Demand and Adverse Selection in a Pooled Annuity Fund

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Abstract

In this paper, we develop a model for examining the demand for annuities together with the possible implications of adverse selection when an individual consumer has access to both a private annuity market and a market with a pooled annuity fund. An earlier paper by Piggott, Valdez, and Detzel (2004) provides a formal analysis of the payout adjustments from a longevity risk-pooling fund, an arrangement referred to in the paper as Group Self Annuitization (GSA). In such a pooled arrangement, the annuitants will bear their own cohorts' systematic risk, but the cohort will share the idiosyncratic risk. The resulting return on the pooled annuity fund can be expressed as the product of a return from an ordinary annuity multiplied by a random variable that accounts for the adjustment that is due to deviations from expectation of mortality and investments. As demonstrated in this paper, a simple analysis of economic choice provides that it is possible to reduce the implications of adverse selection in a pooled annuity fund, thereby helping to lower the costs of annuity. It is well-documented that empirically, individuals do not find private annuity funds an attractive form of investments despite the potential welfare benefits that can be drawn from annuitization. A pooled annuity fund is an alternative to the conventional private annuity fund that can be considered more cost-effective.

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1 Introduction

It is well documented in the aging literature that the uncertainty of an individual's lifetime together with the attitude to risk affect his or her consumption level, investment behavior and demand for annuity and life insurance. Decision making is even further complicated by the fact that we are observing unprecendented longer lifetimes across the globe. The risk of outliving available resources is of prime concern among retirees today, and the early literature [Yaari, 1965] seems to suggest that the risk of longevity can be completely hedged with life annuity products. Yet it remains a puzzle among experts as to why there continues to be very little demand for annuities, where even among the elderly, there appears to be extreme low level of voluntary annuitization. Some surveys have indicated that among retirees, as little as 1.5% have income derived from private annuities.

Some of the more commonly cited reasons for this low level of demand are: (1) poor historical yield performance [Friedman and Warshawsky, 1990]; (2) large loadings and profits from private issuers [Mitchell, et al., 1999]; (3) bequest motives [Kotlikoff and Summers, 1981; Hurd, 1989; Bernheim, 1991]; (4) pre-existing annuities from government pensions [Bernheim, 1991]; and (5) actual design which sometimes discourages full annuitization [Yagi and Nishigaki, 1983].

Milevsky and Robinson (2001) develop analytical techniques that can be used by individuals at retirement who face the choice between voluntary annuitization and self-annuitization. Specifically, they show that one can use the concept of a ruin probability when assessing how much to self-annuitize. Self-annuitization provides greater liquidity than voluntary annuitization; however it does so at the cost of possibly outliving resources. Other interesting work include that of Albrecht and Maurer (2002) which demonstrates how to evaluate that risk by calculating a personal probability of consumption shortfall and show that it is substantial, particularly for high entry ages.

As recommended in Piggott, et al. (2004), a possible response is to separate the systematic from the idiosyncratic risk. The idea is to form groups or cohorts in order to pool idiosyncratic risk where payouts could be conditioned to the mortality experience of the group, subject of course within a clear framework with specified legal rights and obligations. The term Group Self-Annuitization (GSA) has been suggested in Piggott, et al. (2004) whereby a group self-annuity plan will allow retirees to pool together and form a fund to provide for protection against longevity. There are yet implementation issues associated with developing such products, but if properly designed and implemented, GSA products are believed to provide a less expensive form of insurance against the risk of longevity. Although the product appears to obviate the need for an insurance company, it may be sold through a corporate insurer who may act simply as an administrative agent.

Piggott, et al. (2004) provided a formal treatment of how benefit adjustments in a GSA plan should be calculated and under what conditions these adjustments should satisfy. These benefit adjustments provide the means of pricing for the product, and it has been shown that under some regularity conditions, the optimal strategy is to compute the ratio of the expected to actual proportion of survivors that become a central figure in the adjustment calculations. This pricing adjustment is inherited in our model developed in this paper for evaluating demand and the possible presence of adverse selection in the GSA plan. The objective of this paper is to further expand that analysis by considering an economic choice model for the consumption and investment behavior of a consumer individual. Specifically, using a micro-economic utility framework, we assess and analyze the consumption behavior of a rational individual when faced with the choice of having to buy a conventional annuity product or a pooled annuity fund, that is, a GSA account. This similar framework can be used to further analyze and compare the implications of adverse selection in the conventional annuity and in the pooled annuity fund. There is presence of possible adverse selection in the annuity market because individuals use privately known survival probabilities to alter their demand for the product. The presence of the pooled annuity fund does not completely eliminate adverse selection, however, we find that for certain classes of utility functions, as demonstrated and argued in this paper, the presence of the pooled annuity fund reduces adverse selection. Individuals adversely select against the pooled annuity fund to a lesser extent than against a conventional annuity. This can be intituively explained by the fact that there is additional randomness in the rate of return expected from the pooled annuity fund which is derived from the real outcome of the mortality or survivorship probabilities of the pooled individuals. This uncertainty in the payout makes it the more difficult to exercise adverse selection.

For the rest of the paper, we have organized it as follows. First in Section 2, we discuss the mechanics of the pricing or benefit calculations in a pooled annuity fund. This has been well developed in Piggott, et al.(2004) but herein redundantly discussed to justify the premium form of the pooled annuity fund. In Section 3, we examine models using utility framework to assess demand and adverse selection when there is a market allowing the consumer access to both a conventional annuity and a pooled annuity fund. These models are therefore extensions of Abel (1986) and Sinha (1989) who developed economic framework of demand in a market allowing access only to a conventional annuity. In Section 4, we describe how to assess the presence of adverse selection in both types of annuities. In Section 5, we consider the special case where we have the exponential utility function. In this case, it is interestingly demonstrated that the demand is not a function of the individual's current state of health, making it impossible to adversely select against a pooled annuity fund. In the demand model, annuity rate of returns are exogenously given, but in Section 6, we demonstrate how to obtain the equilibrium rates of return from a conventional annuity when demand has been determined among all rational agents in the economy. In this competitive market, the profits for the representative firm issuing the conventional annuity is set to zero at equilibrium. We conclude in Section 7. Appendices follow that detail proofs of results within the paper.

2 Pricing mechanics in a pooled annuity fund

This section documents formalization of developing the actuarial calculation aspects of implementing GSA plans. In a GSA or pooled annuity fund, the operation starts much like an ordinary life annuity purchased in the private market so that much of the initial pricing procedure consists of calculating the annuity payout rate. This annuity benefit payout formula must capture both the annuitant's expected mortality in the future, accounting for possible mortality improvements, and the investment return that is expected to be earned on the invested assets. If these expectations are actually realized over time, the payout rates determined at the point of entry will remain constant.

