Enhancing insurer value through reinsurance optimization ⋆

Yuriy Krvavych [a,1], Michael Sherris [a,1]

[a] *Actuarial Studies, Faculty of Commerce and Economics, University of New South Wales, Sydney 2052, Australia*

Working Paper: August 3, 2004 (final version)

Abstract

The paper investigates the demand for change-loss reinsurance in insurer risk management. It is assumed that the insurer's objective is to maximize shareholder value under a solvency constraint imposed by a regulatory authority. In a one period model of a regulated market where the required solvency level is fixed, an insurer can maintain this level by two control variables: reinsurance and risk capital supplied by shareholders. Two alternatives are considered in the paper. In the first one (conservative model) the required risk capital is determined at the beginning of the period and does not depend on the reinsurance decision. In the second model insurers can reduce the required minimal level of the risk capital taking into account the purchase of reinsurance. It is shown that the first model does not create a demand for reinsurance in a frictionless market, however, there is demand for reinsurance in this model under the presence of corporate tax and financial distress costs. An optimal tradeoff between the required minimal level of the risk capital and purchase of reinsurance occurs in the second model under our assumptions that gross premiums are not dependent on capital or reinsurance of the insurer.

Keywords: optimal reinsurance, change-loss reinsurance, risk capital, costs of financial distress
JEL classification: C61, G22, D81

1 Introduction

Many studies of reinsurance optimization in the classical actuarial literature assume that the insurer objective is to minimize its ruin probability. This assumption is unrealistic from the point of view of the modern theory of integrated risk management for an insurance company, since it focuses on risk minimization only without any explicit regard to the company's economic value. Other more recent studies (e.g. see Taksar (2000 [14]) and references therein) that maximize the expectation of the discounted future dividends (company's value), paid by an insurer to its shareholders, allowing for reinsurance do not take into consideration frictional costs such as corporate tax and costs of financial distress.

In this article, we study the demand for reinsurance in a single period model in the presence of corporate tax and cost of financial distress. In order to construct an objective function of an

⋆ This work was supported by Australian Research Council Discovery Grant DP0345036.
1 *E-mail addresses:* y.krvavych@lviv-actuaries.org (Y. Krvavych), m.sherris@unsw.edu.au (M. Sherris)

insurer we should first understand the nature and economic aspects of an insurance business. In contrast to industrial companies, insurers do not generally leverage themselves via capital markets. They collect insurance premiums (borrow money) by issuing debt in the form of insurance policies that pay the policyholders (lenders) compensation (financial benefits) if pre-specified events occur. To create and then issue insurance contracts, insurers rely on diversification and financial markets. By pooling contracts that are not perfectly correlated, aggregate losses become more predictable. While pooling reduces uncertainty, unexpected losses may still arise, potentially jeopardizing the insurer's ability to meet its obligations. On the other hand, unlike bondholders who can effectively reduce their credit risk exposure by holding a well-diversified portfolio of bonds with different issuers, policyholders generally cannot mitigate insurer default risk in any cost-efficient way. Therefore, policyholders usually accumulate their "credit" exposure with insurers, the financial strength of which is assessed by rating agencies and/or regulators. Insurers satisfy regulatory requirements on solvency/security by holding risk capital in addition to operating capital including a component of premium income. We define the premium, net of administrative expenses, as consisting of the expected value of claim loss and risk loading (or risk premium).

According to Daykin et al (1996 (pp.156-157) [5]), Nakada et al (1999 [11]) and SOA Economic Capital Calculation and Allocation Subgroup (2003 [13]) the sum of the risk premium and risk capital determines the value of economic capital. Like industrial companies, insurers are financed by their principals (shareholders)(see Brealey and Myers (1999 [2]), and Culp (2002 [3])). Shareholders of insurance companies supply risk capital ("equity capital" or "surplus") that is invested on their behalf in financial assets. In so doing, shareholders expect to earn a fair return on invested risk capital. Insurers create shareholders' value through investment in assets and borrowing in the insurance market, rather than in capital markets. It should be noticed here that in the presence of frictional costs such as corporate and individual taxes (double taxation) [2] it is more costly for insurers to create value from investment in financial markets compared to direct investment funds. However, insurers have a competitive advantage in creating value by borrowing in the insurance market since they can directly manage the moral hazard and adverse selection costs of insurance risks. Self risk-pooling arrangements are costly and insurance contracts provide an efficient means of lowering these costs.

We assume that one of the main insurer's objectives is to maximize shareholders value under solvency constraints imposed by a Regulatory Authority. An insurer may traditionally improve its solvency level or reduce insolvency risk, which captures both undesirable large fluctuations and ruin probability (see Gerber (1979 [7]), and Hürlimann (1993 [8])), by buying reinsurance to reduce the unexpected fluctuations in the insurance losses of the insurer. In a regulated insurance market when the solvency level, required by a Regulatory Authority, is fixed the insurer can maintain this level by two control variables: reinsurance and risk capital. In fact there are at least two possibilities to do this: 1) (model M1) risk capital is independent of the future possible buying of reinsurance, and is at least the required minimum value of the risk capital determined at the beginning of the period without reinsurance; 2) (model M2) the required minimum value of the risk capital is determined taking into account future purchasing of reinsurance.

While the first possibility gives two independent control variables, the second one leaves only

[2] The problem of double-taxation in insurance is not present in every country. For instance, in Australia it is reduced due to the use of "tax imputation system", according to which individual shareholders who receive assessable dividends from a company are entitled to a credit for the tax paid by the company on its income.

one control variable, reinsurance, since the required capital is explicitly dependent of the ceding amount of insurance, and thus can be expressed through the reinsurance control variable. Under the second possibility, purchasing reinsurance will normally decrease required risk capital, and decreasing risk capital will increase the demand for reinsurance. By considering shareholders of the insurance company as residual claimants it is natural to consider the measure of performance of risk capital defined as the ratio of the expected payoff to shareholders, allowing for limited liability, to the invested risk capital. And thus an insurer's objective is to maximize this ratio. It is worth noticing that the proposed "return on risk capital" (RRC) is different from the well known "risk adjusted return on capital" (RAROC, e.g. see Nakada et al (1999 [11])), which is defined as a ratio: (risk premium plus investment return) divided by (economic capital). So, RAROC is the type of measure of capital performance that adjusts the returns of an insurer (or usually bank) for risk and expresses this in relation to economic capital (risk premium plus risk capital) employed.
In this article, we study the demand for change-loss reinsurance contracts in single period models M1 and M2 in the presence of corporate tax and costs of financial distress.

2 Demand for reinsurance in shareholder's value creation: one-period frictionless model

Consider a single period model. Let X denote the random aggregate claims of an insurer portfolio $\mathcal{P}_X$ and let X have the distribution function $F(x)$, $x \geq 0$ defined in the probability space $(\Omega, \mathbb{P})$. It is assumed that in perfectly competitive insurance and reinsurance markets, insurers are subject to the risk of insolvency, however to simplify the analysis we assume reinsurers are not subject to the risk of insolvency. At the beginning of the period an insurer should satisfy solvency conditions required by a Regulatory Authority, i.e. an insurer should hold an amount of capital (risk capital), in addition to the premium income (operating capital), such that the insurer's survival probability is equal to, say, α (usually in practice $\alpha \in [0.95, 0.999]$). We will define the measure of required risk capital using value-at-risk (VaR) of X in the following way.

Definition 2.1 *Given some confidence level $\alpha \in (0,1)$, the value-at-risk (VaR) of a portfolio $\mathcal{P}_X$ at the confidence level α is given by the smallest number x such that the probability that the loss X exceeds x is less than or equal to $1-\alpha$:*

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} | \mathbb{P}[X > x] \leq 1 - \alpha\}$$

The total premiums collected at the beginning of the period is equal to $P = (1+\theta)\mathbb{E}[X]$, where $\theta > 0$ is the insurer's risk loading. Note P does not depend on capital or reinsurance (does not allow for liability put). It is assumed that there are no investment earnings. Then the risk capital required by the Regulatory Authority is an amount of capital u such that

$$\mathbb{P}[\{u + P - X > 0\}] \geq \alpha.$$

Therefore,

$$u \geq u_{\min} = \mathrm{VaR}_\alpha[X] - (1+\theta)\mathbb{E}[X].$$

Without reinsurance the return on the risk capital provided by shareholders is equal to

$$\rho(u) = \frac{\mathbb{E}\max\{0, u + P - X\}}{u} - 1.$$

When an insurer takes reinsurance it reduces the premium income, the variance and the value-at-risk of transformed claims (i.e. the value of $u_{\min} + P$ after reinsurance). The main goal of this section is to investigate whether there is a demand for reinsurance in maximizing the return on risk capital. We consider the class of change-loss reinsurance contracts

$$\mathcal{J} = \{J_{a,b}(\cdot) | J_{a,b}(X) = a(X - b)_+,\ a \in [0,1],\ b \in [0,\infty)\}.$$

This class of exogenously pre-specified reinsurance contracts includes ordinary quota share (proportional) and stop-loss (or excess of loss) reinsurance. If $a = 1$ we have stop-loss reinsurance, and if $b = 0$ we have proportional quota share reinsurance. We will investigate the demand for change-loss reinsurance in the following two models:

M1) *two control variables (risk capital and reinsurance)*:

$$\begin{array}{ll} \textbf{maximize} & 1 + \rho(u; a, b) = \frac{\mathbb{E}\max\{0, u + P(a,b) - (X - J_{a,b}(X))\}}{u}, \\ \textbf{subject to} & u \geq u_{\min} \text{ and } (a,b) \in [0,1] \times [0,\infty), \end{array} \tag{1}$$

M2) *one control variable (reinsurance)*:

$$\begin{array}{ll} \textbf{maximize} & 1 + \rho(a, b) = \frac{\mathbb{E}\max\{0, u_{\min}(a,b) + P(a,b) - (X - J_{a,b}(X))\}}{u_{\min}(a,b)}, \\ \textbf{subject to} & (a,b) \in [0,1] \times [0,\infty), \end{array} \tag{2}$$

where $u_{\min}(a, b)$ and $P(a, b)$ are corresponding values of the required minimal risk capital and premium income after reinsurance. The first model is conservative to some extent. It does not allow the insurer to reduce the required minimal risk capital after purchasing reinsurance below the level of required minimal risk capital determined at the beginning of period. However, the direct insurer can change both initial risk capital and the parameters of the change-loss reinsurance to achieve a maximum return ρ on risk capital. In the second model it is assumed that the insurer is allowed to alter (reduce) the required minimal risk capital by taking reinsurance, and moreover, after taking reinsurance the cedent holds exactly such amount of risk capital ($u_{\min}(a, b)$) that just satisfies minimum solvency requirements.

