent* to the investor’s time-horizon.

The paper is arranged as follows. In the following two sections, we outline the essential framework of the model. First, we introduce the binomial growth model, which is extended to incorporate both risky and riskless asset components. In the section thereafter, we introduce log-wealth utility

investors, and examine their propensity to invest between risky and riskless assets. The following section presents the implications for the market risk premium. The penultimate section compares the model's most important predictions with the foundations of the CAPM, before the final section concludes the paper.

2. An Organic Growth Model of Appreciation

(i) *The organic or exponential growth process*

In this subsection, we introduce the concept of exponential or organic growth. A feature of investment in such growth is that the *uncertainty* – which is to say, the *risk* – that surrounds the wealth outcome *of itself* creates an expectation of growth. This observation is consistent with the observation that the potential wealth outcome of an investment is, on the upside, theoretically unbounded, while on the downside it cannot be less than zero. Exponential or organic growth occurs commonly in nature and in the biological sciences. Such growth occurs from the base of the current value, so that the outcome valuation at the end of a time interval is the starting valuation multiplied by $\exp(x)$ where x - the exponential growth rate of the process – is normally distributed.¹ The assumption that stockmarket growth can be modelled by such a process is justified by the evidence of past stock price performance (for example, Fama, 1976, ch. 2; and more recently, Jones and Wilson, 1999).²

An interesting feature of *normally* distributed exponential growth rates over an interval, is that the distribution can be modelled as the outcome of a simple *binomial* distribution of exponential growth rates imposed over a sufficiently large set of sub-intervals. In the binomial framework, the exponential growth rate, x , over each sub-interval, is restricted to being either $\mu + \sigma$ or $\mu - \sigma$ with equal probability; where μ and σ , respectively, are equal to the mean and standard deviation of the assumed underlying normally distributed exponential growth rates over the sub-interval. This observation is useful because the features of exponential growth can be revealed *more readily* in terms of a binomial framework representation. The binomial process is represented in panel 1 of

Table 1, where the entries are the possible wealth outcomes multiplied by the probability of outcome³ (we justify the substitution of a binomial growth process for the actual normally distributed growth process in Appendix A).

- - - Table 1 about here - - -

An interesting example of binomial exponential growth is with $\mu = 0$ and $\sigma = 28.768\%$, so that $\exp(0 + 0.28768)$ and $\exp(0 - 0.28768)$ equal, respectively, $+ 1/3$ and $- 1/4$. That is to say, the investment outlay has an equal chance of increasing by $1/3$ and decreasing by $1/4$ in each investment period.⁴ Commencing with an outlay investment of \$12, the possible outcomes at the end of the first two periods are presented in Table 2 (left-hand side, panel 2). We note that the “expected” wealth outcome of \$12 invested for a single period is determined as $1/2.(12 \times 1^{1/3}) + 1/2.(12 \times 3/4) = 12.50$ (left-hand side, panel 3, Table 2)⁵; which is to say, the expected percentage increase in wealth is $(12.50 - 12)/12 \times 100 = 4.17\%$. Thus we observe that an exponential growth process with underlying mean exponential growth rate μ equal to zero, but with a dispersion of outcome possibilities around such zero rate (σ greater than zero), generates an expected wealth outcome that exceeds the initial investment.

Generally, investors wish to be able to identify the return that equates with an expected wealth outcome. Here, we identify such return, r , as applied *exponentially*, which is to say, applied continuously as opposed to discretely. Thus we define the “expected return” for an investment, r , with the statement:

$\exp(r)$ = the expected wealth outcome of investing \$1 over a single period

which is to say⁶:

$$r = \ln [\text{expected wealth outcome of investing \$1 over a single period}] \quad (1)$$

where \ln represents the natural log function⁶. For the binomial process, we therefore have r (see panel 1, Table 1) as⁷:

$$r = \ln\{ 1/2 [\exp(\mu + \sigma) + \exp(\mu - \sigma)] \}$$

If the period over which r is measured is sub-divided into a sufficient number of intervals, so that the end-of-period outcomes become normally as opposed to binomially distributed, r is expressed:

$$r = \mu + \frac{1}{2} \sigma^2 \quad (2)$$

as utilized in the continuous time framework of the Black-Scholes model.⁸ For example, with $\mu = 0$ and $\sigma = 0.28768$ (as above), we determine, $r = 4.14\%$.

(ii) Exponential growth with risky and riskless assets

In the following sections, we turn to consider how a rational investor might choose to allocate an investment portfolio between the market of risky assets (with mean exponential rate of return μ_m and standard deviation about such rate, σ_m) and a riskless asset such as Government securities (with exponential rate of return, r_f). To this end, we need to be able to describe the possible wealth outcomes of the portfolio in terms of the chosen proportion (ω) allocated to the market of risky assets. In effect, we wish to be able to represent the underlying growth possibilities for the portfolio as a function of ω with a binomial process similar to that of panel 1 of Table 1. It turns out that the appropriate binomial process for the portfolio comprising the risky market and the riskless asset is one with mean exponential growth rate, μ' and standard deviation σ' , where μ' and σ' are related to μ_m , σ_m , r_f , and ω by:

$$\begin{aligned} \mu' = & \frac{1}{2} \{ \ln[\omega \exp(\mu_m + \sigma_m) + (1-\omega) \exp(r_f)] \\ & + \ln[\omega \exp(\mu_m - \sigma_m) + (1-\omega) \exp(r_f)] \} \end{aligned} \quad (3)$$

$$\begin{aligned} \sigma' = & \frac{1}{2} \{ \ln[\omega \exp(\mu_m + \sigma_m) + (1-\omega) \exp(r_f)] \\ & - \ln[\omega \exp(\mu_m - \sigma_m) + (1-\omega) \exp(r_f)] \} \end{aligned} \quad (4)$$

The usefulness of equations 3 and 4 lies in the fact that for any portfolio combination of μ_m , σ_m , r_f , and ω , we can simply substitute such values into these equations, calculate μ' and σ' , and replace μ and σ in Table 1 with μ' and σ' . We demonstrate this outcome in Appendix B. Both the kind of choices that are at stake, and the applicability of equations 3 and 4 to a delineation of such choices, may be illustrated in terms of the above investment opportunity that with equal likelihoods either increases wealth by $1/3$ or decreases it by $1/4$, considered in conjunction with the riskless asset. We

take the approach of firstly, constructing possible outcomes from first-principles, and secondly, observing how such outcomes are generated in terms of equations 3 and 4. Thus consider an investor who is choosing between the following three investment portfolios:

(a) portfolio “100% risky”: investment wealth 100% in the above risky investment opportunity,

(b) portfolio “50% risky”: 50% of investment wealth in the above risky opportunity and 50% in the riskless asset (with zero interest rate), and

(c) portfolio “100% riskless”: investment wealth 100% in the riskless asset (with zero interest rate).