To demonstrate this, consider only a two-period model whereby in its simplest form, assume that at time $t = 0$, a pool of ℓ_x annuitants, assume all aged x, decides on the amount they expect to receive next period if they survive. For treatment of a multiple period framework, see Piggott, et al. (2004). Suppose this is a payment of B so that the starting total fund is

$$F_0 = Bv\ell_{x+1}$$

where ℓ_z denotes the expected number of lives to survive to age z and $v = 1/R$ is the discount factor. See Bowers, et al. (1997) for actuarial pricing of life annuities. If assumptions are realized, then in the following period, the fund will become

$$F_1 = F_0 R$$

and dividing this fund across all surviving lives, we would have the total actual benefits to be received per survivor equal to

$$B^* = \frac{F_1}{\ell_{x+1}} = \frac{F_0 R}{\ell_{x+1}} = \frac{Bv\ell_{x+1}R}{\ell_{x+1}} = B.$$

If the number of actual survivors ℓ^*_{x+1} happened to be different from ℓ_{x+1} and the realized rate of return R^* happened to be different from R, then the ending fund will be

$$F_1^* = F_0 R^*$$

and this fund is to be divided among the actual survivors so that each survivor will then receive

$$\begin{aligned} B^* &= \frac{F_1^*}{\ell^*_{x+1}} = \frac{F_0 R^*}{\ell^*_{x+1}} \\ &= \frac{F_0 R}{\ell_{x+1}} \frac{R^*}{R} \frac{\ell_{x+1}}{\ell^*_{x+1}} = B\frac{R^*}{R}\frac{\ell_{x+1}}{\ell^*_{x+1}}, \end{aligned}$$

where the actual benefit payout then is adjusted by a factor of $\dfrac{R^*}{R}$ to account for deviation in rate of return from expected and additionally by a factor of $\dfrac{\ell_{x+1}}{\ell^*_{x+1}}$ to account for mortality or survivorship deviations.

It is clear from these adjustment factors that a higher realized than expected rate of return on the investments leads to larger benefit payouts, and a lower actual rate of mortality than expected also leads to larger benefit payouts. In the ensuing paragraphs, we will consider these adjustment factors as random variables denoted by Y for which its expectation will exceed 1.

3 A model for consumer demand in the annuity market

To begin the discussion, consider a consumer individual who is expected to live either one or two periods. Following the framework in Abel (1986) and Sinha (1989), we develop a similar model to describe the demand in the annuity market and to examine implications of adverse selection. Death may occur only in the second period with a probability $1 - p$. We additionally assume that consumption and investments are made at the beginning of each period.

A rational consumer is assumed to maximize his utility over the two periods. This utility of consumption function can be expressed in the additive form as

$$U(C_t, C_{t+1}) = u(C_t) + \beta p u(C_{t+1}), \tag{1}$$

where C_t and C_{t+1} are the respective consumption levels at the beginning and the end of the period. Here also, β denotes the discount factor for consumer impatience. Assume that the rational consumer is an investor with $u'(x) > 0$ and $u''(x) < 0$. In other words, the consumer satisfies the principle of non-satiation and is also considered a risk-averse individual. We additionally impose the restriction that the third derivative is positive, that is, $u'''(x) > 0$. This requirement is actually a necessary condition for decreasing absolute risk aversion. See, for example, Levy and Sarnat (1984) for a discussion of this together with its relationship to stochastic dominance. A clear example of a utility function satisfying these requirements is the power utility function expressed by

$$u(c) = \frac{c^{1-\delta} - 1}{1 - \delta}, \tag{2}$$

where δ denotes the consumer's constant relative risk aversion. This is usually restricted to the case where $0 \leq \delta \leq 1$ and when $\delta = 1$, this leads to the logarithmic utility function as a limiting case. The power utility function is considered homothetic; see for example, Abel (1986).

Another class of utility functions satisfying the above requirements is the "exponential" with the form

$$u(c) = -e^{-\delta c}, \tag{3}$$

where δ denotes a measure of the consumer's constant absolute risk aversion assumed to be positive. Clearly, $u'(x) = \delta e^{-\delta c} > 0$, $u''(x) = -\delta^2 e^{-\delta c} < 0$, and $u'''(x) = \delta^3 e^{-\delta c} > 0$.

To continue the discussion, let us first consider the case where the consumer only has access to a conventional annuity product. For simplicity then, assume that the consumer is initially endowed with wealth equal to 1 so that the budget constraints are

$$C_t = 1 - a_t^c \tag{4}$$

and

$$C_{t+1} = a_t^c R_c \tag{5}$$

where a_t^c refers to the proportion of wealth invested in the conventional annuity product at the start of the period and R_c is the return on this annuity investment. If the consumer then survives to the beginning of the second period, he consumes the return from the annuity. Notice that herewith we denote subscripts and superscripts c for the case of the conventional annuity.

Maximizing then (1) subject to the budget constraints leads us to the first-order condition

$$\frac{\partial U}{\partial a_t^c} = -u'(C_t) + \beta p R_c u'(C_{t+1}) = 0.$$

Equivalently, this first-order condition can be expressed as

$$\frac{u'(C_t)}{u'(C_{t+1})} = \beta p R_c, \tag{6}$$

which gives the ratio of marginal utilities. It immediately follows from the concavity assumption of the utility function that the solution to this first-order condition gives the maximum. Furthermore, it can easily be shown, for example, that in the logarithmic utility function, we have the solution

$$a_t^c = \frac{\beta p}{1 + \beta p} \tag{7}$$

giving the demand function for the conventional annuity which in Sinha (1986) has been denoted by $D(R_c; p)$. For the logarithmic utility, this gives a value independent of the expected rate of return promised. Sinha (1986) has an impatience factor of $\beta = 1$ in his analysis.

Adverse selection can be examined by looking at how the annuity demand varies with changes to the mortality or survival rate. Note that p is a privately known subjective probability of survival which can be distinct from the homogeneous rates of mortality used by the insurance company to price for these products. Observe that the first derivative of (7) gives rise to

$$\frac{\partial a_t^c}{\partial p} = \frac{1}{(1 + \beta p)^2} > 0$$

which clearly is always non-negative. Thus, here we see that as the privately known probability of survival increases, the demand for the annuity also increases, indicating a presence of adverse selection.

We now extend the Abel (1986) and the Sinha (1989) models to a market where the individual has the option of purchasing a conventional annuity and investing in a pooled annuity fund, or a GSA account. Following the previous development, an individual is also assumed to maximize his utility function of the form (1) but this time the budget constraints are different.