We will denote the retained risk by $I_{a,b}(X) = X - J_{a,b}(X)$.

2.1 Maximizing return on risk capital by reinsurance and risk capital

Consider the model M1 of "reinsurance-risk capital" optimization

$$\begin{cases} \max\limits_{u,\,a,\,b} \frac{\mathbb{E}\max\{0, u+P(a,b)-I_{a,b}(X)\}}{u}, \\ u \geq u_{\min},\ (a,b) \in [0,1]\times[0,\infty), \end{cases} \tag{3}$$

where $u_{\min} = \mathrm{VaR}_\alpha[X] - (1+\theta)\mathbb{E}[X]$. After taking reinsurance from the class $\mathcal{J}$ the cedent's premium income becomes

$$P(a,b) = P - (1+\eta)\mathbb{E}[J_{a,b}(X)] = (1+\theta)\mathbb{E}[X] - (1+\eta)a\mathbb{E}[(X-b)_+],$$

where $\eta > 0$ is the reinsurer's risk loading. We assume that $\eta > \theta$, i.e. reinsurance loading is higher because it corresponds to a riskier loss. It is a reasonable assumption since it follows from empirical arbitrage constraints imposed by arbitrage avoidance (see Venter (1991 [15])):
1) additivity;
2) a premium calculation principle should produce a higher risk loading, relative to expected losses, for an excess of loss cover than for a primary cover on the same risks.

One of the principles that can meet the above constraints is the mean value premium principle applied to an adjusted (distorted) probability distribution. In our case we have

$$P = (1+\theta)\mathbb{E}_F[X] = \mathbb{E}_G[X],$$

where $G(x) = F(kx)$ and $k = \frac{1}{1+\theta} \in (0.1)$ is a risk adjustment coefficient. Assuming a certain value of risk adjustment coefficient k we can properly determine risk loading for the reinsurer from the following equation

$$\begin{aligned}(1+\eta)\mathbb{E}_F[a(X-b)_+] = \mathbb{E}_G[a(X-b)_+] = \int_b^\infty a(x-b)dG(x) \\ = a\int_b^\infty (1-F(kx))dx = \frac{a}{k}\int_{bk}^\infty (1-F(x))dx.\end{aligned} \tag{4}$$

From (4) we obtain

$$1+\eta(b,\theta) = \frac{1}{k}\frac{\int_{bk}^\infty (1-F(x))dx}{\int_b^\infty (1-F(x))dx} > \frac{1}{k} = 1+\theta.$$

It is obvious that the premium income $P(a,b)$ is positive for all $a \in [0,1]$ and $b \in [0,\infty]$, indeed

$$
\begin{aligned}
P(a,b) &= (1+\theta)\mathbb{E}[X] - (1+\eta(b,\theta))a\mathbb{E}(X-b)_+ \\
&= (1+\theta)\left(\mathbb{E}[X] - a\int\limits_{\frac{b}{1+\theta}}^{\infty}(1-F(x))dx\right) > 0.
\end{aligned} \tag{5}
$$

The cedent's after-reinsurance surplus is equal to

$$
\begin{aligned}
S(u_{\min}+u_1;a,b) &= u_{\min} + u_1 + P(a,b) - I_{a,b}(X)) \\
&= \mathrm{VaR}_\alpha[X] - P + u_1 + P(a,b) - I_{a,b}(X),
\end{aligned}
$$

where $u_1 \geq 0$ is the excess of required minimal risk capital, such that total risk capital is equal to $u = u_{\min} + u_1$.

It follows from this that the return on risk capital after reinsurance is the following function of u_1, a and b

$$
\rho(u_{\min}+u_1;a,b) = \frac{\mathbb{E}\max\{0, S(u_{\min}+u_1;a,b)\}}{\mathrm{VaR}_\alpha[X] - P + u_1} - 1,
$$

and our goal is to find u_1^*, a^* and b^* such that

$$
\rho(u_{\min}+u_1^*;a^*,b^*) = \max_{u_1\in[0,\infty),\, a\in[0,1],\, b\in[0,\infty)} \rho(u_{\min}+u_1;a,b).
$$

We first find the distribution of retained risk $I_{a,b}(X)$ in order to calculate the risk capital and the expected value of the cedent's limited liability.

Consider the case where $a \neq 1$.

$$
\begin{aligned}
F_{I_{a,b}}(y) &= \mathbb{P}[X - a(X-b)_+ \leq y] = \mathbb{P}[\{X - a(X-b)_+ \leq y\} \cap \Omega] \\
&= \mathbb{P}[\{X - a(X-b)_+ \leq y\} \cap \{X > b\}] + \mathbb{P}[\{X - a(X-b)_+ \leq y\} \cap \{X \leq b\} \\
&= \mathbb{P}[\{(1-a)X + ab \leq y\} \cap \{X > b\}] + \mathbb{P}[\{X \leq y\} \cap \{X \leq b\}] \\
&= \mathbb{P}\left[\left\{X \leq \frac{y-ab}{1-a}\right\} \cap \{X > b\}\right] + \mathbb{P}[X \leq \min\{y,b\}] \\
&= \mathbf{1}_{\left\{b < \frac{y-ab}{1-a}\right\}} \mathbb{P}\left[b < X \leq \frac{y-ab}{1-a}\right] + F(\min\{y,b\}) \\
&= \mathbf{1}_{\left\{b < \frac{y-ab}{1-a}\right\}} \left(F\left(\frac{y-ab}{1-a}\right) - F(b)\right) + F(\min\{y,b\}) \\
&= \begin{cases} F(y), & y < b; \\ F\left(\frac{y-ab}{1-a}\right), & y \geq b. \end{cases},
\end{aligned}
$$

where $\mathbf{1}_A$ is indicator of an event A. As we can see under $a < 1$ the distribution function $F_{I_{a,b}}$ is continuous. In the trivial case, when $a = 1$ (stop-loss reinsurance) we have

$$F_{I_{1,b}}(y) = \mathbf{1}_{\{b<y\}}\,(1-F(b)) + F(\min\{y,b\}) = \begin{cases} F(y), & y<b; \\ 1, & y \ge b. \end{cases}$$

Theorem 2.1 *There is no demand for change-loss reinsurance in the model M1, and moreover*

$$\{u^* = u_{\min};\ a^* = 0 \vee b^* = \infty\} = \arg\left\{\max_{u\in[u_{\min},\infty),\, a\in[0,1],\, b\in[0,\infty)} \rho(u;a,b)\right\}.$$

Proof. Let us calculate the cedent's expected terminal value.

$$\begin{aligned}
\mathbb{E}\left[0,\, S(u_{\min}+u_1;a,b)\right] &= \mathbb{E}\left[0,\, \mathrm{VaR}_\alpha[X] - P + u_1 + P(a,b) - I_{a,b}(X)\right] \\
&= \mathbb{E}\left[0,\, \mathrm{VaR}_\alpha[X] + u_1 - (1+\eta(b,\theta))a\int_b^\infty (1-F(x))dx - I_{a,b}(X)\right] \\
&= \int_0^{\delta(u_1,\,a,\,b)} (\delta(u_1,\,a,\,b)-y)dF_{I_{a,b}}(y),
\end{aligned} \tag{6}$$

where $\delta(u_1,\,a,\,b) = \mathrm{VaR}_\alpha[X] + u_1 - (1+\eta(b,\theta))a\int_b^\infty (1-F(x))dx > \mathrm{VaR}_\alpha[X] - P > 0$.

In order to calculate the latter integral we will use a result from probability.

Lemma 1 *For any random variable Z with continuous and almost everywhere differentiable cdf G the following equality holds*

$$\forall c\in[0,\infty) \quad \int_0^c (c-z)dG(z) = \int_0^c G(z)dz.$$

So, for $a<1$

$$\begin{aligned}
&\int_0^{\delta(u_1,\,a,\,b)} (\delta(u_1,\,a,\,b)-y)dF_{I_{a,b}}(y) = \int_0^{\delta(u_1,\,a,\,b)} F_{I_{a,b}}(y)dy \\
&= \mathbf{1}_{\{\delta(u_1,\,a,\,b)<b\}} \int_0^{\delta(u_1,\,a,\,b)} F(y)dy + \mathbf{1}_{\{\delta(u_1,\,a,\,b)\ge b\}}\left(\int_0^b F(y)dy + \int_b^{\delta(u_1,\,a,\,b)} F\left(\frac{y-ab}{1-a}\right)dy\right) \\
&= \mathbf{1}_{\{\delta(u_1,\,a,\,b)<b\}} \int_0^{\delta(u_1,\,a,\,b)} F(y)dy + \mathbf{1}_{\{\delta(u_1,\,a,\,b)\ge b\}}\left(\int_0^b F(y)dy + (1-a)\int_b^{\frac{\delta(u_1,\,a,\,b)-ab}{1-a}} F(y)\,dy\right).
\end{aligned}$$