- - - Table 2 about here - - -

Comparing the three portfolios, we can make the following observations. With initial investment of, say, \$12, the outcome wealth for portfolio “riskless” remains at \$12. The outcome wealth possibilities for portfolios “100% risky” and “50% risky” over the first couple of intervals are represented in panel 2 of Table 2 (for portfolio “100% risky”, left-hand side; portfolio “50% risky”, right-hand side). For example, commencing with an initial investment of \$12, the upper outcome of \$14 in panels 2 at the end of the first step for portfolio “50% risky” is calculated as the outcome of \$6 invested at 0% in the riskless asset, = \$6, in conjunction with \$6 which increases by $\frac{1}{3}$, = \$8, yielding $\$6 + \$8 = \$14$. The entry $\$16^{\frac{1}{3}}$ at the end of the second step is then calculated as the outcome of \$7 invested at 0% in the riskless asset, = \$7, in conjunction with \$7 which increases by $\frac{1}{3}$, = $\$9^{\frac{1}{3}}$, yielding $\$16^{\frac{1}{3}}$. The expected wealth outcome, $E(W_N)$ (that is, the probability-weighted sum of outcomes) at the end of period N is presented in panel 3 of Table 2. The per-period expected return, r , is then calculated as $r = \ln[E(W_N)/E(W_{N-1})]$, yielding constant expected returns, 4.08% and 2.06%, respectively, for portfolios “100% risky” and “50% risky” (panel 4, Table 2). For portfolio “100% riskless”, for comparison, r is trivially zero. In addition, we observe that the wealth outcomes for portfolio “100% risky” remain 'centred' through time on the initial investment outlay, \$12; which is to say, the median outcome remains as \$12 (see end-of-second period, left-hand side panel 2). This follows because sequences of a gain of $\frac{1}{3}$ followed by a loss of $\frac{1}{4}$, and a

loss of $1/4$ followed by a gain of $1/3$, both bring us back to \$12. In the case of portfolio “50% risky”, however, both these sequences have created an ‘upward drift’, meaning that the ‘centred’ or median outcome increases through time, which outcome at the end of the second period is \$12.25 (see end-of-second period, right-hand side panel 2).

In Appendix C (parts i – iii), we demonstrate the applicability of equations 3 and 4 by showing how these equations reproduce the above example in a straight-forward manner.

3. The Log-Wealth Utility Function and Portfolio Selection

A log-wealth utility function is supported as descriptive of investor wealth choices by Markowitz (1991) who considers that such a function captures the mean-variance investment criterion. The log-wealth utility function has the properties of (i) decreasing absolute risk aversion (meaning that equal absolute losses decline in importance as wealth increases) and (ii) constant relative risk aversion (meaning that one is equally averse to proportional losses in wealth) (for an introduction to utility, see Kritzman, 1992). Economists generally agree that investors have decreasing absolute risk aversion. Interestingly, Copeland and Weston (1988) argue that a “decreasing marginal utility of wealth is probably genetically coded because without it we would exhibit extreme impulsive behavior. We would engage in the activity with the highest marginal utility to the exclusion of all other choices. Addictive behavior would be the norm” (p. 88). As the inverse of the exponential function of growth, a natural log-outcome utility has the effect of balancing the instinct to pursue growth opportunities with the need to safeguard current wealth.⁹ We point out, however, that there is less consensus that investors maintain strict constant relative risk aversion at the prospect of an increasingly severe loss at an increasingly low probability: “prospect theory” suggests that investors are typically *less* averse to such down-side losses than is implied by log-wealth utility. Nevertheless, we shall in the remainder of the paper take it as a worthwhile hypothesis that investors can be described by a natural log-wealth utility. We shall return briefly to the assumption of log-wealth utility in the conclusion to the paper.

For a natural log-wealth utility investor, the per-period utility (U) provided by investment in organic or exponential growth with mean per-period exponential growth rate, μ , is equal to μ . That is, we

have:

$$U = \text{mean exponential growth rate} \quad (5)$$

which may be interpreted as *either* the utility gain over a singly-identified period, or the utility gain over N periods divided by N . In effect, equation 5 follows from the observation that the natural log function, ‘ln’ is the *inverse* of the exponential function, ‘exp’ (that is, $\ln[\exp(x)] = x$, for any x), as demonstrated in Appendix D.¹⁰ Thus an investor with a natural log-wealth utility function enjoys per-period utility $U = \mu$ from investment in exponential growth *independent* of the dispersion of possible outcomes at the termination of each stage of the investment process. Only the mean exponential growth rate, μ , is important. We thereby observe that it is the case that the expectation of return generated by the dispersion or volatility of returns is just sufficient to compensate the investor for the risk to wealth outcome implied by the self-same dispersion. In other words, in the expression for an investment’s expected return, r (equation 2), the contribution generated by the dispersion of outcomes ($\frac{1}{2}\sigma^2$) is just sufficient to compensate the investor for bearing such dispersion (σ). In a fundamental sense, risk creates its own reward. Alternatively, in the context of exponential growth, we may observe that a log-wealth utility investor is not “risk-averse”, but “risk-neutral”.

We are now in a position to apply the utility concept to an investor’s selection between portfolios (a): “100% riskless”, (b): “100% risky”, and (c): “50% risky” as considered in the previous section. The log-wealth utility at the end of each period is calculated as the sum of the utilities afforded by each outcome, with each utility multiplied by the probability of the outcome’s occurrence (this is the Von Neumann and Morgenstern theorem expressed as equation A in Appendix D). Thus we have:

(a) Portfolio 100% riskless:

The utility remains as $\ln(12) = 2.4849$.

The per-period utility gain is therefore zero.

(b) Portfolio “100% risky” (left-hand side panel 2, Table 2):

$$\begin{aligned} \text{original wealth utility} &= \ln(12) &&= 2.4849, \\ \text{end-of-period 1 utility} &= 0.5 \ln(16) + 0.5 \ln(9) &&= 2.4849, \\ \text{end-of-period 2 utility} &= 0.25 \ln(21^{1/3}) + 0.5 \ln(12) + 0.25 \ln(6^3/4), &&= 2.4849. \end{aligned}$$

The per-period utility gain is therefore zero.