With the introduction of the pooled annuity product into the market, the individual's consumptions at periods t and $t + 1$, respectively, now become

$$C_t = 1 - a_t^c - a_t^g \tag{8}$$

and

$$C_{t+1} = a_t^c R_c + a_t^g R_g Y \tag{9}$$

where a_t^c, as before, denotes the proportion of wealth invested in a conventional annuity which pays a one-period return of R_a, and a_t^g denotes the proportion invested in a pooled annuity fund with a one-period return of R_g. Notice that we are adopting the convention of using superscripts and subscripts of g to denote the pooled annuity fund. Furthermore, observe that the consumption level in the next period is now a random variable, whose exact value is known only at the beginning of the second period. The random variable Y is the adjustment factor earlier discussed in the previous section, which for our purposes is assumed to have finite mean

$$1 \leq E(Y) < \infty$$

and finite variance $Var(Y) < \infty$. The requirement that the mean $E(Y)$ must be at least equal to 1 follows from the intuitive explanation that a higher rate of return is to be expected from the pooled annuity account because both the risks of investments and mortality are now borne by the consumer individual. The institution issuing the pooled annuity fund is simply acting as a manager of the accounts and does not bear any of these risks. This is the very essence of a pooled annuity fund whereby the longevity risk is now borne by the participants of the pool. In effect, the expected utility to be maximized can be re-expressed then as the expectation

$$E\left[U(C_t, C_{t+1})\right] = u(C_t) + \beta p E\left[u(C_{t+1})\right]. \tag{10}$$

The rate of return R_c is exogenously determined from the prespective of the consumer, although such a return can be determined from a steady state equilibrium that is reached when all the firms issuing conventional annuities have maximized their profits and the individuals in the economy have maximized their utility functions. The rate of return R_g initially promised to the consumer at the beginning of the period can thus be determined from the return on the conventional annuity through certainty-equivalent arguments. For example, R_g can be solved using the certainty-equivalent formula

$$u(R_c) = E\left[u(R_g Y)\right]. \tag{11}$$

The following two examples illustrate the explicit forms of this relationship for the power and the exponential utility functions.

Example 1: *Power utility* Using the power utility function in (2), it can be shown that this relationship between R_c and R_g is given by

$$R_g = R_c \left\{ \left[E\left(Y^{1-\delta}\right) \right] \right\}^{-(1-\delta)}. \blacksquare$$

Example 2: *Exponential utility* Using the exponential utility function in (3), we have the relationship

$$R_c = -\frac{1}{\delta} \log\left[m_Y(-\delta R_g)\right],$$

where $m_Y(\cdot)$ denotes the moment generating function of Y.$\blacksquare$

By the concavity of the utility function, we know that

$$u(R_c) = E\left[u(R_g Y)\right] \le u\left[R_g E(Y)\right],$$

so that by the increasing property of the utility function, we must have $R_g E(Y) \ge R_c$. There is a clear intuitive explanation to this. Because of the additional risks of longevity borne by the individuals in the pooled annuity fund, the expected return from this pooled annuity must then be at least equal to the expected return offered by the conventional annuity.

To find the optimal proportions a_t^c and a_t^g, we take the first-order condition and this yields to the following system of equations:

$$\frac{\partial E\left[U(C_t, C_{t+1})\right]}{\partial a_t^c} = -u'(C_t) + \beta p R_c E[u'(C_{t+1})] = 0$$

and

$$\frac{\partial E\left[U(C_t, C_{t+1})\right]}{\partial a_t^g} = -u'(C_t) + \beta p R_g E[u'(C_{t+1})Y] = 0.$$

These equations can be re-written as

$$\frac{u'(C_t)}{E\left[u'(C_{t+1})\right]} = \beta p R_c \tag{12}$$

and

$$\frac{u'(C_t)}{E\left[u'(C_{t+1})Y\right]} = \beta p R_g. \tag{13}$$

Dividing (12) by (13) and re-arranging the terms, we succinctly write these conditions as

$$R_c E[u'(C_{t+1})] = R_g E[u'(C_{t+1})Y]. \tag{14}$$

In the appendix, we show that because of the concavity property of the utility function, these first-order conditions, if the solution exists, indeed give the optimal solutions. The proof shows that the Hessian matrix is indeed a negative semi-definite.

Example 3: *Logarithmic utility* In the case of the logarithmic utility which is a limiting case of the power utility function, it is straightforward to show that the conditions in (12) and (13) yield to the following system of equations:

$$1 = \beta p R_c E\left[\frac{1 - a_t^c - a_t^g}{a_t^c R_c + a_t^g R_g Y}\right]$$

and

$$1 = \beta p R_g E\left[\frac{(1 - a_t^c - a_t^g)\, Y}{a_t^c R_c + a_t^g R_g Y}\right]. \blacksquare$$

Now, to illustrate the model, let us consider a numerical example. Suppose we assume the logarithmic utility function and assume the parametric values $R_a = 2.0$, $\beta = 2.5$, and $p = 0.7$. In the case where we have only access to a conventional annuity, according to (7), the optimal proportion of annuity to purchase is given by

$$a_t^c = 63.64\%$$

so that the optimal consumption levels are

$$C_t = 0.3636 \text{ and } C_{t+1} = 1.2727.$$

To be able to compare this in the case where the individual has access to a pooled annuity fund, we need assumption for the adjustment factor Y. Suppose there are three possible states of the world as

$$Y = \begin{cases} 0.60, & \text{w.p. } 0.15 \\ 1.00, & \text{w.p. } 0.40 \\ 1.15, & \text{w.p. } 0.45 \end{cases},$$

where w.p. means "with probability". In this case, it can be computed using (12) and (13) that the optimal proportions of annuities to purchase are given by

$$a_t^c = 29.96\% \text{ and } a_t^g = 33.68\%$$

so that the expected optimal consumption levels are

$$C_t = 0.3636 \text{ and } E\left(C_{t+1}\right) = 1.2872.$$

In the presence of conventional annuities alone, the maximum expected utility is

$$U(C_t, C_{t+1}) = -0.5896$$

whereas in the presence of both conventional and pooled annuities, the maximum expected utility is

$$E\left[U(C_t, C_{t+1})\right] = -0.5698$$

which is clearly larger than that with conventional annuities alone. Thus, this demonstrates that the presence of pooled annuity funds allow the individuals to have larger expected utility. As a matter of fact, the beginning consumption level in this case is also increased, although the ending consumption level is lower on an expectation basis, this level is subject to some variability and a possibility that the actual consumption level is much larger than this expectation.

4 Comparing adverse selection in a conventional annuity and a pooled annuity fund

Despite the welfare benefits to annuitizing, as alluded in the introduction, it is surprising to find that in well-developed countries like the United States and Australia, the annuity market is still considered "thin". The market is also subject to possible adverse selection from the consumers whereby annuity purchasers use their private information of their own survival probabilities in making annuity purchase decisions. This has also been explicitly captured in our previous development. Observe that the demand for annuities, both in the case where there is only a market for conventional annuities and in the case where there is market for both the conventional and the pooled annuities.

In this section, we demonstrate following from the model development previously described, that there is lesser form of adverse selection in a pooled annuity fund than in a conventional annuity. To see this, what we do is compare how the respective demands for both annuities change when there is additional information of a change in the survival probabilities. In mathematical terms, we compare the partial derivatives $\partial a_t^c / \partial p$ and $\partial a_t^g / \partial p$ to assess the presence of adverse selection. Explicit expressions for these partial derivatives are shown in details in the appendix.