For every fixed u_1 and b we have

$$\frac{\partial}{\partial a}\int\limits_0^{\delta(u_1,\,a,\,b)} F(y)dy = -(1+\eta(b,\theta))F(\delta(u_1,\,a,\,b))\int\limits_b^{\infty}(1-F(y))dy < 0$$

and

$$\frac{\partial}{\partial a}\left(\int\limits_0^{b} F(y)dy + (1-a)\int\limits_b^{\frac{\delta(u_1,\,a,\,b)-ab}{1-a}} F\left(y\right)dy\right) = (1-a)\frac{\partial}{\partial a}\left(\frac{\delta(u_1,\,a,\,b)-ab}{1-a}\right)$$

$$\times F\left(\frac{\delta(u_1,\,a,\,b)-ab}{1-a}\right) - \int\limits_b^{\frac{\delta(u_1,\,a,\,b)-ab}{1-a}} F(y)dy = \frac{\delta(u_1,\,a,\,b)-b}{1-a}F\left(\frac{\delta(u_1,\,a,\,b)-ab}{1-a}\right)$$

$$-\int\limits_b^{\frac{\delta(u_1,\,a,\,b)-ab}{1-a}} F(y)dy - (1+\eta(b,\theta))F\left(\frac{\delta(u_1,\,a,\,b)-ab}{1-a}\right)\int\limits_b^{\infty}(1-F(y))dy$$

$$\leq F\left(\frac{\delta(u_1,\,a,\,b)-ab}{1-a}\right)\left(\frac{\delta(u_1,\,a,\,b)-b}{1-a} - \int\limits_b^{\frac{\delta(u_1,\,a,\,b)-ab}{1-a}} F(y)dy - (1+\eta(b,\theta))\int\limits_b^{\infty}(1-F(y))dy\right)$$

$$= F\left(\frac{\delta(u_1,\,a,\,b)-ab}{1-a}\right)\left(\int\limits_b^{\frac{\delta(u_1,\,a,\,b)-ab}{1-a}}(1-F(y))dy - (1+\eta(b,\theta))\int\limits_b^{\infty}(1-F(y))dy\right)$$

$$= F\left(\frac{\delta(u_1,\,a,\,b)-ab}{1-a}\right)\left(\int\limits_b^{\frac{\delta(u_1,\,a,\,b)-ab}{1-a}}(1-F(y))dy - (1+\theta)\int\limits_{\frac{b}{1+\theta}}^{\infty}(1-F(y))dy\right) < 0.$$

The latter inequality holds, since for any $\theta > 0$ $\left(b, \frac{\delta(u_1,\,a,\,b)-ab}{1-a}\right) \subset \left(\frac{b}{1+\theta}, \infty\right)$. Therefore, the function $\rho(\cdot\,,\,a,\,\cdot)$ decreases on $[0,1)$. So, for every fixed excess of required minimal risk capital $u_1 \geq 0$ the return ρ on risk capital takes its maximum value on $[0,1)$ when $a = a^* = 0$ or equivalently when $b = b^* = \infty$. Moreover, the integral in (6) is a continuous function of a on $[0,1]$, since

$$\lim_{a\to 1}\int\limits_0^{\delta(u_1,\,a,\,b)}(\delta(u_1,\,a,\,b)-y)dF_{I_{a,b}}(y) = \int\limits_0^{\delta(u_1,\,1,\,b)}(\delta(u_1,\,1,\,b)-y)dF_{I_{1,b}}(y).$$

We can verify the latter equality by considering the following two cases:

1) If for some fixed b $\delta(u_1,1,b) < b$ then $\exists a_0 > 0$ such that $\forall a > a_0$ $\delta(u_1,a,b) < b$ and

$$\int\limits_0^{\delta(u_1,\,a,\,b)}(\delta(u_1,\,a,\,b)-y)dF_{I_{a,b}}(y) = \int\limits_0^{\delta(u_1,\,a,\,b)}(\delta(u_1,\,a,\,b)-y)dF(y)$$

$$= \int_0^{\delta(u_1,a,b)} F(y)dy \to \int_0^{\delta(u_1,1,b)} F(y)dy, \quad \text{when} \quad a \to 1.$$

2) If for some fixed b $\delta(u_1,1,b) \geq b$ then $\forall a$ $\delta(u_1,a,b) \geq b$ and

$$\lim_{a\to1} \int_0^{\delta(u_1,a,b)} F_{I_{a,b}}(y)dy = \lim_{a\to1} \left\{ \int_0^b F(y)dy + (1-a) \int_b^{\frac{\delta(u_1,a,b)-ab}{1-a}} F(y)dy \right\}$$

$$= \lim_{a\to1} \left\{ \int_0^b F(y)dy + \frac{\int_b^K F(y)dy}{\frac{1}{1-a}} \right\},$$

where $K = \frac{1}{1-a}\left(\mathrm{VaR}_\alpha[X] + u_1 - a\left((1+\eta(b,\theta))\int_b^\infty (1-F(x))dx + b\right)\right)$. The latter limit can be found using the L'Hôpital rule.

So,

$$\lim_{a\to1} \int_0^{\delta(u_1,a,b)} F_{I_{a,b}}(y)dy = \lim_{a\to1} \left\{ \int_0^b F(y)dy + \frac{\delta(u_1,\,1,\,b)-b}{(1-a)^2} \frac{F\left(\frac{\delta(u_1,a,b)-ab}{1-a}\right)}{\frac{1}{(1-a)^2}} \right\}$$

$$= \int_0^b F(y)dy + \delta(u_1,\,1,\,b) - b = \int_0^b (\delta(u_1,\,1,\,b)-y)dF(y) + (\delta(u_1,\,1,\,b)-b)(1-F(b))$$

$$= \int_0^{\delta(u_1,1,b)} (\delta(u_1,\,1,\,b)-y)dF_{I_{1,b}}(y).$$

Therefore, the function $\rho(\cdot,\,a,\,\cdot)$ decreases on $[0,1]$, and there is no demand for reinsurance.

Finally

$$\frac{\partial}{\partial u_1}\rho(u_{\min}+u_1,0,\infty) = \frac{\partial}{\partial u_1} \frac{\int_0^{\mathrm{VaR}_\alpha[X]+u_1} F(y)dy}{\mathrm{VaR}_\alpha[X]+u_1-(1+\theta)\mathbb{E}[X]}$$

$$= \frac{F\left(\mathrm{VaR}_\alpha[X]+u_1\right)\left(\mathrm{VaR}_\alpha[X]+u_1-(1+\theta)\mathbb{E}[X]\right) - \int_0^{\mathrm{VaR}_\alpha[X]+u_1} F(y)dy}{\left(\mathrm{VaR}_\alpha[X]+u_1-(1+\theta)\mathbb{E}[X]\right)^2}$$

$$\leq \frac{F\left(\mathrm{VaR}_\alpha[X]+u_1\right)\left(\int_0^{\mathrm{VaR}_\alpha[X]+u_1}(1-F(y))dy - (1+\theta)\int_0^\infty(1-F(y))dy\right)}{\left(\mathrm{VaR}_\alpha[X]+u_1-\mathbb{E}[X]\right)^2} < 0,$$

and we conclude that the return ρ on risk capital attains its maximum value

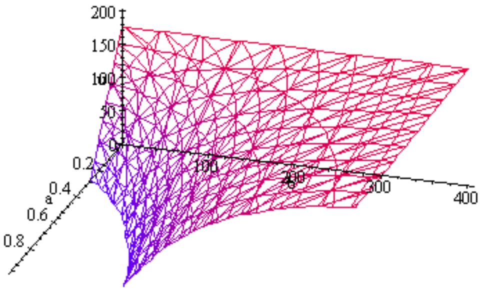


Fig. 1. *Graphical illustration of excess of required minimal risk capital* $u_1(a,b)$ *under fixed level* 10% *of return on equity: exponential case*

$$\frac{\int\limits_0^{\mathrm{VaR}_\alpha[X]} F(y)dy}{\mathrm{VaR}_\alpha[X]-P} - 1 = \frac{P - \int\limits_0^{\mathrm{VaR}_\alpha[X]} (1-F(y))dy}{\mathrm{VaR}_\alpha[X]-P} > 0$$

at $(u^* = u_{\min}; a^* = 0 \vee b^* = \infty)$.

$\square$

Even for simple forms of distribution function F it is impossible to express u_1 as a simple function of a and b. However we can use some numerical examples to explicitly plot the excess $u_1(a,b)$ of the required minimal risk capital. Here we consider two examples:
1) aggregate loss X is exponentially distributed with cdf $F(x) = 1 - \mathrm{e}^{-0.01x}$ (light tail distribution);
2) aggregate loss X is Pareto distributed with cdf $F(x) = 1 - \left(\frac{100}{100+x}\right)^2$ (heavy tail distribution);
For both distributions the mean is equal to 100. It is assumed for these examples that the required solvency level is $\alpha = 97.5\%$, the insurer's risk loading $\theta = 0.4$ (i.e. risk adjustment coefficient $k = 0.7143$). In the Figures 1 and 2 we can see the surface of all indifference points $(u_1,\ a,\ b)$ under which the return on equity is the same fixed value (10%). We can see that the less change-loss reinsurance the cedent takes (a decreases or/and b increases) more risk capital is needed to provide the predetermined fixed return, and vice versa.

2.2 Maximizing return on risk capital by reinsurance.

Consider the model M2 of "reinsurance" optimization

$$\begin{cases} \max\limits_{a,\,b} \frac{\mathbb{E}\max\{0, u_{\min}(a,b)+P(a,b)-(X-J_{a,b}(X))\}}{u_{\min}(a,b)}, \\ (a,b) \in [0,1]\times[0,\infty), \end{cases} \tag{7}$$

where the required minimal risk capital

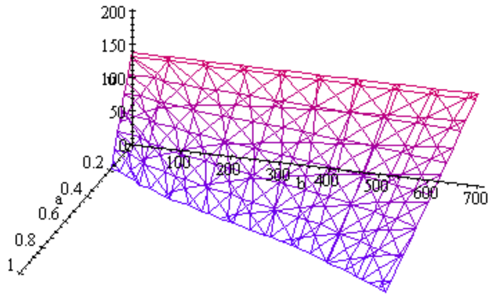


Fig. 2. *Graphical illustration of excess of required minimal risk capital $u_1(a,b)$ under fixed level 10% of return on equity: Pareto case*

$$u_{\min}(a,b) = \mathrm{VaR}_\alpha[X - J_{a,b}(X)] - P(a,b),$$

In this model the direct insurer is allowed, under a fixed solvency level, to reduce minimal risk capital by taking into account the purchase of change-loss reinsurance. The required minimal risk capital will then depend on the parameters of the change-loss reinsurance.