(c) Portfolio “50% risky” (right-hand side panel 2, Table 2):

$$\begin{aligned} \text{original wealth utility} &= \ln(12) &&= 2.4849, \\ \text{end-of-period 1 utility} &= 0.5 \ln(14) + 0.5 \ln(10.5) &&= 2.4952, \\ \text{end-of-period two utility} &= 0.25 \ln(16^{1/3}) + 0.5 \ln(12^{1/4}) + 0.25 \ln(9^3/16) &&= 2.5055. \end{aligned}$$

The per-period utility gain is therefore:

$$2.5055 - 2.4952 = 2.4952 - 2.4849 = 0.0103$$

or 1.03%.¹¹

Thus we observe that the per-period utility 1.03% offered by the hybrid portfolio “50% risky” actually *exceeds* that offered either by portfolio “100% risky” (per period utility = 0) or by portfolio “100% riskless” (per period utility = 0). That is, a log-wealth utility investor chooses the hybrid portfolio “50% risky”. In Appendix C (part iv), the above outcomes are revealed immediately in terms of equation 5.

We can consider whether an even greater utility might be offered by some other combination of the risky and riskless assets. To this end, we observe that the per-period log-wealth utility (U) offered by a portfolio with proportion ω in the risky asset may be expressed (with equations 3 and 5) as:

$$\begin{aligned} U = \mu' &= \frac{1}{2} \{ \ln[\omega \exp(\mu + \sigma) + (1 - \omega) \exp(r_f)] \\ &+ \ln[\omega \exp(\mu - \sigma) + (1 - \omega) \exp(r_f)] \} \end{aligned} \quad (6)$$

With $\mu = r_f = 0$, the value of U as derived above is maximized with $\omega = 0.5$. We demonstrate this

outcome in Appendix E. Thus we observe that portfolio “50% risky” ($\omega = 0.5$) offers higher utility than any other portfolio ($\omega \neq 0.5$). That is, portfolio “50% risky” remains the preferred choice for log-wealth utility investors. Utilities for a range of values of ω are computed in the first row of Table 3.

- - - Table 3 about here - - -

It is interesting to close this section by applying the utility model of equation 6 to the case of a “depression” economy characterized as one offering only market investment opportunities with a zero mean exponential growth rate ($\mu_m = r_f = 0$). In such case, our above analysis implies that a log-wealth utility individual continues to allocate half of resources in growth ($\omega = 0.5$). Thus even with the economy so reduced, investment persists. With each successive outcome, half of the outcome resources (either enhanced or depleted) are recommitted to the possibility of growth. Further, the *greater* the volatility of outcomes as represented by the market standard deviation, σ_m , the greater is the propensity to invest. That is, the utilities peak more pronounced at $\omega = 0.5$ with increasing σ_m . To see this, we present in Table 3 the utilities for $\mu_m = r_f = 0$ as a function of ω for the case $\sigma_m = 15.0\%$ (annualized) as measured over US stock performances through 1980-1994, and for $\sigma_m = 33.6\%$ (annualized) as measured over the period 1926-1939.

In the following section, we extend our analysis to consider a riskless return greater than zero, and thereby to a consideration of the market risk premium as it is required to induce investors to arrange their investment portfolios between such riskless asset and the risky market in proportion to the actual market availability of these assets.

4. Portfolio Allocation and the Market Premium

We commence with a rational log-wealth utility investor who seeks to optimise utility by balancing his investment portfolio between the market of risky assets (with $\mu = \mu_m$, $\sigma = \sigma_m$) and the riskless asset. Such investor therefore seeks to optimise the utility function U that is the outcome of equations 3 and 5:

$$U = \frac{1}{2} \{ \ln[\omega \exp(\mu_m + \sigma_m) + (1-\omega) \exp(r_f)] + \ln[\omega \exp(\mu_m - \sigma_m) + (1-\omega) \exp(r_f)] \} \quad (7)$$

In the particular case $\mu_m = r_f$, the value of U as derived above is maximized as in the previous section when $\omega = 0.5$ (as demonstrated in Appendix E). In other words, if it were the case that $\mu_m = r_f$, log-wealth investors would choose to allocate their investment wealth *equally* between the market of risky assets and the riskless asset. We present this outcome in panel 1 of Table 4, where with $\mu_m = r_f = 2.0\%$ (annualized, reflecting a real return on Government bonds, data from Ibbotson Associates, see Brealey and Myers, 1996, p. 146), and standard deviation of market returns, $\sigma_m = 15\%$ (annualized, reflecting the performance of US stocks over the period 1980-1994, see Brealey and Myers, 1996, p. 153), we simulate the utility U (equation 7) over the range of ω . We observe that in all cases, the utility of a portfolio that is long in both the market of risky assets and the riskless asset, is greater than the utility offered by investing exclusively in either the risky market or the riskless asset, and, as we have observed, is maximized by investing equally between the risky market and the riskless asset. Further, going short in either the risky market or the riskless asset offers utilities that are markedly less than those offered by investing in any long combination of the risky market and riskless asset, dropping to zero at roughly 50% leverage and becoming negative at higher levels of leverage. Finally, and consistent with Table 1 (panel 4), the per-period utility is determined *independently* of the number of periods over which the simulation is run.

- - - **Table 4 about here** - - -

The supply of the riskless asset, however, is less than the supply of assets constituting the market of risky assets. Panel 1 in Table 4 cannot therefore be considered as representing equilibrium. Equilibrium requires that investors allocate their portfolios between the market of risky assets and the riskless asset as to be consistent with the relative *availability* of these asset classes. We therefore require that investors choose to allocate a greater part of their portfolios in the market of risky assets. To this end, it must be the case that the risky market is priced so that μ_m is *greater than* r_f . Setting μ_m equal to 2.5% (while maintaining $r_f = 2.0\%$), leads to the outcomes in panel 2 of

Table 4. As in panel 1, a portfolio invested 100% in the riskless asset ($\omega = 0$) has utility equal to r_f ($= 2.0\%$) while a portfolio constituted 100% in the market now has utility equal to μ_m ($= 2.5\%$). In this case, the utilities offered in the range $\omega = 55\%$ to 90% remain essentially flat at 2.6% . That is, investors with log-wealth utilities now choose to hold between 10 and 45% of their portfolio in the riskless asset. Thus we observe that the ‘flatness’ of the outcome utilities implies a fair degree of investment flexibility consistent with log-wealth utility. Setting μ_m equal to 3.0% (while maintaining $r_f = 2.0\%$), leads to the outcomes in panel 3 of Table 4. In this case, the utilities offered in the range $\omega = 65\%$ to 100% remain essentially flat at 3.0% , implying that investors with log-wealth utilities choose to hold between zero and 35% of their portfolio in the riskless asset.