Summarizing the results in the appendix, we have

$$\frac{\partial a_t^c}{\partial p} = -\frac{\beta R_c E\left[u'\left(C_{t+1}\right)\right] \times R_g E\left[u''\left(C_{t+1}\right) Y\left(R_c - R_g Y\right)\right]}{u''\left(C_t\right) \times E\left[u''\left(C_{t+1}\right)\left(R_c - R_g Y\right)^2\right]} = \frac{C \times A}{u''\left(C_t\right) \times (A + B)}$$

and that

$$\frac{\partial a_t^g}{\partial p} = -\frac{\beta R_c E\left[u'\left(C_{t+1}\right)\right] \times R_c E\left[u''\left(C_{t+1}\right)\left(R_c - R_g Y\right)\right]}{u''\left(C_t\right) \times E\left[u''\left(C_{t+1}\right)\left(R_c - R_g Y\right)^2\right]} = \frac{C \times B}{u''\left(C_t\right) \times (A+B)}$$

where the terms A, B, and C have explicit expressions given in the appendix. Because the second derivative of the utility function is negative, we notice that

$$A + B = E\left[u''\left(C_{t+1}\right)\left(R_c - R_g Y\right)^2\right] < 0.$$

Thus we have the following proposition.

Proposition 1 *Assume that the individual consumer has utility function u satisfying $u' > 0$ and $u'' < 0$. In a market where there is access to both conventional annuity and pooled annuity funds, we have the following:*

$$\frac{\partial}{\partial p}\left(a_t^c + a_t^g\right) > 0,$$

that is, the individual consumer will in the aggregate adversely select against both annuity products.

Proof. Using results (25) and (26) in the appendix, we find that

$$\frac{\partial}{\partial p}\left(a_t^c + a_t^g\right) = \frac{\partial a_t^c}{\partial p} + \frac{\partial a_t^g}{\partial p} = \frac{C \times (A+B)}{u''\left(C_t\right) \times (A+B)} = \frac{C}{u''\left(C_t\right)}.$$

The result then immediately follows from the fact that the terms $C = -\beta R_c E\left[u'\left(C_{t+1}\right)\right]$ and $u''\left(C_t\right)$ are both strictly negative. ■

The result is true in the aggregate. It is not clear whether either or both of $\frac{\partial a_t^c}{\partial p}$ and $\frac{\partial a_t^g}{\partial p}$ are positive, so that adverse selection will be present in both situations. However, if we additionally impose the condition that the third derivative $u''' > 0$ which holds true for example in the case where preferences satisfy decreasing absolute risk aversion, together with the condition that the consumer adversely select against each product simultaneously, then there is less severe adverse selection in the pooled annuity fund than in a conventional annuity. We express this result as the following proposition.

Proposition 2 *Assume that the individual consumer has utility function u satisfying $u' > 0$, $u'' < 0$, and $u''' > 0$. Let*

$$\frac{\partial a_t^c}{\partial p} > 0 \quad \text{and} \quad \frac{\partial a_t^g}{\partial p} > 0.$$

In a market where there is access to both conventional annuity and pooled annuity funds, we have the following result:

$$\frac{\partial a_t^c}{\partial p} > \frac{\partial a_t^g}{\partial p},$$

that is, the individual consumer will less adversely select against the pooled annuity fund than against a conventional annuity.

Proof. The proof of this uses the result that if $g(z)$ and $h(z)$ are either both non-decreasing or both non-increasing functions of z, then

$$E[g(Z)h(Z)] \geq E[g(Z)]E[h(Z)].$$

For a discussion of this result, see Casella and Berger (2002). Now, consider the ratio

$$\begin{aligned}
\frac{R_g^2 E\left[u''(C_{t+1})Y^2\right]}{R_c^2 E\left[u''(C_{t+1})\right]} &= \left(\frac{R_g}{R_c}\right)^2 \frac{E\left[Y\cdot(u''(C_{t+1})Y)\right]}{E\left[u''(C_{t+1})\right]} \\
&\geq \left(\frac{R_g}{R_c}\right)^2 \frac{E(Y)E\left[(u''(C_{t+1})Y)\right]}{E\left[u''(C_{t+1})\right]} \\
&\geq \left(\frac{R_g}{R_c}\right)^2 \frac{E(Y)^2 E\left[u''(C_{t+1})\right]}{E\left[u''(C_{t+1})\right]} \\
&= \left(\frac{R_g E(Y)}{R_c}\right)^2 > 1.
\end{aligned}$$

The last inequality follows because we know that $R_g E(Y) > R_c$. These results lead us to

$$R_g^2 E\left[u''(C_{t+1})Y^2\right] < R_c^2 E\left[u''(C_{t+1})\right]$$

and deducting both sides by $R_c R_g E\left[u''(C_{t+1})Y\right]$, we have

$$R_g^2 E\left[u''(C_{t+1})Y^2\right] - R_c R_g E\left[u''(C_{t+1})Y\right] < R_c^2 E\left[u''(C_{t+1})\right] - R_c R_g E\left[u''(C_{t+1})Y\right]$$

or equivalently

$$-R_g E\left[u''(C_{t+1})Y(R_c - R_g Y)\right] < R_c E\left[u''(C_{t+1})(R_c - R_g Y)\right].$$

By assumption that both $\frac{\partial a_t^c}{\partial p} > 0$ and $\frac{\partial a_t^g}{\partial p} > 0$, it can be seen that both the right-hand side and the left-hand side of the previous inequality are non-positive. Therefore, we have

$$-\frac{R_g E\left[u''(C_{t+1})Y(R_c - R_g Y)\right]}{R_c E\left[u''(C_{t+1})(R_c - R_g Y)\right]} > 1$$

and the result immediately follows because the left-hand side of the above inequality is the ratio of the partial derivatives. ■

5 The case of the exponential utility

In this section, we examine the special case of the exponential utility because with this utility form, we are able to derive explicit expressions for the proportions invested in the conventional and the pooled annuity fund.

The form of the utility function is expressed in (3). Because $u'(c) = \delta e^{-\delta c}$, the first-order conditions can then be written as

$$\delta e^{-\delta(1-a_t^c-a_t^g)} = \beta p R_c \delta E\left[e^{-\delta\left(a_t^c R_c + a_t^g R_g Y\right)}\right]$$

and

$$\delta e^{-\delta(1-a_t^c-a_t^g)} = \beta p R_g \delta E\left[Y e^{-\delta\left(a_t^c R_c + a_t^g R_g Y\right)}\right].$$

Dividing these two conditions, we have

$$R_c E\left(e^{-\delta a_t^g R_g Y}\right) = R_g E\left(Y e^{-\delta a_t^g R_g Y}\right).$$

Now, consider the special case where the random variable adjustment Y has an exponential with mean 1. Then, we have

$$R_c = \frac{R_g}{1+\delta a_t^g R_g}$$

so that the proportion of wealth invested in the pooled annuity fund will be

$$a_t^g = \frac{1}{\delta}\frac{R_g - R_c}{R_c R_g}. \tag{15}$$

The corresponding proportion invested in the conventional annuity will be

$$a_t^c = \frac{\log\left(\beta p R_c^2/R_g\right) + \delta\left(1-a_t^g\right)}{\delta\left(1+R_c\right)}. \tag{16}$$

Interestingly, notice that the proportion invested in a pooled annuity account, a_t^g, is independent of the consumer's survival probability p. This implies that, in this situation, the adverse selection problem does not exist at all for the pooled annuity account. If the consumer, for instance, believes that he or she has a higher survival probability so that this will also prolong his or her average lifetime, then this same consumer will not alter the amount of conventional annuity purchases. This is not the case of the conventional annuity as one observes that the proportion invested remains to be a function of p.