After reinsurance the cedent's surplus is equal to

$$S(a,b) = u_{\min}(a,b) + P(a,b) - (X - J_{a,b}(X)) = \mathrm{VaR}_\alpha[J_{a,b}(X)] - (X - J_{a,b}(X)).$$

In order to calculate this, first we have to find the value-at-risk of the transformed retained risk after reinsurance, i.e. $\mathrm{VaR}_\alpha[I_{a,b}(X)]$. According to Definition 2.1

$$\begin{aligned}\alpha &= \mathbb{P}\left[I_{a,b}(X) < \mathrm{VaR}_\alpha[I_{a,b}(X)]\right] \\ &= \begin{cases} F(\mathrm{VaR}_\alpha[I_{a,b}(X)]), & \mathrm{VaR}_\alpha[I_{a,b}(X)] < b; \\ F\left(\frac{\mathrm{VaR}_\alpha[I_{a,b}(X)] - ab}{1-a}\right), & \mathrm{VaR}_\alpha[I_{a,b}(X)] \geq b, \end{cases}\end{aligned}$$

or equivalently

$$\mathrm{VaR}_\alpha[X] = \begin{cases} \mathrm{VaR}_\alpha[I_{a,b}(X)], & \mathrm{VaR}_\alpha[I_{a,b}(X)] < b; \\ \frac{\mathrm{VaR}_\alpha[I_{a,b}(X)] - ab}{1-a}, & \mathrm{VaR}_\alpha[I_{a,b}(X)] \geq b. \end{cases}$$

We then have

$$\mathrm{VaR}_\alpha[I_{a,b}(X)] = \begin{cases} \mathrm{VaR}_\alpha[X], & \mathrm{VaR}_\alpha[X] < b; \\ ab + (1-a)\mathrm{VaR}_\alpha[X], & \mathrm{VaR}_\alpha[X] \geq b. \end{cases}$$

and by Definition 2.1

$$\mathrm{VaR}_\alpha[I_{1,b}(X)] = \begin{cases} \mathrm{VaR}_\alpha[X], & \mathrm{VaR}_\alpha[X] < b; \\ b, & \mathrm{VaR}_\alpha[X] \geq b. \end{cases}$$

Summarizing, we conclude that $\forall a \in [0,1]$:

$$\mathrm{VaR}_\alpha[I_{a,b}(X)] = \begin{cases} ab + (1-a)\mathrm{VaR}_\alpha[X], & b \leq \mathrm{VaR}_\alpha[X]; \\ \mathrm{VaR}_\alpha[X], & b > \mathrm{VaR}_\alpha[X]. \end{cases}$$

The required minimal risk capital under change-loss reinsurance is equal to

$$u(a,b) = \\ = \begin{cases} ab + (1-a)\mathrm{VaR}_\alpha[X] + (1+\eta(b,\theta))a \int\limits_b^\infty (1-F(x))dx - (1+\theta)\mathbb{E}[X], & b \leq \mathrm{VaR}_\alpha[X]; \\ \mathrm{VaR}_\alpha[X] + (1+\eta(b,\theta))a \int\limits_b^\infty (1-F(x))dx - (1+\theta)\mathbb{E}[X], & b > \mathrm{VaR}_\alpha[X]. \end{cases} \tag{8}$$

The cedent's terminal value is equal to

$$S(a,b) = \begin{cases} ab + (1-a)\mathrm{VaR}_\alpha[X] - I_{a,b}(X), & b \leq \mathrm{VaR}_\alpha[X]; \\ \mathrm{VaR}_\alpha[X] - I_{a,b}(X), & b > \mathrm{VaR}_\alpha[X], \end{cases}$$

and the expected value of its limited liability at the end of the period is

$$V(a,b) = \mathbb{E}[\max\{0, S(a,b)\}] = \int\limits_0^{\Delta(a,b)} (\Delta(a,b) - y)\, dF_{I_{a,b}}(y) = \int\limits_0^{\Delta(a,b)} F_{I_{a,b}}(y)dy, \tag{9}$$

where

$$\Delta(a,b) = \mathrm{VaR}_\alpha[I_{a,b}(X)] = \begin{cases} ab + (1-a)\mathrm{VaR}_\alpha[X], & b \leq \mathrm{VaR}_\alpha[X]; \\ \mathrm{VaR}_\alpha[X], & b > \mathrm{VaR}_\alpha[X], \end{cases}$$

For $a \in [0,1)$ cdf $F_{I_{a,b}}$ is continuous, and thus

$$V(a,b) = \begin{cases} \int\limits_0^b F(y)dy + \int\limits_b^{\Delta(a,b)} F\left(\frac{y-ab}{1-a}\right) dy, & b \leq \mathrm{VaR}_\alpha[X] \\ \int\limits_0^{\mathrm{VaR}_\alpha[X]} F(y)dy, & b > \mathrm{VaR}_\alpha[X] \end{cases} \\ = \begin{cases} \int\limits_0^{\mathrm{VaR}_\alpha[X]} F(y)dy - a \int\limits_b^{\mathrm{VaR}_\alpha[X]} F(y)dy, & b \leq \mathrm{VaR}_\alpha[X] \\ \int\limits_0^{\mathrm{VaR}_\alpha[X]} F(y)dy, & b > \mathrm{VaR}_\alpha[X]. \end{cases}$$

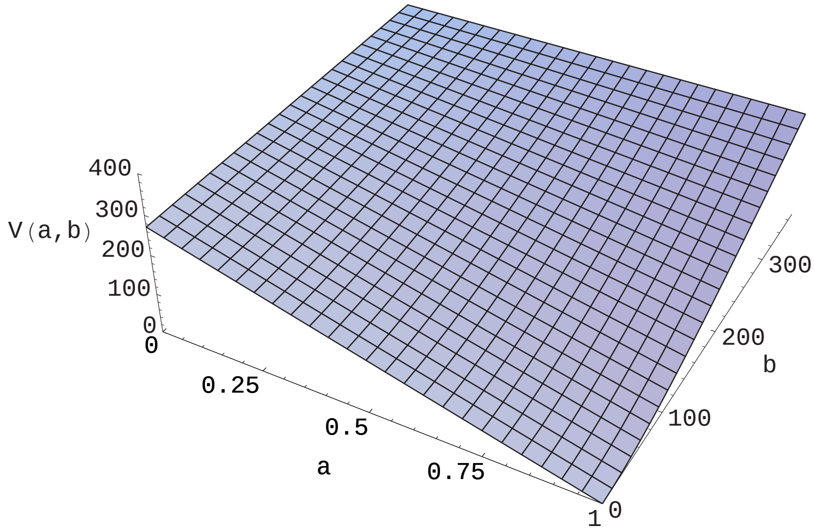


Fig. 3. *Graphical illustration of function $V(a,b)$ defined on* $\{a \in [0,1]\} \cap \{b \leq \mathrm{VaR}_\alpha[X]\}$

In the case where $a = 1$ $I_{1,b}(X)$ is a stop-loss transformation of the loss X, and thus the cdf $F_{I_{1,b}}(y)$ of transformed loss has a jump at point $y = b$. For $b \leq \mathrm{VaR}_\alpha[X]$: $\Delta(a,b) = b$ and according to (9)

$$V(a,b) = \int_0^b (b-y)dF_{I_{1,b}}(y) = \int_0^b (b-y)dF(y) + (b-b)(1 - F_{I_{1,b}}(b-)) = \int_0^b F(y)dy,$$

for $b > \mathrm{VaR}_\alpha[X]$: $\Delta(a,b) = \mathrm{VaR}_\alpha[X]$ and $V(a,b) =$

$$= \int_0^{\mathrm{VaR}_\alpha[X]} (\mathrm{VaR}_\alpha[X] - y)dF_{I_{1,b}}(y) = \int_0^{\mathrm{VaR}_\alpha[X]} (\mathrm{VaR}_\alpha[X] - y)dF(y) = \int_0^{\mathrm{VaR}_\alpha[X]} F(y)dy.$$

Therefore, $\forall a \in [0,1]$

$$V(a,b) = \begin{cases} \int_0^{\mathrm{VaR}_\alpha[X]} F(y)dy - a \int_b^{\mathrm{VaR}_\alpha[X]} F(y)dy, & b \leq \mathrm{VaR}_\alpha[X] \\ \int_0^{\mathrm{VaR}_\alpha[X]} F(y)dy, & b > \mathrm{VaR}_\alpha[X]. \end{cases} \tag{10}$$

We see that the global maximum of V is equal to $\int_0^{\mathrm{VaR}_\alpha[X]} F(y)dy$. Moreover, if the cedent's objective is to maximize the expected value of its limited liability at the end of period, then there is no demand for reinsurance contracts from the class of change-loss reinsurance contracts, since $V(a,b)$ attains its local maximum when $a = 0$ or/and $b = \mathrm{VaR}_\alpha[X]$ (see Figure 3).

However, there may be a demand for change-loss reinsurance in the case where the cedent's objective is to maximize the return $\rho(a,b) = \frac{V(a,b)}{u(a,b)} - 1$ (or gross return $1+\rho(a,b)$) on risk capital supplied by shareholders at the beginning of period. First of all the required risk capital decreases on $\{a \in [0,1]\} \cap \{b > \mathrm{VaR}_\alpha[X]\}$ when a tends to 0 or/and b tends to ∞. This means that the return on equity ρ attains its local maximum when $a = 0$ or $b = \infty$, i.e.

$$\max_{\{a\in[0,1]\}\cap\{b>\mathrm{VaR}_\alpha[X]\}} (1+\rho(a,b)) = \frac{\int\limits_0^{\mathrm{VaR}_\alpha[X]} F(y)dy}{\mathrm{VaR}_\alpha[X] - P}$$

and thus there is no demand for reinsurance contracts from the subclass $\{a \in [0,1]\} \cap \{b > \mathrm{VaR}_\alpha[X]\}$.