Thus we observe that the differential between μ_m and r_f is determined by market supply and demand. The relationship however is dependent on σ_m . This is because with increasing σ_m , investors choose to allocate the riskless asset to their portfolios at increasingly lower returns relative to the expectation of return on the risky market. To see this, we consider the standard deviation of the market at the higher value of 20% (as measured over the extended historical period 1926-1994, data from Ibbotson Associates, see Brealey and Myers, p.152). To achieve the same profiles for investor allocations between risky and riskless assets as in panels 2 and 3 of Table 4, we need to increase the differential between μ_m and r_f . Thus setting $\mu_m = 3.0\%$ (while maintaining $r_f = 2.0\%$), generates the utilities in panel 4 of Table 4, where again the utility remains essentially flat in the range $\omega = 55\%$ to 90% (as for $\sigma = 15\%$, panel 2), implying that investors will again invest between 10 and 45% of their investment portfolios in the riskless asset; while setting $\mu_m = 3.5\%$ generates the utilities in panel 5, where the utility remains essentially flat in the range $\omega = 65\%$ to 100% (as for $\sigma = 15\%$, panel 3), implying that investors again choose to invest between zero and 35% in the riskless asset. Finally, if we consider the standard deviation of the market as high as 30% (a value closer to the 33.6% actually measured for US stock returns over the depression period, 1926-1939), and again set the return on Government bonds (r_f) at 2.0% (it was closer to 1.85% during the years of the depression), we can estimate that the risky market must be priced to offer a mean exponential rate of return (μ_m) of 4.5% before investors allocate their portfolios in the range 65% to 90% in the market (panel 6, Table 4). Again, consistent with Table 1 (panel 4), the per-period utilities are observed to be *independent* of the number of periods over which the simulation is run.

We may summarise the implications of the above outcomes as follows. The market risk premium (*MRP*) per period over and above the rate offered by Government bonds (r_f) is determined with equation 2 as:

$$MRP = [\mu_m + \frac{1}{2} \sigma_m^2] - r_f \quad (8)$$

With an annualized standard deviation of market returns, σ_m , equal to 15.0% (as measured over the period 1980-94), and a rate on Government bonds (r_f) equal to 2.0%, we have estimated μ_m as 3.0%, as being necessary to induce investors to allocate less than 35% of their portfolio in Government bonds (panel 3, Table 4). That is, $(\mu_m - r_f)$ is estimated as approximately 1.0%. In addition, the market investor enjoys the additional expectation of return ($\frac{1}{2} \sigma_m^2$) that is generated by the riskiness of the market itself; which with $\sigma_m = 0.15$, is also approximately 1.0%. The annualized market risk premium (*MRP*) over and above the rate offered by Government bonds (equation 8) is thereby estimated as $(\mu_m - r_f) + \frac{1}{2} \sigma_m^2 = 1.0\% + 1.0\% = 2\%$. A market risk premium of perhaps less than two percent is clearly much less than has been supposed on the basis of the historical performance of US stocks. For example, average premiums for US stocks over Government bonds over the period 1926-1994 are closer to 7.0% (see Brealey and Myers, 1996, p. 146). On setting the standard deviation of the market (σ_m) as high as 30% (while maintaining $r_f = 2.0\%$), we have calculated a mean exponential rate of return (μ_m) of 4.5% as necessary to induce investors to allocate between 65% and 90% of their portfolio in the risky market (panel 6, Table 4). The expectation of return generated by the riskiness of the market itself, $\frac{1}{2} \sigma_m^2$, is then approximately 4.5% ($\sigma_m = 0.30$). In this case, the annualized market risk premium (*MRP*) is estimated as $(\mu_m - r_f) + \frac{1}{2} \sigma_m^2 = 2.5\% + 4.5\% = 7.0\%$. The kind of returns actually enjoyed in later years by US stock markets therefore appear more consistent with a risk premium calculated with a level of stock market volatility more typical of the period 1926-1939.

Schwert (1991) has observed that although the data might now indicate that by the late 1980s and early 1990s, the volatility of US stock market returns had become closer to 15%, the widespread impression at this time remained that of especially volatile stock prices (also, Brealey and Myers,

1996, p. 153). In this case, historical US market returns may be interpreted as in part due to investor perceptions of volatility. Alternatively, following Jorion and Goetzmann (1999), the high level of historical US stockmarket returns may be interpreted as the phenomenon that US stocks have continued to surpass investor expectations. That is to say, historical US market returns are not actually representative of investor expectations.

5. The Organic Growth Model with Log-wealth Utility Investors versus the Foundations of the CAPM

In this section, we consider the implications of our organic growth model of appreciation in conjunction with log-wealth utility investors, as they compare with the elements of the capital asset pricing model (CAPM). To this end, we consider the foundations of the CAPM as presented by Sharpe (1964). Sharpe identifies the availability of portfolios facing the investor over a single period as the “broken egg” combinations of risk (σ) and expected return (r) in Figure 1. The introduction of a risk-free asset with return r_f , allows for the risk (σ)-return (r) portfolio combinations on the dashed line in Figure 1. The portfolio combinations are formed by combining the risk-free asset at point r_f with the portfolio that is at point M (points above M are obtained by borrowing the risk-free asset at the same rate). The portfolios on the dashed line - the capital market line - constitute the efficient set on account of that they offer a superior risk-return relationship for the risk-averse investor than any alternative combination of risky assets below the line (which all offer less return for equal risk). Since all investors must now be assumed to choose portfolio M in combination with the risk-free asset, portfolio M must represent the portfolio of all available risky assets. A rational investor over a single period will choose from the capital market line in Figure 1, the particular efficient portfolio that offers maximum utility. To this end, Sharpe considers investors “A”, “B” and “C” whose rates of trade-off between risk and reward are represented, respectively, by the indifference curves A, B and C in Figure 1. That is to say, the risk-return combinations for each curve provide the investor with the same utility (see Kritzman, 1992, for example). Higher combinations, however, provide greater utilities than lower ones, since for a given level of risk (σ), all investors are assumed to prefer a higher expected return (r) to a lesser one. The risk-averse

investor is therefore deduced to maximise utility by finding the point of tangency between the efficient set represented by the capital market line and his/her highest risk-return indifference curve. For example, investor “A” is the most risk averse of the three pictured in Figure 1 and will choose to invest all of his portfolio in the risk-free asset. Investor “C”, who is the least risk averse, will borrow at the risk-free rate to invest more than 100% of his portfolio in the market portfolio. Investor “B” has chosen to invest 100% in the market portfolio, M. Nevertheless, in the context of the model advanced within the present paper, we have observed that investors rationally choose to be long in both the risky and non-risky components of the market. Thus we observe that our model actually precludes all three investors “A”, “B” and “C” as considered by Sharpe (!)