According to the certainty equivalence between the returns R_c and R_g, we have the relationship

$$R_g = \frac{1}{\delta}\left(e^{\delta R_c} - 1\right). \tag{17}$$

This relationship is depicted in Figure 1. Here in Figure 1, we show the relationship between the returns from the conventional annuity and the pooled annuity fund for four different levels of risk aversion parameter δ: $\delta = 0.1$, 0.2, 0.3, 0.4, and 0.5. As expected, the higher the value of this risk aversion parameter, the larger the return on the pooled annuity fund will the consumer demand to be indifferent with the corresponding return from the conventional annuity.

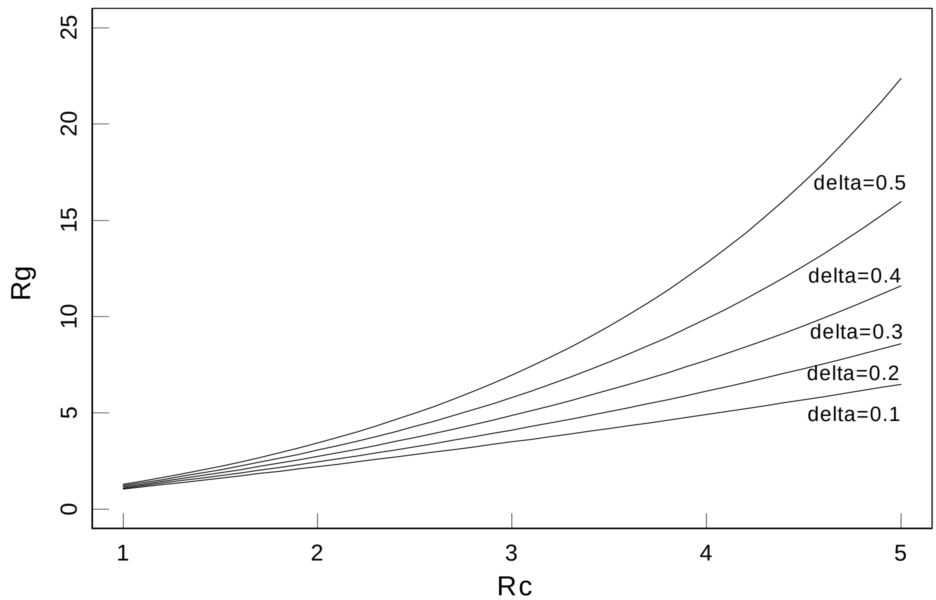

Figure 1: Certainty Equivalence between R_c and R_g

Let us consider a numerical example to illustrate this consumer's behavior. We set risk aversion parameter by $\delta = 0.3$, the return for the conventional annuity $R_c = 7$, and the impatience or discount factor by $\beta = 2.5$. Then, by considering different survival probabilities, say $p = 0.5$, 0.6, and 0.7, we list in Table 1 the optimal consumptions and optimal combination of conventional annuity and pooled annuity accounts.

Table 1: Optimal Consumption for Different Levels of Survival Probabilities

Survival Probability	Proportions		Consumption	
	conventional	pooled (GSA)	Beginning	Ending
p	a_t^c	a_t^g	C_t	C_{t+1}
0.5	0.4753	0.3366	0.1881	11.3684
0.6	0.5512	0.3366	0.1121	11.9001
0.7	0.6155	0.3366	0.0479	12.3497

From Table 1, we find that regardless of the level of survival probabilities, the amount of investment in a pooled annuity account remains the same level at 0.3366. In the case of the conventional annuity, however, it is not surprising to find that the proportion of investment made increases with increasing survival probability. Figure 2 further depicts the effect of varying the survival probability p on the proportions of annuity investments.

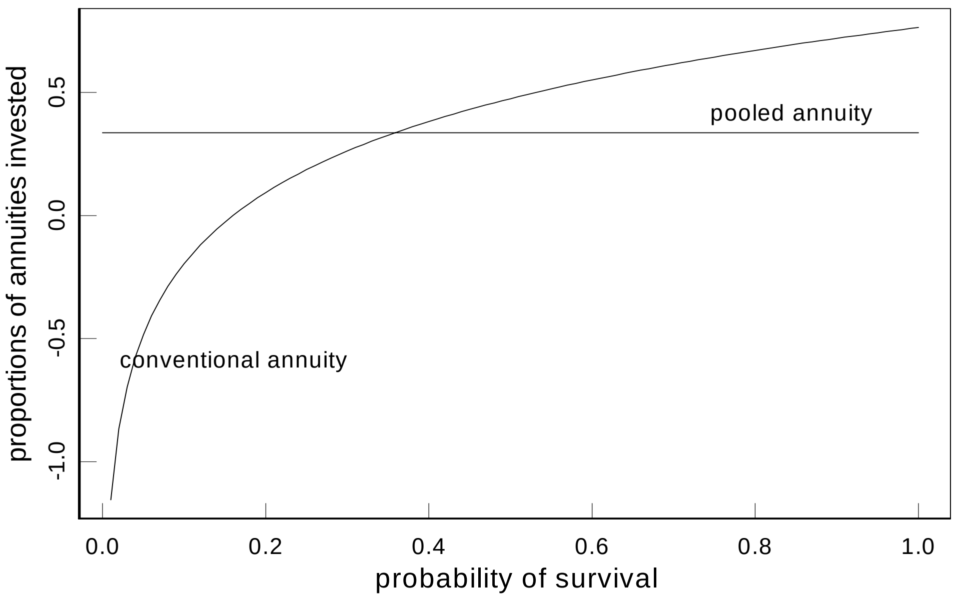

Figure 2: The Effect of Probability of Survival p on the Demand for Annuities

Suppose for some reason, an individual believes he has a higher survival probability than many other people, then he is expected to increase his purchases on conventional annuity than many others. However, if he also increases his purchases pooled annuity accounts, then he will not get as good a return of an investment as in a conventional counterpart. The explanation for this has to do with the fact that in the pooled annuity account, he is expected to absorb the systematic portion of the longevity risk himself. As time progresses, he would receive less and less of these payments despite that he is expected to survive many other annuitants. In the conventional annuity, the longevity risk is entirely borne by the issuing insurance company, and he is expected to receive a pre-determined level of annuity payments from this issuing company.

A pooled annuity account is thus somewhere between the conventional annuity and the individual self-annuitization. A conventional annuity provides an individual complete protection of the longevity risk because the entire portion of the risk is transfered to the insurer. In contrast, with individual self-annuitization, all the longeivity risk is borne by the individual consumer himself. In a pooled annuity account, however, the pool can only diversify away the idiosyncratic portion of the longevity risk. The systematic portion longevity risk remains to be absorbed by annuitant himself.