When the threshold of change-loss reinsurance $b \le \mathrm{VaR}_\alpha[X]$, the value of the required risk capital changes in the following way: for any fixed $a \in [0,1]$

$$\frac{\partial}{\partial b} u(a,b) = aF\left(\frac{b}{1+\theta}\right) \ge 0, \tag{11}$$

that is $u(a,\cdot)$ increases on $(0, \mathrm{VaR}_\alpha[X])$, and on $\{a \in [0,1]\} \cap \{b > \mathrm{VaR}_\alpha[X]\}$

$$\frac{\partial}{\partial b} u(a,b) = -a\left(1 - F\left(\frac{b}{1+\theta}\right)\right) \le 0, \tag{12}$$

thus $u(a,\cdot)$ decreases on $(\mathrm{VaR}_\alpha[X], \infty)$.

Moreover,

$$\lim_{b\to 0} u(a,b) = (1-a)(\mathrm{VaR}_\alpha[X] - P) \quad \text{and} \quad \lim_{b\to\infty} u(a,b) = (\mathrm{VaR}_\alpha[X] - P). \tag{13}$$

It follows from (11)-(13) that the global minimum of $u(a,b)$ is attained at $a = 1$ and $b = 0$ (full reinsurance) and equal to 0. In other words the purchase of full reinsurance reduces the required risk capital to 0. But in this case insurance premium income and the expected value of its limited liability at the end of the period are equal to 0, and thus the insurer is out of business (or it is replaced by the reinsurer). To avoid this degenerate situation we restrict the quota share in the domain of all change-loss reinsurance contracts by an upper bound $a_1 < 1$. This will guarantee the existence of both the insurer and reinsurer in the market.

The main aim of this subsection is to show that in contrast to the model M1 under specific conditions there may be a demand for reinsurance in the model M2. Moreover, it is difficult to tackle the problem of maximization of the return ρ on the risk capital supplied by shareholders in the case of a general form of the distribution function F of clams size. Therefore, to provide intuition about the results we will restrict ourself to the case of exponentially distributed claims size.

Let $F(x) = 1 - \mathrm{e}^{-\gamma x}$, $\gamma > 0$, $x \ge 0$. Then $x = F^{-1}(y) = -\frac{\ln(1-y)}{\gamma}$, and thus

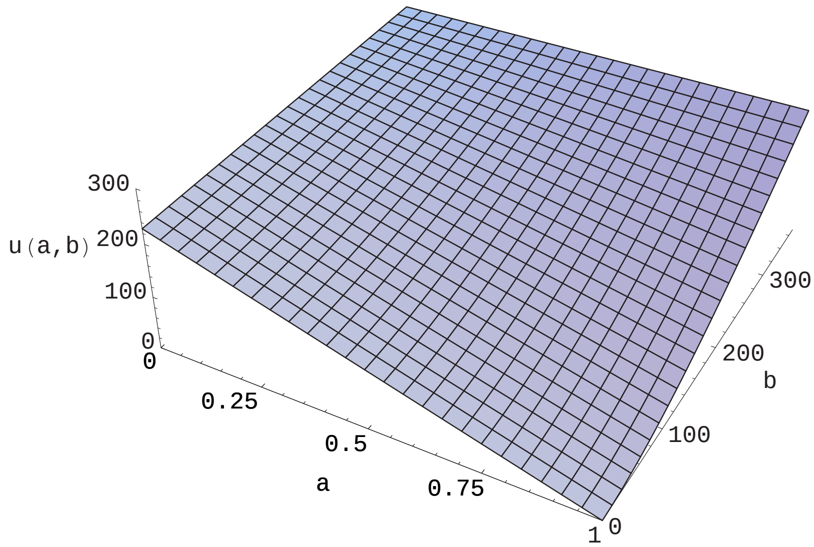


Fig. 4. *Graphical illustration of function* $u(a,b)$ *defined on* $\{a \in [0, a_1]\} \cap \{b \leq \mathrm{VaR}_\alpha[X]\}$

$$\mathrm{VaR}_\alpha[X] = F^{-1}(\alpha) = -\frac{\ln(1-\alpha)}{\gamma}$$

Using (5) we obtain

$$P(a,b) = \frac{1+\theta}{\gamma} - \frac{1+\eta(b,\theta)}{\gamma} a\mathrm{e}^{-\gamma b} = \frac{1+\theta}{\gamma}\left(1 - a\mathrm{e}^{-\frac{\gamma b}{1+\theta}}\right). \tag{14}$$

The cedent's terminal value defined in (10) becomes

$$V(a,b) = \\ = \begin{cases} a\left(b + \frac{\ln(1-\alpha)}{\gamma} + \frac{1}{\gamma}\left(\mathrm{e}^{-\gamma b} - (1-\alpha)\right)\right) - \frac{\ln(1-\alpha)}{\gamma} - \frac{\alpha}{\gamma}, & b \in [0, \mathrm{VaR}_\alpha[X]]; \\ -\frac{\ln(1-\alpha)}{\gamma} - \frac{\alpha}{\gamma}, \quad b \in (\mathrm{VaR}_\alpha[X], \infty). \end{cases}$$

The required risk capital on $\{a \in [0, a_1]\} \cap \{b \in [0, \mathrm{VaR}_\alpha[X]]\}$ defined in (8) becomes

$$\begin{aligned} u(a,b) &= ab - (1-a)\frac{\ln(1-\alpha)}{\gamma} + \frac{1}{\gamma}\left((1+\eta(b,\theta))a\mathrm{e}^{-\gamma b} - (1+\theta)\right) \\ &= a\left(b + \frac{\ln(1-\alpha)}{\gamma} + \frac{1}{\gamma}(1+\theta)\mathrm{e}^{-\gamma\frac{b}{1+\theta}}\right) - \frac{\ln(1-\alpha)}{\gamma} - \frac{1+\theta}{\gamma}. \end{aligned} \tag{15}$$

We now investigate the question as to whether there is a demand for change-loss reinsurance in shareholders value creation at all. In order to do this we examine the ratio $1 + \rho(a,b)$ (total return on equity) in the case of exponentially distributed cedent's aggregate claims size X. To illustrate we assume that $\alpha = 0.975$; $a_1 = 0.92$ (the upper bound of quota share); $\theta = 0.4$

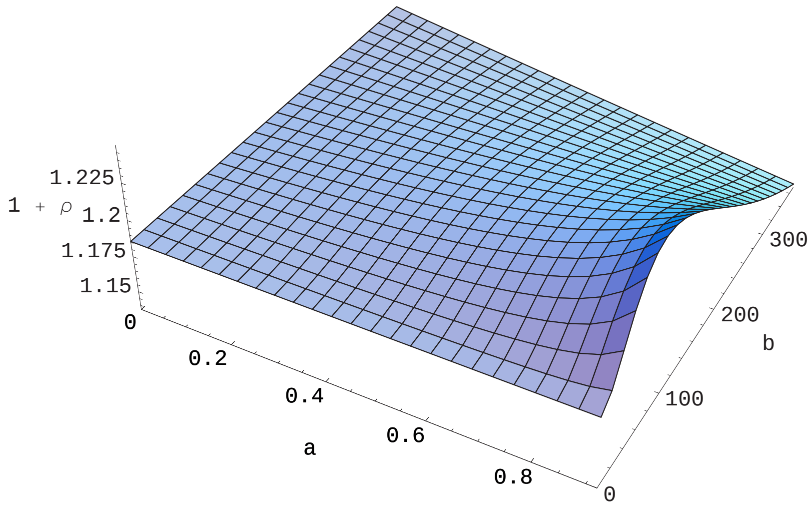
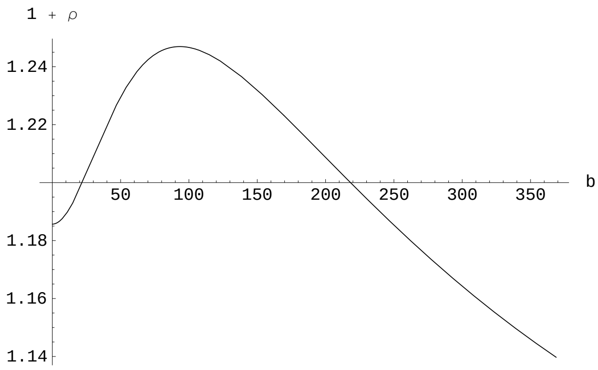


Fig. 5. *Graphical illustration of the total return on equity as the function* $1+\rho(a,b) = \frac{V(a,b)}{u(a,b)}$ *defined on* $\{a \in [0,a_1]\} \cap \{b \leq \mathrm{VaR}_\alpha[X]\}$

(the risk adjustment coefficient $k = 0.7143$) and $\gamma = 0.01$ ($\mathbb{E}[X] = \frac{1}{\gamma} = 100$). For this particular case the total return on risk capital as the function $1+\rho(a,b) = \frac{V(a,b)}{u(a,b)}$, defined on $\{a \in [0,a_1]\} \cap \{b \leq \mathrm{VaR}_\alpha[X]\}$, attains its local maximum 1.24693 at the point (a=0.92; b=95.11) (see Figure 5). On the other hand the local maximum of $1+\rho(a,b)$ on $\{a \in [0,a_1]\} \cap \{b > \mathrm{VaR}_\alpha[X]\}$ is equal to 1.1857 for $b = \infty$ (no reinsurance). So, the change-loss reinsurance contract with $b^* = 95.11$ and $a^* = a_1 = 0.92$) is an optimal contract under which the cedent's return on risk capital is maximal.



Graphical illustration of the total return on equity as the function $1+\rho(a_1,b) = \frac{V(a_1,b)}{u(a_1,b)}$ *on the interval* $b \in [0, \mathrm{VaR}_\alpha[X]]$.