- - - Figures 1 and 2 about here - - -

Samuelson over many years has argued that the investment horizon can have no effect on the preferred riskiness of a portfolio (for example, Samuelson, 1963, 1969, 1989, 1994; Merton and Samuelson, 1974): “Then it is an exact theorem that *The Investment Horizon Can Have No Effect on Your Portfolio Proportions*” (Samuelson’s emphasis and capitals, 1994). Kritzman (1994) observes quoting Samuelson: “A rising mean does *not* overcome the increase in dispersion”. Our model is consistent in this respect with Samuelson. Nevertheless, this outcome is not necessarily consistent with the foundations of the CAPM. To see this, consider a possible portfolio over a single period which is characterized in Figure 1 by an expected return r , with level of risk σ . Over N periods, the portfolio is characterized by a compounded expected return Nr , with level of risk $\sqrt{N}\sigma$.¹² The risk-return relationship offered by the single-period portfolios in Figure 1 when taken over N periods is therefore represented on a per-period basis as in Figure 2, which is Figure 1 with the x -axis compressed by dividing each standard deviation by \sqrt{N} (thus the gradient of the capital market line is \sqrt{N} times the gradient allowing a single investment period).¹³ Now, consider investor “A” in Figure 1 (who chooses rationally to abstain from the market over a single period on account of that the highest attainable position of utility is achieved with 100% investment in the risk-free portfolio). Following Sharpe, this investor construes risk as the variability of outcome returns for which he requires an enhanced expectation of return. On this basis, he has constructed his utility curves as an indifference between such risk and return. He therefore imposes these indifference curves as were presented in Figure 1 on the capital market line offered by an N -period investment

as represented by the solid line in Figure 2. In this case, he clearly achieves a *higher* utility indifference curve by investing a proportion of investment wealth in the market portfolio (at clarified by point A_N , Figure 2). For such an investor, the portfolio allocation between risky and riskless assets is thereby observed to be *dependent* on the time horizon. Thus we observe that the model presented in the present paper both (i) restricts investor portfolio allocation choices as compared with the CAPM, and (ii) unlike the CAPM, denies the possibility that longer investment horizons might induce investors to greater risk-taking.

6. Conclusion

The paper has advanced a Dempsey organic growth model of appreciation (*Dogma*) for the market. A key feature of the model is that growth occurs continuously with rates that are normally distributed and independent of each other through time. The model represents a simplification of reality. For example, the model does not allow for a consideration of serial correlation - mean reversion, for example - of stock returns. In the model, investors in the market are characterized as having a log-wealth utility. We have attempted to justify a log-wealth utility as representative of investor preferences. Nevertheless, we have noted that although a decreasing marginal utility of wealth appears reasonable, it is possible that investors are actually less averse to down-side losses than log-utility suggests. It is reasonable, therefore, to consider that investors are, on balance, actually *less* risk-averse than is implied by log-wealth utility. But in such case, the required risk premium for rational stock market investment is likely to be even *less* than we have supposed. Thus we re-confirm that our model predicts a much smaller than previously supposed market risk premium.

A feature of the model is that “risk creates its own reward”, meaning that the potential growth rates that derive from an investment’s volatility, are potentially sufficient to compensate log-wealth utility investors for bearing such volatility. With a volatility of outcome returns represented by a 15% standard deviation, a risk premium of perhaps less than two percent over and above the rate on Government bonds appears consistent with a market dictated by investors with a log-wealth utility. In terms of the model, a market risk premium of seven percent – which is closer to the historical

average of US stock returns – appears more consistent with a standard deviation of returns equal to 30%, which is more representative of the volatility of US markets through the depression years of the 1930s.

The model predicts that log-wealth utility investors choose rationally to go long in both the risky market and riskless assets. That is, they choose to short neither the market nor the riskless asset. Further, this outcome pertains even in the case that underlying mean exponential growth rates are reduced to zero. The implication is that even with investment opportunities so reduced, risky investment in the economy continues to endure. Finally, within the constraints of the model, a rational investor is indifferent to the investment time horizon when assessing the risk profile of an investment portfolio.

Footnotes

1. By $exp(x)$ we mean “ e to the power of x ”, where e is the exponential number, approximately equal to 2.7183. The value of $exp(x)$ is equal to the limit of $[1 + x/N]^N$ as N approaches infinity; that is, the limit as the growth rate x is applied *continuously* as opposed to *discretely* over the time interval. Thus, for example, when \$1 is subject to an exponential growth rate of, say, 10% over an interval, the outcome at the end of the interval is $exp(0.10) = \$1.105$, which is observed to be somewhat greater than the outcome of simply increasing \$1 by 10% at the end of the interval ($= \$1.10$).
2. Fama (1976) observes that an *a priori* expectation for such a process is reinforced by the mathematics of selection as captured by the Central Limit Theorem. Nevertheless, more recent studies indicate that stock returns are more accurately described as exhibiting leptokurtosis, meaning that values of the exponential growth rate x both near to the mean and highly divergent from the mean appear more likely than predicted by a normal distribution for x (Turner and Weigel, 1992; Jackwerth and Rubinstein, 1996).
3. \$1 subject to exponential growth rates x_1, x_2 , over successive intervals has an outcome equal to $exp(x_1).exp(x_2)$. In Table 1, we use the relationship:

$$exp(x_1).exp(x_2) = exp(x_1 + x_2).$$
4. This particular example has been considered in this journal by Kritzman (1994) and by Kritzman and Rich (1998). These authors, however, choose not to draw attention to the fact that $exp(0.28768) = 1.3333$, and $exp(-0.28768) = 0.75$. They therefore miss the point that the example is actually one of binomial exponential growth (with $\mu = 0, \sigma = 0.28768$). The outcome is that both Olsen and Khaki (1998) and Van Eaton and Conover (1998) interpret the numbers $1/3$ and $1/4$ as being merely opportunistic in Kritzman’s (1994) argument that a log-wealth utility function implies non-increasing utility with extended investment horizon (as we discuss below). Again, Bierman (1998) uses the example of a \$100 investment with equally likely outcomes \$130 and \$76.9 (to illustrate investor portfolio allocations), but does not draw attention to the example as one of binomial exponential

growth (with $\mu = 0$, $\sigma = 0.262364$, so that $\exp(0.262364) = 1.3$, and $\exp(-0.262364) = 0.769$).

5. By the “expected” wealth outcome, we mean the sum of the possible wealth outcomes each multiplied by its probability of outcome. Thus the expected wealth outcome of \$12 invested for a single period is $\frac{1}{2}(12 \times 1\frac{1}{3}) + \frac{1}{2}(12 \times \frac{3}{4}) = \12.50 .

6. The natural log, or ‘ln’ function, is the *inverse* of the ‘exp’ function. That is:

$$\ln[\exp(y)] = y, \text{ for all } y.$$

This is consistent with the *definition* of “the natural logarithm of x”, written $\ln(x)$, which is, that number y such that $\exp(y) = x$. That is, $\ln(x) = y$, and $\exp(y) = x$ are equivalent statements. Combining these statements, we have:

$\ln[\exp(y)] = y$, as above. Hence equation 1 follows from the preceding expression for r in the text.