In Figures 3 and 4, we display the effect of varying both the return R_c on the conventional annuity and the risk aversion parameter δ on the proportion of annuity investments. We observe from Figure 3 that there is exponential decline of the demand for a pooled annuity for increasing return on the conventional annuity because it becomes more attractive to increase investment in the conventional annuity. However, it is interesting to observe that for the conventional annuity, the demand increases rapidly for low levels of return but slowly declines after some point. This decline can be further explained from the fact that because returns become more attractive thereby providing more consumption in the next period, it becomes more optimal for the consumer to invest less of both the conventional and the pooled annuity because the possible increase in wealth provided by the increase in return allows the consumer to do so. There is now more available wealth that can be consumed for the beginning of the period.

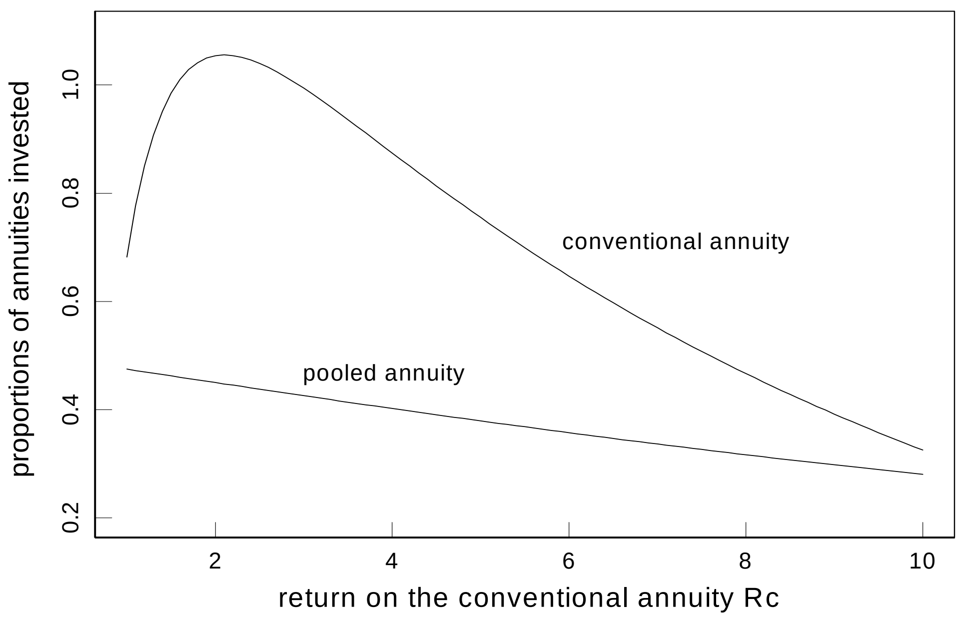

Figure 3: The Effect of the R_c on the Demand for Annuities

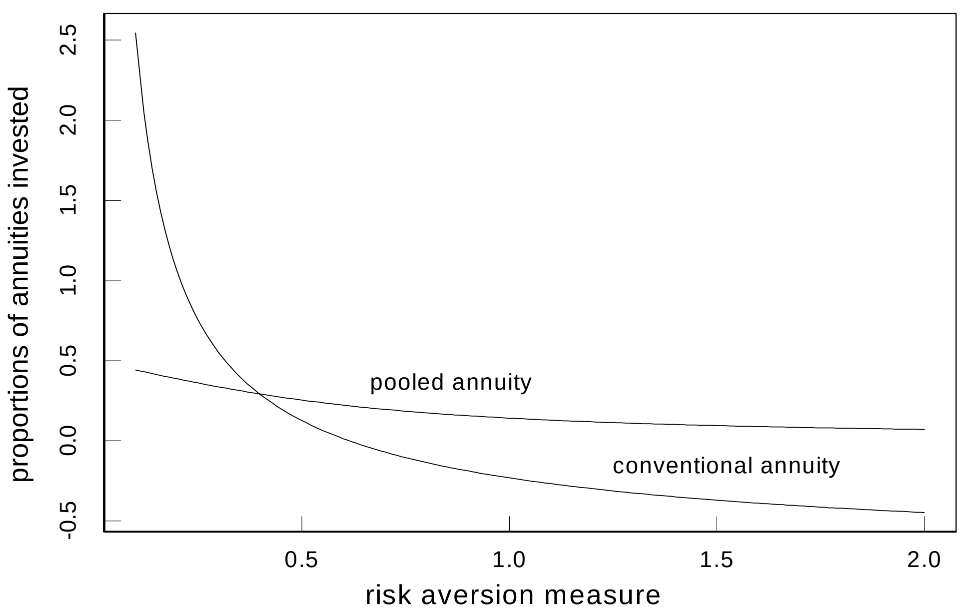

Figure 4: The Effect of the Risk Aversion δ on the Demand for Annuities

6 The equilibrium rate of return

In our earlier development, when maximizing the expected utility of the consumer, we assumed that the return on the conventional annuity was exogenously determined and is given and known to the consumer. However, this return can be solved using economic equilibrium arguments. In this section, we show how to solve for this equilibrium return on the conventional annuity when the annuity market also includes available investment in a pooled annuity account, or a Group Self-Annuitization account.

Assume a perfectly competitive market that provides for both a conventional annuity and a pooled annuity account. The equilibrium can be reached when the profits for all the representative firms in this competitive market become zero. Observe that because there is zero risk for the issuing organization of the GSA account, it follows that zero profits is immediate in this case.

Suppose that, for the moment, the consumers in the competitive market can be characterized according to their level of health which is therefore reflected in their probability of survival. Suppose this characterization classifies each consumer in the economy according to healthy (H) or not-so-healthy (L). Each of these types of consumer therefore carries with it a probability of p_H or p_L. The profit function, therefore, for the issuing company of the conventional annuity can be expressed as

$$\pi(R_c; p_H, p_L, \alpha, \delta) = \alpha(R - p_H R_c) D_H(R_c, p_H, \delta) + (1-\alpha)(R - p_L R_c) D_L(R_c, p_L, \delta) \quad (18)$$

where D_H, D_L denote the demand function from the healthy and not-so-healthy consumers, respectively; α denotes the proportion of healthy consumers; and R denotes a risk-free rate of return. The equilibrium rate of return R_c^* for the conventional annuity can therefore be solved from the equilibrium equation

$$\pi(R_c^*; p_H, p_L, \alpha, \delta) = 0. \quad (19)$$

If all the consumers have the same exponential utility with constant relative risk aversion δ as in Section 5, the demand function for the conventional annuity for consumer type H can be written as

$$D_H(R_c, p_H, \delta) = a_t^{c,H} = \frac{\log\left(\beta p_H R_c^2 / R_g\right) + \delta\left(1 - a_t^{g,H}\right)}{\delta\left(1 + R_c\right)}.$$

Similarly, the demand function for the conventional annuity for consumer type L is written as

$$D_L(R_c, p_H, \delta) = a_t^{c,L} = \frac{\log\left(\beta p_L R_c^2 / R_g\right) + \delta\left(1 - a_t^{g,L}\right)}{\delta\left(1 + R_c\right)}.$$

Now, substitute these demand functions into the profit equation (18), we would then have

$$\pi(R_c; p_H, p_L, \alpha, \delta) = \frac{1}{\delta\left(1 + R_c\right)} \times \left[\begin{array}{l} \alpha(R - p_H R_c)\left(\log\left(\beta p_H R_c^2/R_g\right) - \delta a_t^{g,H}\right) \\ +(1-\alpha)(R - p_L R_c)\left(\log\left(\beta p_L R_c^2/R_g\right) - \delta a_t^{g,L}\right) \\ +\delta R + \delta R_c\left(\alpha p_H + (1-\alpha) p_L\right) \end{array}\right],$$

so that the equilibrium return R_c^* can then be solved by setting the profit function to zero.