Summarizing one may conclude that there may be demand for reinsurance in the model M2. This means, that in principle, an insurer might create value for shareholders by altering its capital structure after issuing insurance using reinsurance. Indeed, due to peculiarities of the insurance business an insurer is generally leveraged itself via the insurance market. That is, it resembles a leveraged investment fund in which debt is raised through the sale of insurance policies rather than via capital markets (although in addition an insurer can issue additional debt in a capital market). Purchasing reinsurance effectively reduces the insurer's debt and risk capital required by the regulator, and thus changes the financial leverage of the insurance company. Therefore, the decision to reinsure can be treated as both a risk-management and a capital-structure tool in shareholders value creation.

In contrast, the model M1 is conservative since it does not allow the insurer to reduce risk capital after taking reinsurance. Therefore, holding extra risk capital offsets the demand for reinsurance.

Remark. In this paper we consider actuarial approach of insurer risk management. This means that the gross premium P does not reflect the effect of insolvency on policy payoff. Using economic approach of insurance asset-liability modelling we can redefine single period models M1 and M2 in the following way. As it was earlier, all premiums are collected at the beginning of the period and all insurance claims are paid at the end of the period. At the beginning of the period the insurer's assets A_0 consist of premiums P_0 and risk capital (equity) E_0 supplied by shareholders. All assets at time 0 are invested in financial instruments with time-1-payoff $A_1 = (1 + r_A)A_0$. The terminal value of insurance claims (losses) is a random variable L_1.

The main economic (natural) assumption is to assume that an insurer cannot pay insurance indemnities to its policyholders at the end of the period at the level higher than the terminal value of its assets. This is the assumption of the limited liability of the insurer against its policyholders.

At the end of the period the shareholders' value (terminal equity value) is

$$E_1 = \begin{cases} A_1 - L_1, & A_1 > L_1 \\ 0, & A_1 \le L_1, \end{cases}$$

and the terminal value of insurer's liability is

$$\Lambda_1 = \begin{cases} L_1, & A_1 > L_1 \\ A_1, & A_1 \le L_1. \end{cases}$$

So, the 'fair insurance premium' is

$$\begin{aligned} P_0 &= \mathrm{e}^{-r}\mathbb{E}_{\mathbb{Q}}\left[\Lambda_1\right] = \mathrm{e}^{-r}\mathbb{E}_{\mathbb{Q}}\left[L_1\,\mathbf{1}_{\{A_1>L_1\}} + A_1\,\mathbf{1}_{\{A_1<L_1\}}\right] \\ &= \mathrm{e}^{-r}\mathbb{E}_{\mathbb{Q}}\left[L_1 - (L_1 - A_1)\,\mathbf{1}_{\{A_1<L_1\}}\right], \end{aligned}$$

where r is the risk-free interest rate, $\mathbb{Q}$ is the risk-neutral risk measure. In the latter equality the second term represents the value of insolvency exchange option, and we can see that the premium under new economic assumption of limited liability is less than one calculated using actuarial approach.

Under the fair (equilibrium) pricing the value of equity is

$$E_0 = \mathrm{e}^{-r}\mathbb{E}_{\mathbb{Q}}\left[(A_1 - L_1)\,\mathbf{1}_{\{A_1 > L_1\}}\right].$$

Summarizing, we conclude that the equilibrium insurance premium is a solution to the following system of two equations

$$\begin{aligned} P_0 &= \mathrm{e}^{-r}\mathbb{E}_{\mathbb{Q}}\left[L_1 - (L_1 - (1 + r_A)(P_0 + E_0))\,\mathbf{1}_{\{(1+r_A)(P_0+E_0) < L_1\}}\right] \\ E_0 &= \mathrm{e}^{-r}\mathbb{E}_{\mathbb{Q}}\left[((1 + r_A)(P_0 + E_0) - L_1)\,\mathbf{1}_{\{(1+r_A)(P_0+E_0) > L_1\}}\right]. \end{aligned}$$

By introducing a change-loss reinsurance we transform the company's liability from L_1 to $\widehat{L}_1 = L_1 - a(L_1 - b)^+$, the premium P_0 to $\widehat{P}_0 = \mathrm{e}^{-r}\mathbb{E}_{\mathbb{Q}}\left[\widehat{L}_1\right]$, and solve the same system with respect to $\widehat{P}_0$ and $\widehat{E}_0$. This new economic equilibrium model of the insurer should not impose any demand for reinsurance in the maximization of the return on equity, unless frictional costs such as taxes and costs of financial distress are included.

In this paper, in both models M1 and M2 the gross premium is determined using mean value premium principle without any adjustment with respect to the value of insolvency put. This possibly cause the situation where the model M2 imposes demand for reinsurance in frictionless environment.

3 Demand for reinsurance in shareholder's value creation: one-period model under the presence of corporate tax

In this section the problem of demand for change-loss reinsurance in a single-period model under the presence of a corporate tax is studied.

As it was shown in the previous section the decision to reinsure is an important tool of altering a company's capital structure, which in turn gives an opportunity to create (enhance) shareholders value. However, in the paper by Garven (1987 [6]) the author suggests that in order for insurer capital-structure decisions (including reinsurance) to matter in any meaningful sense, factors such as frictional capital costs, including tax shields, agency and financial distress costs, must be considered. Indeed, unlike investment funds, insurers may be subject to additional corporate tax and operate in a highly regulated environment where regulations are designed to protect policyholders. These frictions generate a need to provide shareholders with an additional return on the risk capital they supply over and above the base cost of capital. There are essentially three sources of frictional capital costs: costs of double taxation, costs of financial distress and agency costs.

In this section we will consider costs of double taxation only. Insurance companies are taxed on their investment return and underwriting profit before it is distributed to shareholders. This generates an additional cost component relative to an investment fund. We assume that the cost of taxes arising out of the insurance transactions and insurers investment should be included in the risk loading (risk premium) paid by policyholders. The reason is as follows:

when writing a policy the insurer commits equity capital (risk capital required by the regulator) to the insurance business. The owners (principals) of the insurance company always have the alternative of not writing insurance and investing their capital directly in financial assets (shares and bonds) in a capital market. They will not enter into an insurance transaction if by doing so they subject income on their investment to an additional layer of taxation. Therefore, the policyholders must pay the tax to provide a fair after-tax return on equity capital. According to the set up of the Myers-Cohn model of determining the fair insurance premium (see Myers and Cohn (1987 [10])), the premium is defined as fair if the insurer is indifferent between selling the policy and not selling it. The insurer will be indifferent if the market value of the insurer's equity is not changed by writing the policy.

We reconsider the models M1 and M2 of the reinsurance optimization by taking into account
1) the possibility to reinvest both risk capital (equity capital) and premium income at the beginning of the period;
2) corporate tax on underwriting profits and investment income

The return on investment i is a random variable. Claim costs are assumed to be independent of return on investment. Underwriting profits and investment income are taxed at the end-of period, at the rate τ, if taxable earnings are positive, and the residual funds are distributed to shareholders [3] .

So, again, using an explicit formula we can define the cedent's expected value of after-tax limited liability. Before tax, the shareholders have a valuable claim, if the terminal value of cash flows derived from the cedent's underwriting, reinsurance and investment activities is non-negative only. The expected value of this valuable claim

- within the model M1 is equal to

$$\begin{aligned} V_S(u,a,b) &= \mathbb{E}_i\left[\mathbb{E}_{I_{a,b}(X)}\left[\max\left\{(1+i)\,(u+P(a,b)) - I_{a,b}(X);\,0\right\} \mid\, i\right]\right] \\ &= \mathbb{E}_i\left[\int_0^{\delta_1} (\delta_1 - y)\, dF_{I_{a,b}}(y)\right], \end{aligned}$$

  where

$$\begin{aligned} \delta_1 &= (1+i)\,(u+P(a,b)) \\ &= (1+i)\left(\mathrm{VaR}_\alpha[X] + u_1 - (1+\eta(b,\theta))a\int_b^\infty (1-F(x))dx\right), \end{aligned}$$

- within the model M2 is equal to

$$\begin{aligned} \widetilde{V}_S(a,b) &= \mathbb{E}_i\left[\mathbb{E}_{I_{a,b}(X)}\left[\max\left\{(1+i)\,(u(a,b)+P(a,b)) - I_{a,b}(X);\,0\right\} \mid\, i\right]\right] \\ &= \mathbb{E}_i\left[\int_0^{\Delta_1} (\Delta_1 - y)\, dF_{I_{a,b}}(y)\right], \end{aligned}$$

[3] In a multi-period model, if taxable earnings are negative but the direct insurer is still solvent, then it receives a tax shield equal $q\tau$, where $q \leq 1$. In other words, an insurer carries taxable (at rate $q\tau$) losses forward to offset future income. If the insurer is insolvent, then the tax shield on losses is equal to zero. Here in this section we consider a single period model, and thus assume that $q = 0$

where

$$\Delta_1 = (1+i)\Delta = (1+i)\left(u(a,b) + P(a,b)\right)$$
$$= (1+i)\mathrm{VaR}_\alpha[I_{a,b}(X)] = \begin{cases} (1+i)\left(ab + (1-a)\mathrm{VaR}_\alpha[X]\right), & b \le \mathrm{VaR}_\alpha[X]; \\ (1+i)\mathrm{VaR}_\alpha[X], & b > \mathrm{VaR}_\alpha[X]. \end{cases}$$

If taxable total amount of investment income and underwriting profit after reinsurance is non-negative, then the government will have a valuable claim. The expected value of this taxable amount is equal to:

- within the model M1

$$V_T(u,a,b) = \tau\mathbb{E}_i\left[\mathbb{E}_{I_{a,b}(X)}\left[\max\left\{i\left(u + P(a,b)\right) + P(a,b) - I_{a,b}(X);\, 0\right\} \mid\, i\right]\right]$$
$$= \tau\mathbb{E}_i\left[\int\limits_0^{\delta_2} (\delta_2 - y)\, dF_{I_{a,b}}(y)\right],$$

  where

$$\delta_2 = iu + (1+i)P(a,b) = \delta_1 - u$$
$$= i\left(\mathrm{VaR}_\alpha[X] + u_1 - (1+\eta(b,\theta))a\int\limits_b^\infty (1-F(x))dx\right)$$
$$+ (1+\theta)\mathbb{E}[X] - (1+\eta(b,\theta))a\int\limits_b^\infty (1-F(x))dx,$$

- within the model M2 this taxable amount is equal to

$$\widetilde{V}_T(a,b) = \tau\mathbb{E}_i\left[\mathbb{E}_{I_{a,b}(X)}\left[\max\left\{i\left(u(a,b) + P(a,b)\right) + P(a,b) - I_{a,b}(X);\, 0\right\} \mid\, i\right]\right]$$
$$= \tau\mathbb{E}_i\left[\int\limits_0^{\Delta_2} (\Delta_2 - y)\, dF_{I_{a,b}}(y)\right],$$

  where

$$\Delta_2 = iu(a,b) + (1+i)P(a,b) = \Delta_1 - u(a,b) = i\mathrm{VaR}_\alpha[I_{a,b}(X)] + P(a,b).$$

The total shareholders expected after-tax terminal value is equal to:

- within the model M1

$$V_\tau(u,a,b) = V_S(u,a,b) - V_T(u,a,b)$$
$$= \mathbb{E}_i\left[\int\limits_0^{\delta_1} (\delta_1 - y)\, dF_{I_{a,b}}(y) - \tau\int\limits_0^{\delta_2} (\delta_2 - y)\, dF_{I_{a,b}}(y)\right], \tag{16}$$

- within the model M2 it is equal to

$$\widetilde{V}_\tau(a,b) = \widetilde{V}_S(a,b) - \widetilde{V}_T(a,b)$$
$$= \mathbb{E}_i \left[\int_0^{\Delta_1} (\Delta_1 - y)\, dF_{I_{a,b}}(y) - \tau \int_0^{\Delta_2} (\Delta_2 - y)\, dF_{I_{a,b}}(y) \right]. \tag{17}$$

From here we obtain the total return on risk capital u is equal to:

- within the model M1

$$1 + \rho(u,a,b) = \frac{V_\tau(u,a,b)}{u} = \mathbb{E}_i \left[\Psi_{u,a,b}(i)\right]$$
$$= \mathbb{E}_i \left[\frac{\int_0^{\delta_1} (\delta_1 - y)\, dF_{I_{a,b}}(y) - \tau \int_0^{\delta_2} (\delta_2 - y)\, dF_{I_{a,b}}(y)}{u} \right], \tag{18}$$

- within the model M2

$$1 + \rho(a,b) = \frac{\widetilde{V}_\tau(a,b)}{u(a,b)} = \mathbb{E}_i \left[\Psi_{a,b}(i)\right]$$
$$= \mathbb{E}_i \left[\frac{\int_0^{\Delta_1} (\Delta_1 - y)\, dF_{I_{a,b}}(y) - \tau \int_0^{\Delta_2} (\Delta_2 - y)\, dF_{I_{a,b}}(y)}{u(a,b)} \right] \tag{19}$$

To avoid the degenerate situation in the revised model M2, where purchase of full reinsurance offsets the required risk capital $u(a,b)$ and underwriting liability to zero (insurer assumes no insurance risk), we restrict quota share in the class of change-loss reinsurance by an upper bound $a_1 < 1$. We investigate the value of the return $\rho(a,b)$ on risk capital in the model M2 on the following three regions $\{a \in [0,a_1]\} \cap \{b \le \mathrm{VaR}_\alpha[X]\}$, $\{a \in [0,a_1]\} \cap \{\mathrm{VaR}_\alpha[X] < b \le (1+i)\mathrm{VaR}_\alpha[X]\}$ and $\{a \in [0,a_1]\} \cap \{b > (1+i)\mathrm{VaR}_\alpha[X]\}$, since

$$\Delta_1 = \begin{cases} (1+i)\left(ab + (1-a)\mathrm{VaR}_\alpha[X]\right) \quad (> b), & \text{if } b \le \mathrm{VaR}_\alpha[X]; \\ (1+i)\mathrm{VaR}_\alpha[X] \quad (> b), & \text{if } \mathrm{VaR}_\alpha[X] < b \le (1+i)\mathrm{VaR}_\alpha[X]; \\ (1+i)\mathrm{VaR}_\alpha[X] \quad (< b); & \text{if } b > (1+i)\mathrm{VaR}_\alpha[X]. \end{cases}$$

For $a \in [0,1)$

$$\int_0^{\Delta_1} (\Delta_1 - y)\, dF_{I_{a,b}}(y) =$$
$$= \begin{cases} \int_0^b F(y)dy + (1-a) \int_b^{\frac{\Delta_1 - ab}{1-a}} F(y)dy, & \text{if } b \le \mathrm{VaR}_\alpha[X]; \\ \int_0^b F(y)dy + (1-a) \int_b^{\frac{\Delta_1 - ab}{1-a}} F(y)dy, & \text{if } \mathrm{VaR}_\alpha[X] < b \le (1+i)\mathrm{VaR}_\alpha[X]; \\ \int_0^{\Delta_1} F(y)dy, & \text{if } b > (1+i)\mathrm{VaR}_\alpha[X] \end{cases}$$

$$= \begin{cases} \int_0^b F(y)dy + (1-a) \int_b^{\frac{iab+(1+i)(1-a)\mathrm{VaR}_\alpha[X]}{1-a}} F(y)dy, & \text{if } b \le \mathrm{VaR}_\alpha[X]; \\ \int_0^b F(y)dy + (1-a) \int_b^{\frac{(1+i)\mathrm{VaR}_\alpha[X]-ab}{1-a}} F(y)dy, & \text{if } \mathrm{VaR}_\alpha[X] < b \le (1+i)\mathrm{VaR}_\alpha[X]; \\ \int_0^{(1+i)\mathrm{VaR}_\alpha[X]} F(y)dy, & \text{if } b > (1+i)\mathrm{VaR}_\alpha[X]; \end{cases}$$

and

$$\int_0^{\Delta_2} (\Delta_2 - y)\, dF_{I_{a,b}}(y) =$$

$$= \mathbf{1}_{\{\Delta_2 \le b\}} \int_0^{\Delta_2} F(y)dy + \mathbf{1}_{\{\Delta_2 > b\}} \left(\int_0^b F(y)dy + (1-a) \int_b^{\frac{\Delta_2 - ab}{1-a}} F(y)dy \right).$$

We should notice that, in fact,

$$\lim_{a\to 1} \int_0^{\Delta_1} (\Delta_1 - y)\, dF_{I_{a,b}}(y) = \int_0^{\Delta_1} (\Delta_1 - y)\, dF_{I_{1,b}}(y)$$

$$= \begin{cases} \int_0^b F(y)dy + (\Delta_1 - b), & \text{if } b \le \mathrm{VaR}_\alpha[X]; \\ \int_0^b F(y)dy + (\Delta_1 - b), & \text{if } \mathrm{VaR}_\alpha[X] < b \le (1+i)\mathrm{VaR}_\alpha[X]; \\ \int_0^{\Delta_1} F(y)dy, & \text{if } b > (1+i)\mathrm{VaR}_\alpha[X]. \end{cases}$$

If we let $y = \frac{1}{1-a}$, then for $b \le (1+i)\mathrm{VaR}_\alpha[X]$, using L'Hôpital rule, we obtain

$$\lim_{a\to 1} (1-a) \int_b^{\frac{\Delta_1 - ab}{1-a}} F(y)dy = \lim_{y\to +\infty} \frac{\int_b^{(y\Delta_1 - (y-1)b)} F(y)dy}{y}$$

$$= \lim_{y\to +\infty} (\Delta_1 - b) F\left(y(\Delta_1 - b) + b\right) = \Delta_1 - b.$$

The following equality also holds for $b \in \{\Delta_2 > b\}$

$$\lim_{a\to 1} \int_0^{\Delta_2} (\Delta_2 - y)\, dF_{I_{a,b}}(y) = \int_0^{\Delta_2} (\Delta_2 - y)\, dF_{I_{1,b}}(y).$$

To proceed further we will consider the model under the assumption that the aggregate amount of insurance claims X is exponentially distributed, that is $F(x) = 1 - \mathrm{e}^{-\gamma x},\ x \ge 0,\ \gamma > 0$. As it was shown in the preceding section $\mathrm{VaR}_\alpha[X] = -\frac{\ln(1-\alpha)}{\gamma}$. We will also use the same formulae for the premium income $P(a,b)$ and the required risk capital $u(a,b)$ defined earlier in (14) and (15) respectively.