7. We have the general relationships:

$$\exp(A+B) = \exp(A).\exp(B), \text{ and}$$

$$\ln(A \times B) = \ln(A) + \ln(B).$$

Hence

$$r = \ln\left\{ \frac{1}{2} [\exp(\mu+\sigma) + \exp(\mu-\sigma)] \right\}$$

may also be expressed as:

$$r = \mu + \ln\left\{ \frac{1}{2} [\exp(\sigma) + \exp(-\sigma)] \right\}$$

Provided σ is greater than 0, $\frac{1}{2} [\exp(\sigma) + \exp(-\sigma)]$ is always greater than 1; in which case, $\ln\left\{ \frac{1}{2} [\exp(\sigma) + \exp(-\sigma)] \right\}$ is always greater than zero. Thus we observe that provided μ is not actually negative, and σ is greater than zero, the exponential growth rate for expected wealth, r , is greater than zero.

8. We have r as the \ln [expected value of $\exp(x)$], where x is normally distributed with mean μ and standard deviation σ . We also have the general relationship between the logarithm of

an expectation and the expectation of a logarithm where z is a random variable:

$$\ln[\text{expected value of } z] = \text{expected value of } [\ln z] + \frac{1}{2} \text{ variance of } [\ln z]$$

On substituting $z = \exp(x)$ in the above, the left-hand side is r , while the first term on the right-hand side is μ , and the final term is $\frac{1}{2} \sigma^2$. Hence equation 2.

9. The log-wealth utility function can be traced back to Daniel Bernoulli (whose work anticipates the work of Von Neumann and Morgenstern) and his ‘St Petersburg Paradox’ published in 1738. The paradox involves a coin tossing game. The game ends when the player tosses a ‘head’. At this point, the player is awarded $\$2^N$ where N is the number of tosses required to achieve a ‘head’. The probability of achieving $\$2^N$ for any N is $1/2^N$. The probability-weighted value of any possible outcome is therefore \$1. Thus the probability-weighted sum of all possible outcomes is $\$1 + \$1 + \$1 + \dots = \$$ infinite. Hence the theoretical value of being allowed to play the game should be infinite. As Bernoulli recognises, however, no player in practice can be expected to offer more than a modest sum for the privilege of being allowed to play the game. He introduced the concept of utility of wealth to explain the paradox. Application of a log-wealth utility allows the game to be valued at \$4.

10. If we assume that investors have an alternative log-wealth (W) utility function - to base 10, for example, expressed $\log_{10}(W)$ - outcomes are not materially affected. This follows because the natural log of W - expressed above as $\ln(W)$ - is related to $\log_{10}(W)$ by :

$$\ln(W) = \log_{10}(W) / \log_{10}(e)$$
 for all W . Thus taking utility as log-wealth to base 10 has the effect of applying a fixed scaling factor $\log_{10}(e)$ to all utilities. Since it is the *differences* between utilities that concern us, rather than their absolute value, a fixed scaling factor applied to all calculated utilities has no material effect on our conclusions (see footnote 11).

11. The designation “per cent” for utility $\times 100$ is arbitrary. The effect of employing the designation here, is that the per-period utility from investing an amount $\$W$ in a riskless investment with per-period exponential growth rate, r_f , is normalized

as r_f per cent. [To see this, observe that if we have an initial investment of $\$W$ subject to exponential growth rate, r_f , the outcome at the end of the period is $\$W \exp(r_f)$. The gain in log-wealth utility is therefore

$\ln [W \exp(r_f)] - \ln (W) = \ln (W) + \ln [\exp(r_f)] - \ln (W) = r_f.]$ The outcome is that alternative utilities are more directly comparable with the utility derived from the riskless asset.

12. The compounded exponential growth rate, $N\mu$, and standard deviation about such rate, $\sqrt{N}\sigma$, over N periods are as in panels 2 and 3 of Table 1. The expected return, Nr , compounded over N periods is then determined from equation 2 with $N\mu$ substituted for μ , and $\sqrt{N}\sigma$ substituted for σ :

$$N\mu + \frac{1}{2} [\sqrt{N}\sigma]^2 = N [\mu + \frac{1}{2} \sigma^2] = Nr$$

13. We could, alternatively, have multiplied the standard deviations (σ) on the x-axis by \sqrt{N} , and multiply the expected returns (r) on the y-axis by N .

Appendix A

The binomial framework as a substitution for normally distributed returns

If a binomial process over a single interval with equally likely exponential growth rates $\mu+\sigma$, $\mu-\sigma$ (as Table 1), is extended over N intervals, the outcome exponential growth rates over such extended period approach a *normal* distribution with mean $N\mu$ and standard deviation $\sqrt{N}\sigma$ (as indicated in panels 2 and 3, Table 1). To see this, consider, as an example, that the mean exponential growth rate taken over two periods is determined as the probability-weighted average of the possible rates (see panel 1, Table 1):

$$\begin{aligned} & \frac{1}{4}(2\mu + 2\sigma) + \frac{1}{2}(2\mu) + \frac{1}{4}(2\mu - 2\sigma) \\ & = 2\mu \end{aligned}$$

The standard deviation is then determined as the square-root of the probability-weighted average of the squares of the differences between each possible exponential growth rate and the mean rate (2μ):

$$\begin{aligned} & \sqrt{\left\{ \frac{1}{4}[(2\mu + 2\sigma) - 2\mu]^2 + \frac{1}{2}[(2\mu) - 2\mu]^2 + \frac{1}{4}[(2\mu - 2\sigma) - 2\mu]^2 \right\}} \\ & = \sqrt{2}\sigma. \end{aligned}$$

Similarly, in the case of normally distributed rates over each interval (with mean μ and standard deviation σ), the mean and standard deviation over N intervals are $N\mu$ and $\sqrt{N}\sigma$, respectively (for example, Fabozzi and Modigliani, 1992). Thus, we observe that the binomial structure reproduces the mean and standard deviation structure assuming normally distributed rates over each individual interval. By allocating a sufficient number of intervals to the investment period under consideration, the distribution of binomially generated exponential growth rates in Table 1 thereby converges to what we would generate with a normal distribution of rates per interval (see, for example, Cox and Rubinstein, 1985, p. 190-200). For these reasons, we are justified in substituting the binomial framework of Table 1 for the assumed underlying growth process with normally distributed exponential growth rates.