It is not straightforward to solve the equilibrium equation (19) without resorting to numerical computation. To demonstrate how to solve it numerically, we set risk aversion $\delta = 0.3$, the risk-free interest rate $R = 1.1$, the impatience or discount factor $\beta = 2.5$. Furthermore, assume that the proportion of healthy H consumers is 70% and that the survival probability of group H and L are 0.7 and 0.4, respectively. In Figure 5, we display the profit as a function of the return on the conventional annuity. As shown in this figure, there exists equlibirium return on the conventional annuity that clears the market and sets the insurance company's profit to zero. In this specific example, the equilibrium return turns out to be $R_c^* = 1.683432$.

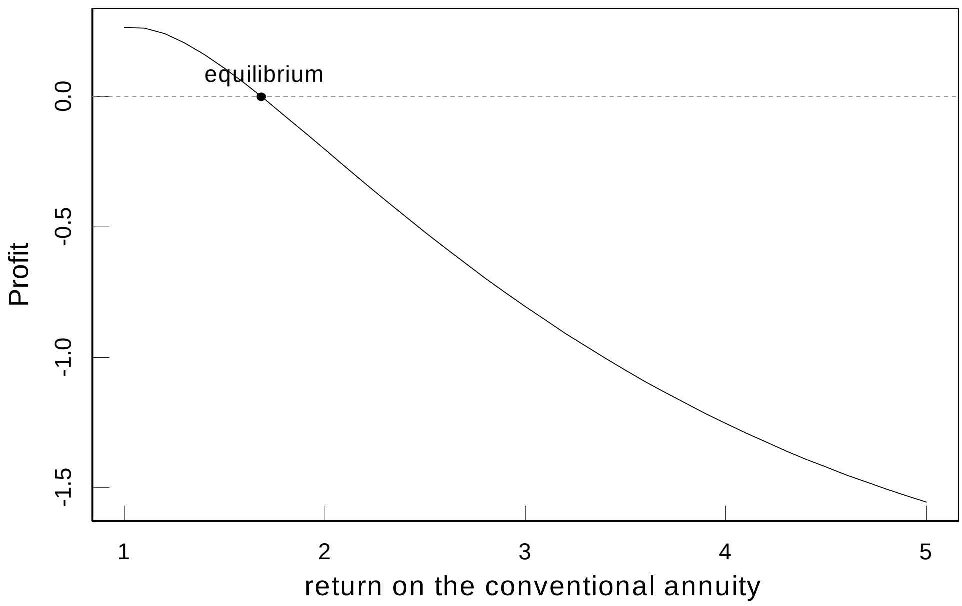

Figure 5: The Insurer's Profit Function in Determining Equilibrium Return

One can also examine the implications of changing some of the parameter values on the equlibrium return of the conventional annuity. For example, we examine the effect of α,the proportion of group H (healthy consumers) in the economy. Figure 6 displays this effect. As expected, the equilibrium return decreases when the α increases. This result reasonably coincides with that of Sinha's (1989) conclusion in a market without the pooled annuity account. The reason for this is that α has no effect on the demand of a conventional annuity as well as that of the pooled annuity account, but a rise in α increases the total annuity payments made by the insurer and therefore decreases the gain of the insurer at the same time. As a result, this reduces the investment return promised by the insurer so that the equilibrium return of the conventional annuity also reduces.

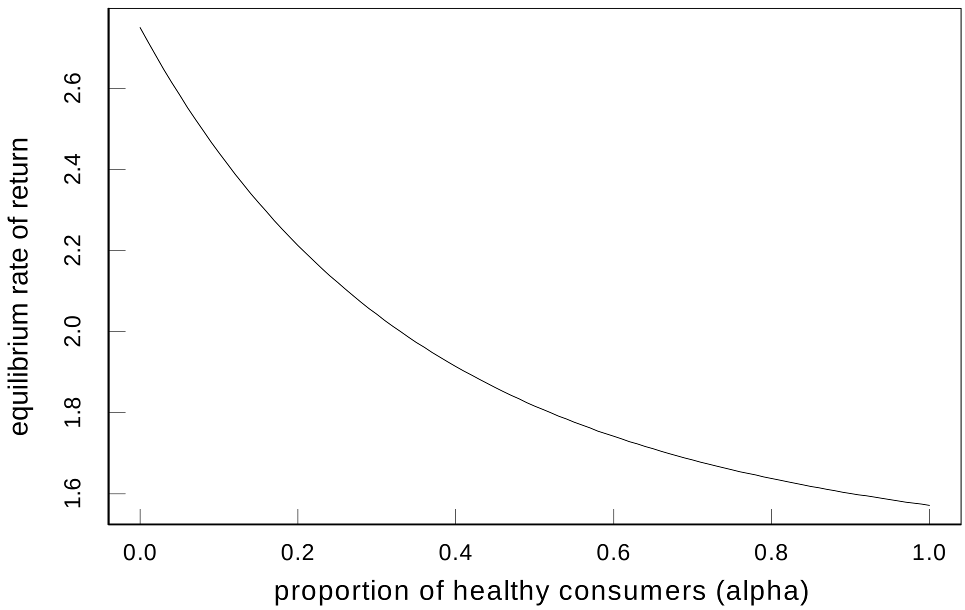

Figure 6: The Effect of Varying the Proportion α on the Equilibrium Return

7 Concluding Remarks

This paper examines demand and adverse selection in a market where consumers have the choice of purchasing either or both a conventional annuity and a pooled annuity account. The primary difference between these two types of annuity has to do with the idiosyncratic portion of longevity risks. Unlike that of a conventional annuity, the idiosyncratic portion of longevity risk is absorbed by the participants of a pooled annuity fund. In effect, the benefit payments are adjusted for each period to reflect deviations of actual experience to that expected. The benefit payouts are therefore random variables, and this paper reflects these adjustments by an adjustment random variable Y which is then used to reflect the uncertainty in the payout from the pooled annuity account. In determining demand then for these annuities, we develop models similar to Abel (1986) and Sinha (1989) which maximizes the individual expected utility. We find that demand for both types of annuity involves a function of the survival probability p which is subjectively known only to the consumer. Now, in assessing the possible presence of adverse selection, we examined how these demand functions change when there are possible changes to these subjective probabilities p. What we find is that there continues to be presence of adverse selection in the aggregate, that is, if the consumer believes there is increase in life expectancy, or equivalently in survival probability, then the consumer will demand a greater amount of annuities in the aggregate. However, there is ambiguity as to how the separate demands for the conventional and the pooled annuity funds change. What is unambiguous is the result that when there is presence of adverse selection in each of the two types of annuity, there is usually a lower level of adverse selection present in the pooled annuity account. This result further justifies offering pooled annuity accounts into the market.