Appendix B

*The binomial framework for a portfolio comprising
the risky market with a riskless asset*

An investor who is long in both the market of risky assets as depicted in Table 1 and ‘riskless’ assets such as Government securities, has equally likely wealth outcomes at the end of the first period (S_u, S_d) per dollar of investment, which may be represented as:

$$S_u = \omega \exp(\mu_m + \sigma_m) + (1 - \omega) \exp(r_f)$$

(A)

and

$$S_d = \omega \exp(\mu_m - \sigma_m) + (1 - \omega) \exp(r_f)$$

where ω is the proportion of the portfolio invested in the market of risky assets (with mean exponential growth rate μ_m and standard deviation about such rate, σ_m) and $(1 - \omega)$ is the proportion invested in the riskless asset (with exponential growth rate, r_f). The above outcomes can be expressed:

$$S_u = \exp(\mu' + \sigma')$$

(B)

and:

$$S_d = \exp(\mu' - \sigma')$$

for which, combining equations A and B, we have:

$$\begin{aligned} \mu' = & \frac{1}{2} \{ \ln[\omega \exp(\mu_m + \sigma_m) + (1 - \omega) \exp(r_f)] \\ & + \ln[\omega \exp(\mu_m - \sigma_m) + (1 - \omega) \exp(r_f)] \} \end{aligned}$$

(3)

$$\begin{aligned} \sigma' = & \frac{1}{2} \{ \ln[\omega \exp(\mu_m + \sigma_m) + (1 - \omega) \exp(r_f)] \\ & - \ln[\omega \exp(\mu_m - \sigma_m) + (1 - \omega) \exp(r_f)] \} \end{aligned}$$

(4)

Allowing that the investor’s portfolio is rebalanced at the end of each period to maintain the proportion of the portfolio in risky assets as ω , the above outcomes per dollar of investment (S_u, S_d) are repeated over successive intervals. Further, since μ_m, σ_m, r_f and ω are assumed to be fixed

across time, μ' , σ' are also fixed across time. The significance of these observations is that the growth behavior of a portfolio of the market of risky assets and the riskless asset continues to be characterized across time by the binomial framework of Table 1 with $\mu = \mu'$ and $\sigma = \sigma'$.

Appendix C

Table 2 constructed algebraically (with equations 3 and 4)

The “100% risky” portfolio has been characterized above by:

$$\begin{aligned}\mu &= 0 \text{ and} \\ \sigma &= 0.28768\end{aligned}$$

(left-hand side panel 1, Table 2). Substituting these values, respectively, for μ_m and σ_m in equations 3 and 4, the “50% risky” portfolio (with $\omega = 0.5$ and $r_f = 0$), is characterized by:

$$\begin{aligned}\mu' &= 0.01031, \text{ and} \\ \sigma' &= 0.14384\end{aligned}$$

(right-hand side panel 1, Table 2). We are now in a position to re-construct algebraically the entries of Table 2:

(i) the binomial structure of Table 2 (panel 2)

For example, the upper outcome of $\$16^{1/3}$ at the end of the second step for portfolio “50% risky” is derived as $\$12 \cdot \exp[2(\mu' + \sigma')] = \$12 \cdot \exp[2(0.01031 + 0.14384)] = \$16^{1/3}$, as determined in the text.

(ii) the expected return in Table 2 (panel 4)

The expected return, r , may be calculated generally for portfolios “100% risky” and “50% risky” with equation 1. Thus for portfolio “100% risky”, $r = \ln\{^{1/2}[\exp(\mu+\sigma) + \exp(\mu-\sigma)]\}$, which (with $\mu = 0$, $\sigma = 0.28768$) determines $r = 4.08\%$ (left-hand side, panel 4), as determined in the text. For portfolio “50% risky”, $r = \ln\{^{1/2}[\exp(\mu'+\sigma') + \exp(\mu'-\sigma')]\}$, which (with $\mu' = 0.01031$, $\sigma' = 0.14384$) determines $r = 2.06\%$ (right-hand side, panel 4), as determined in the text.

(iii) upward drift in Table 2 (panel 2)

In the case of portfolio “100% risky”, the median outcome at the end of two periods, is calculated as: $12 \cdot \exp(\mu + \sigma) \cdot \exp(\mu - \sigma) = 12 \cdot \exp(2\mu) = 12$ (since $\mu = 0$), as determined in the text. That is, the condition for such ‘centred’ growth is that the mean exponential growth rate equals zero. In the case of portfolio “50% risky”, the median outcome at the end of two periods is calculated as: $12 \cdot \exp(\mu' + \sigma') \cdot \exp(\mu' - \sigma') = 12 \cdot \exp(2\mu') = \12.25 (since $\mu' = 0.01031$), as determined in the text.

(iv) utility in Table 2 (panel 5)

For portfolio “100% risky”, we have $\mu = 0$. With equation 5, the per-period utility (U) is therefore equal to zero (left-hand side panel 5, Table 2), as determined in the text. For portfolio “50% risky”, we have $\mu' = 0.01031$. With equation 5, the per-period utility (U) is therefore equal to 0.01031, or 1.031%, as determined in the text.

Appendix D

Log-wealth utility identified as the mean of the distribution of exponential growth rates

We have the Von Neumann and Morgenstern (1947) theorem that the utility U derived from an investment’s possible outcomes is determined as:

$$U = \sum_{i=1}^N p(x_i) u(x_i) \quad (A)$$

where $p(x_i)$ is the probability of each possible wealth outcome x_i , and $u(x_i)$ is the corresponding utility. Thus, for example, the natural log-wealth utility at the end of two periods in panel 1 of Table 1, is determined as:

$$\begin{aligned} & \frac{1}{4} \ln[\exp(2\mu+2\sigma)] + \frac{1}{2} \ln[\exp(2\mu)] + \frac{1}{4} \ln[\exp(2\mu-2\sigma)] \\ & = 2\mu. \end{aligned}$$

More generally, we can observe that the natural log-wealth utility offered by investment of \$1 in an N-period investment horizon is $N\mu$ (panel 4, Table 1). Thus we observe that the per-period natural log-wealth utility (U) offered by such investment is determined as:

$$U = \text{mean exponential growth rate } (\mu) \quad (5)$$

Appendix E

Demonstration of the proposition:

when the mean exponential growth rate for risky assets (μ) is equal to the riskless rate (r_f), portfolio utility is maximized with equal investments in risky and riskless assets

We have the general rule that if $y = \ln [f(x)]$, where $f(x)$ is a continuous function of x , then:

$$\frac{\delta y}{\delta x} = \frac{\delta f(x) / \delta x}{f(x)}$$

Using the rule to differentiate per-period utility U (equation 6):

$$\begin{aligned} U = & \frac{1}{2} \{ \ln[\omega \exp(\mu + \sigma) + (1 - \omega) \exp(r_f)] \\ & + \ln[\omega \exp(\mu - \sigma) + (1 - \omega) \exp(r_f)] \} \end{aligned} \quad (6)$$

with respect to ω , we have:

$$\frac{\delta U}{\delta \omega} = \frac{1}{2} \frac{\exp(\mu + \sigma) - \exp(r_f)}{\omega \exp(\mu + \sigma) + (1 - \omega) \exp(r_f)} + \frac{\exp(\mu - \sigma) - \exp(r_f)}{\omega \exp(\mu - \sigma) + (1 - \omega) \exp(r_f)}$$

The value of U is maximized with respect to ω at the point that $\delta U / \delta \omega$ equals zero. With $\mu = r_f$, the right-hand side of the above equation equals zero when $\omega = 1/2$.