Appendix A: Proof of negative semi-definite hessian

For the solution to be considered optimal, consider the Hessian matrix expressed by

$$H = \begin{pmatrix} \dfrac{\partial^2 U}{\partial a_t^{c2}} & \dfrac{\partial^2 U}{\partial a_t^c \partial a_t^g} \\ \\ \dfrac{\partial^2 U}{\partial a_t^c \partial a_t^g} & \dfrac{\partial^2 U}{\partial a_t^{g2}} \end{pmatrix}$$

where here U is simplified to denote the expected utility expressed in (10). This Hessian matrix must be negative semi-definite. To prove that H is indeed negative semi-definite, we take any non-negative vector $\mathbf{x}^T = (x_1, x_2)$ and show that

$$\mathbf{x}^T H \mathbf{x} = \frac{\partial^2 U}{\partial a_t^{c2}} x_1^2 + 2 \frac{\partial^2 U}{\partial a_t^c \partial a_t^g} x_1 x_2 + \frac{\partial^2 U}{\partial a_t^{g2}} x_2^2 \leq 0. \tag{20}$$

Note that we have

$$\frac{\partial^2 U}{\partial a_t^{c^2}} = u''(C_t) + \beta p R_c^2 E\left[u''(C_{t+1})\right],$$

$$\frac{\partial^2 U}{\partial a_t^{g^2}} = u''(C_t) + \beta p R_g^2 E[u''(C_{t+1}) Y^2],$$

and

$$\frac{\partial^2 U}{\partial a_t^c \partial a_t^g} = \frac{\partial^2 U}{\partial a_t^g \partial a_t^c} = u''(C_t) + \beta p R_c R_g E[u''(C_{t+1}) Y].$$

Thus, we can immediately show that $\mathbf{x}^T H \mathbf{x}$ simplifies to

$$\mathbf{x}^T H \mathbf{x} = 2u''(C_t)(x_1 + x_2)^2 + \beta p E[u''(C_{t+1})(R_c x_1 + R_g Y x_2)^2]$$

for which it is always non-positive because $u'' < 0$. Therefore, H is negative semi-definite.

Appendix B: Derivation of partial derivatives

In this appendix, we show the detailed calculation of the partial derivatives $\partial a_t^c / \partial p$ and $\partial a_t^g / \partial p$ that are used to assess the presence of adverse selection. We first suppose that the optimal solution to the expected utility maximization exists so that the first-order conditions (12) and (13) are satisfied.

Re-writing (12) as $u'(C_t) = \beta p R_c E[u'(C_{t+1})]$ and differentiating this implicitly both sides of the equation with respect to p, we would have

$$\begin{aligned} & u''(C_t) \left(\frac{\partial C_{t+1}}{\partial a_t^c} \frac{\partial a_t^c}{\partial p} + \frac{\partial C_{t+1}}{\partial a_t^g} \frac{\partial a_t^g}{\partial p} \right) \\ = \; & \beta R_c \left\{ E\left[u'(C_{t+1})\right] + pE\left[u''(C_{t+1}) \left(\frac{\partial C_{t+1}}{\partial a_t^c} \frac{\partial a_t^c}{\partial p} + \frac{\partial C_{t+1}}{\partial a_t^g} \frac{\partial a_t^g}{\partial p} \right) \right] \right\}. \end{aligned}$$

We know that $\dfrac{\partial C_t}{\partial a_t^c} = \dfrac{\partial C_t}{\partial a_t^g} = -1$, $\dfrac{\partial C_{t+1}}{\partial a_t^c} = R_c$, and $\dfrac{\partial C_{t+1}}{\partial a_t^g} = R_g Y$, so that this leads us to

$$-u''(C_t)\left(\frac{\partial a_t^c}{\partial p} + \frac{\partial a_t^g}{\partial p}\right) = \beta R_c \left\{E\left[u'(C_{t+1})\right] + pE\left[u''(C_{t+1})\left(R_c \frac{\partial a_t^c}{\partial p} + R_g Y \frac{\partial a_t^g}{\partial p}\right)\right]\right\}. \tag{21}$$

Now, from (14) therefore, it also follows by differentiating implicitly both sides of the equation with respect to p that we have

$$R_c E\left[u'(C_t)\left(\frac{\partial C_{t+1}}{\partial a_t^c}\frac{\partial a_t^c}{\partial p} + \frac{\partial C_{t+1}}{\partial a_t^g}\frac{\partial a_t^g}{\partial p}\right)\right] = R_G E\left[u'(C_t)\, Y\left(\frac{\partial C_{t+1}}{\partial a_t^c}\frac{\partial a_t^c}{\partial p} + \frac{\partial C_{t+1}}{\partial a_t^g}\frac{\partial a_t^g}{\partial p}\right)\right].$$

By further simplifying this, we derive the ratios of the partial derivatives

$$\begin{aligned}
\frac{\partial a_t^c/\partial p}{\partial a_t^g/\partial p} &= \frac{R_g^2 E\left[u''(C_{t+1})\, Y^2\right] - R_c R_g E\left[u''(C_{t+1})\, Y\right]}{R_c^2 E\left[u''(C_{t+1})\right] - R_c R_g E\left[u''(C_{t+1})\, Y\right]} \\
&= -\frac{R_g}{R_c}\frac{E\left[u''(C_{t+1})\, Y(R_c - R_g Y)\right]}{E\left[u''(C_{t+1})(R_c - R_g Y)\right]}
\end{aligned} \tag{22}$$

Now combining (21) and (22), it can be shown that

$$\frac{\partial a_t^c}{\partial p} = -\frac{\beta R_c E\left[u'(C_{t+1})\right] \times R_g E\left[u''(C_{t+1})\, Y(R_c - R_g Y)\right]}{u''(C_t) \times E\left[u''(C_{t+1})(R_c - R_g Y)^2\right]} \tag{23}$$

and that

$$\frac{\partial a_t^g}{\partial p} = -\frac{\beta R_c E\left[u'(C_{t+1})\right] \times R_c E\left[u''(C_{t+1})(R_c - R_g Y)\right]}{u''(C_t) \times E\left[u''(C_{t+1})(R_c - R_g Y)^2\right]}. \tag{24}$$

By defining the terms

$$A = R_g E\left[u''(C_{t+1})\, Y(R_c - R_g Y)\right],$$

$$B = R_c E\left[u''(C_{t+1})(R_c - R_g Y)\right],$$

and

$$C = -\beta R_c E\left[u'(C_{t+1})\right],$$

we notice that

$$A + B = E\left[u''(C_{t+1})(R_c - R_g Y)^2\right]$$

so that in fact, the partial derivaties in (23) and (24) can be succinctly written as

$$\frac{\partial a_t^c}{\partial p} = \frac{C \times A}{u''(C_t) \times (A + B)} \tag{25}$$

and that

$$\frac{\partial a_t^g}{\partial p} = \frac{C \times B}{u''(C_t) \times (A + B)}. \tag{26}$$

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