REFERENCES

Bierman Jr. H., 1998. "A Utility Approach to the Portfolio Allocation Decision and the Investment Horizon." *Journal of Portfolio Management*, vol. 25, no. 1 (Fall), 81-87.

Brealey, R. A. and S. C. Myers, 1996. *Principles of Corporate Finance* (International Edition). New York: McGraw-Hill.

Copeland, T. E. and J. F. Weston, 1988. *Financial Theory and Corporate Policy*. New York: Addison Wesley.

Cox, J. C. and M. Rubinstein, 1985. *Options Markets*. Prentice-Hall.

Jackwerth, C. P. and M. Rubinstein, 1996. "Recovering Probability Distributions from Contemporaneous Security Prices." *Journal of Finance*, vol. 51, no. 5, 1611-1631.

Fabozzi, F. and F. Modigliani, 1992. *Capital Markets: Institutions and Instruments*. New Jersey: Prentice-Hall International Editions.

Fama, E. F., 1976. *Foundations of Finance*. New York: Basic Books.

Jones, P. J. and J. W. Wilson, 1999. "Expectations about Real Returns." *Journal of Portfolio Management*, vol. 25, no. 2 (Winter), 45-52.

Jorion, P. and W. N. Goetzmann, 1999. "Global Stock Markets in the Twentieth Century." *Journal of Finance*, vol. 54, no. 3, 953-980.

Kritzman, M., 1992. "What Practitioners Need to Know About Utility." *Financial Analysts Journal*, vol. 48, no. 3 (May/June), 17-20.

Kritzman, M., 1994. "What Practitioners Need to Know About Time Diversification." *Financial*

Analysts Journal, vol. 50, no. 1 (January/February), 14-18.

Kritzman, M., and D. Rich, 1998. "Beware of Dogma." *Journal of Portfolio Management*, vol. 24, no. 4 (Summer), 66-77.

Markowitz, H. M., 1991. "Foundations of Portfolio Theory." *Journal of Finance*, vol. 46, no. 2, 469-477.

Merton, R. and P. Samuelson, 1974. "Fallacy of the Log-normal Approximation to Portfolio Decision-making over Many Periods." *Journal of Financial Economics*, vol. 1, no. 1 (May), 67-94.

Olsen, R. A. and M. Khaki, 1998. "Risk, Rationality and Time Diversification." *Financial Analysts Journal*, vol. 54, no. 5 (September/October), 58-63.

Samuelson, P., 1963. "Risk and Uncertainty: A Fallacy of Large Numbers." *Scientia*, 6th series, 57th year (April/May), 108-113.

Samuelson, P., 1969. "Lifetime Portfolio Selection by Dynamic Stochastic Programming." *Review of Economics and Statistics*, vol. 51, no. 3 (August), 239-246.

Samuelson, P., 1989. "The Judgement of Economic Science on Rational Portfolio Management: Indexing, Timing, and Long-Horizon Effects." *Journal of Portfolio Management*, vol. 16, no. 1 (Fall), 4-12.

Samuelson, P., 1994. "The Long Term Case for Equities and How It Can Be Oversold." *Journal of Portfolio Management*, vol. 21, no. 1 (Fall), 15-24.

Sharpe, W., 1964. "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk." *Journal of Finance*, vol. 19, no. 3 (September), 425-442.

Schwert, W. G, 1991. "Market Volatility: Review." *Journal of Portfolio Management*, vol. 17, no. 4 (Summer), 74-78.

Siegel, L. B., 1997. "Are Stocks Risky? Two Lessons." *Journal of Portfolio Management*, vol. 23, no. 3 (Spring), 29-34.

Turner, A. L and E. J. K. Weigel, 1992. "Daily Stock Market volatility: 1928-1989." *Management Science*, vol. 38, 1586-1609.

Van Eaton, R. D. and J. A. Conover, 1998. "Misconceptions about Optimal Equity Allocation and Investment Horizon." *Financial Analysts Journal*, vol. 54, no. 2 (March/April), 52-59.

Von Neumann, J. and O. Morgenstern, 1947. *Theory of Games and Economic Behavior*. Princeton N.J: Princeton University Press.

TABLE 1: Exponential Growth Characteristics

Panel 1	Probability-weighted wealth outcomes after:		
Starting Wealth	One interval	Two intervals	Three intervals
			$1/8. \exp[3\mu+3\sigma]$
		$1/4. \exp[2\mu+2\sigma]$ <	
	$1/2. \exp[\mu+\sigma]$ <		$3/8. \exp[3\mu+\sigma]$
\$1.00 <		$1/2. \exp[2\mu]$ <	
	$1/2. \exp[\mu-\sigma]$ <		$3/8. \exp[3\mu-\sigma]$
		$1/4. \exp[2\mu-2\sigma]$ <	
			$1/8. \exp[3\mu-3\sigma]$
Panel 2	Mean exponential rate of return:		
	μ	2μ	3μ
Panel 3	Standard deviation of exponential rates of return:		
	σ	$\sqrt{2}\sigma$	$\sqrt{3}\sigma$
Panel 4	Log-wealth utility:		
0	μ	2μ	3μ

TABLE 2: Exponential Growth Characteristics

Portfolio 100% in risky investment, which has equal chance of increasing by $\frac{1}{3}$ and decreasing by $\frac{1}{4}$			Portfolio 50% in prior risky investment, 50% in the riskless asset		
Panel 1			Binomial Characteristics:		
$m = 0$; $s = 0.28768$			$m\bar{c} = 0.01031$; $s\bar{c} = 0.14384$		
Panel 2			Wealth outcomes after:		
Starting Wealth	One interval	Two intervals	Starting Wealth	One interval	Two intervals
		$21\frac{1}{3}$			$16\frac{1}{3}$
	$16\langle$			$14\langle$	
$\$12\langle$		12	$\$12\langle$		$12\frac{1}{4}$
	$9\langle$			$10\frac{1}{2}\langle$	
		$6\frac{3}{4}$			$9\frac{3}{16}$
Panel 3			Expected wealth, $E(W_N)$ at end of period:		
$\$12$	$\$12.50$	$\$13.021$	$\$12$	$\$12.25$	$\$12.505$
Panel 4			$r = \ln [E(W_N)/E(W_{N-1})]$ per period (%):		
	4.08%			2.06%	

Panel 5 **Change in log-wealth utility per period (%):**

0

1.03%