Now, we can determine explicit forms of $\Psi_{a,b}(i)$ from (19) on the following two ranges:

1) $b \in \mathcal{D} \cap \{b \leq (1+i)\mathrm{VaR}_\alpha[X]\}$

$$\begin{aligned}\Psi_{a,b}(i) = \Bigg[& \left(\Delta_1 - \frac{1}{\gamma}\left(1-\mathrm{e}^{-\gamma b}\right) - \frac{1-a}{\gamma}\left(\mathrm{e}^{-\gamma b} - \mathrm{e}^{-\gamma \frac{\Delta_1 - ab}{1-a}}\right)\right) \\ & -\tau \left(\left\{ \Delta_2 - \frac{1}{\gamma}\left(1-\mathrm{e}^{-\gamma b}\right) - \frac{1-a}{\gamma}\left(\mathrm{e}^{-\gamma b} - \mathrm{e}^{-\gamma \frac{\Delta_2 - ab}{1-a}}\right) \right\} \times \mathbf{1}_{\{\Delta_2 \geq b\}} \right. \\ & \left. + \left\{ \Delta_2 - \frac{1}{\gamma}\left(1 - \mathrm{e}^{-\gamma \Delta_2}\right) \right\} \times \mathbf{1}_{\{\Delta_2 < b\}} \right) \Bigg] \frac{1}{u(a,b)},\end{aligned}$$

where $\mathbf{1}_{\{A\}}$ is an indicator function of event A,

$$\Delta_1 = \begin{cases} (1+i)\left(ab - (1-a)\frac{\ln(1-\alpha)}{\gamma}\right), & \text{if } b \leq \mathrm{VaR}_\alpha[X] = -\frac{\ln(1-\alpha)}{\gamma}; \\ -(1+i)\frac{\ln(1-\alpha)}{\gamma}, & \text{if } \mathrm{VaR}_\alpha[X] < b \leq (1+i)\mathrm{VaR}_\alpha[X]; \end{cases}$$

and $\Delta_2 = \Delta_1 - u(a,b) = i\Delta + P(a,b) =$

$$= \begin{cases} i\left(ab - (1-a)\frac{\ln(1-\alpha)}{\gamma}\right) + \frac{1}{\gamma}\left((1+\theta) - a(1+\eta(b,\theta))\mathrm{e}^{-\gamma b}\right), & \text{if } b \leq \mathrm{VaR}_\alpha[X]; \\ -i\frac{\ln(1-\alpha)}{\gamma} + \frac{1}{\gamma}\left((1+\theta) - a(1+\eta(b,\theta))\mathrm{e}^{-\gamma b}\right), & \text{if } \mathrm{VaR}_\alpha[X] < b \leq (1+i)\mathrm{VaR}_\alpha[X]; \end{cases}$$

2) $b \in \mathcal{D} \cap \{b > (1+i)\mathrm{VaR}_\alpha[X]\}$ (that is $b > \Delta_1 > \Delta_2$)

$$\Psi_{a,b}(i) = \left[\left(\Delta_1 - \frac{1}{\gamma}\left(1-\mathrm{e}^{-\gamma\Delta_1}\right)\right) - \tau\left(\Delta_2 - \frac{1}{\gamma}\left(1-\mathrm{e}^{-\gamma\Delta_2}\right)\right)\right]\frac{1}{u(a,b)},$$

where $\Delta_1 = -(1+i)\frac{\ln(1-\alpha)}{\gamma}$ and $\Delta_2 = -i\frac{\ln(1-\alpha)}{\gamma} + \frac{1}{\gamma}\left((1+\theta) - a(1+\eta(b,\theta))\mathrm{e}^{-\gamma b}\right)$.

We can get an analogous formulae to calculate $\Psi_{u,a,b}(i)$ in the model M1.

$$\begin{aligned}\Psi_{u,a,b}(i) = \Bigg[& \left(\left\{ \delta_1 - \frac{1}{\gamma}\left(1-\mathrm{e}^{-\gamma b}\right) - \frac{1-a}{\gamma}\left(\mathrm{e}^{-\gamma b} - \mathrm{e}^{-\gamma \frac{\delta_1 - ab}{1-a}}\right) \right\} \times \mathbf{1}_{\{\delta_1 \geq b\}} \right. \\ & \left. + \left\{ \delta_1 - \frac{1}{\gamma}\left(1-\mathrm{e}^{-\gamma\delta_1}\right) \right\} \times \mathbf{1}_{\{\delta_1 < b\}} \right) \\ & -\tau \left(\left\{ \delta_2 - \frac{1}{\gamma}\left(1-\mathrm{e}^{-\gamma b}\right) - \frac{1-a}{\gamma}\left(\mathrm{e}^{-\gamma b} - \mathrm{e}^{-\gamma \frac{\delta_2 - ab}{1-a}}\right) \right\} \times \mathbf{1}_{\{\delta_2 \geq b\}} \right. \\ & \left. + \left\{ \delta_2 - \frac{1}{\gamma}\left(1-\mathrm{e}^{-\gamma\delta_2}\right) \right\} \times \mathbf{1}_{\{\delta_2 < b\}} \right) \Bigg] \frac{1}{u},\end{aligned}$$

where

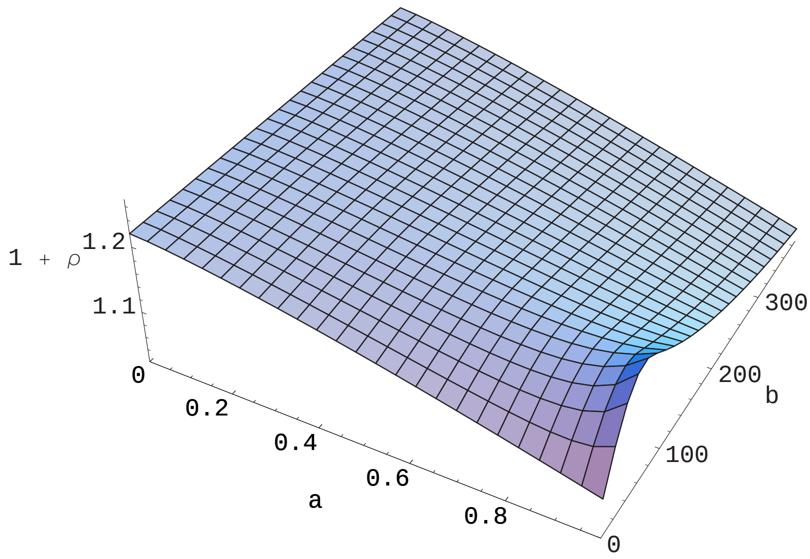


Fig. 6. *Graphical illustration of the total return* $1+\rho(u_{\min},\,a,\,b)$ *in the model M1 with corporate tax* $\tau=30\%$

$$\delta_1 = (1+i)\left(-\frac{\ln(1-\alpha)}{\gamma} + u_1 - \frac{1+\theta}{\gamma}\,a\,\mathrm{e}^{-\gamma\frac{b}{1+\theta}}\right),$$
$$\delta_2 = i\left(-\frac{\ln(1-\alpha)}{\gamma} + u_1 - \frac{1+\theta}{\gamma}\,a\,\mathrm{e}^{-\gamma\frac{b}{1+\theta}}\right) + \frac{1+\theta}{\gamma}\left(1-a\mathrm{e}^{-\gamma\frac{b}{1+\theta}}\right)$$

and $\delta_2 = \delta_1 - u$.

We consider numerical examples using the same parameters as in the previous subsection, i.e. $\alpha = 0.975$; $\theta = 0.4$, $a_1 = 0.92$ (upper bound of quota share in the class of admissible change-loss reinsurance contracts) and $\gamma = 0.01$ ($\mathbb{E}[X] = \frac{1}{\gamma} = 100$). We assume that i is a deterministic value and is equal to $i = 10\%$. Let us further consider, the corporate tax $\tau = 30\%$ for the model M2. For the model M1 we will consider the range of corporate tax τ from 15% to 40%.

The Figure 6 represents the graph of total return $1+\rho(u_{\min},\,a,\,b)$ ($u_1 = 0$) on risk capital in the revised model M1 under $\tau = 30\%$. In this graph we can see that there is demand for stop-loss reinsurance.
The following two figures represents graph $1+\rho(u_{\min},\,1,\,b)$ on intervals $b \in [0, \mathrm{VaR}_\alpha[X]]$ and $b \in [\mathrm{VaR}_\alpha[X], \infty)$.

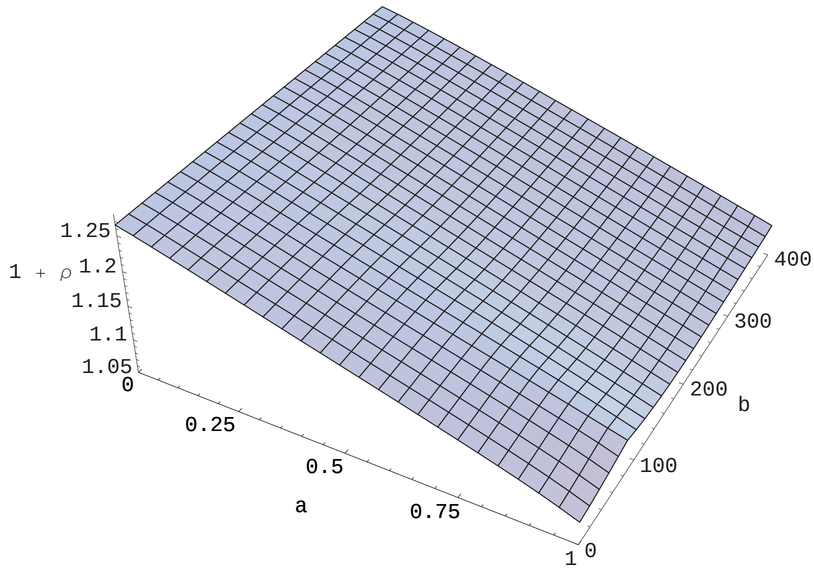
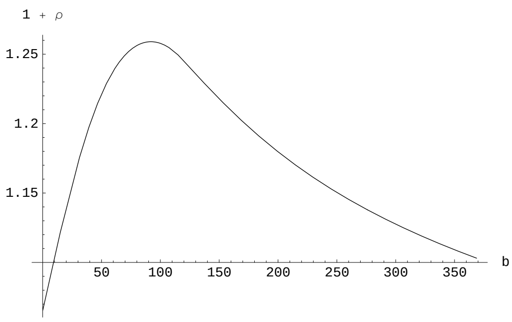
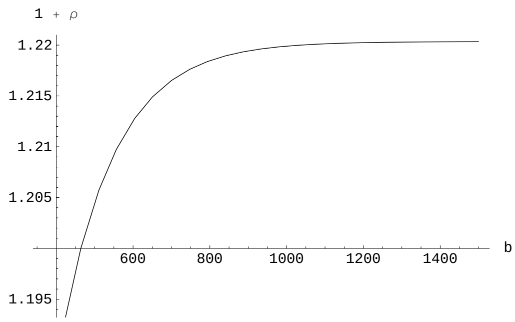


Fig. 7. *Graphical illustration of the total return* $1+\rho(u_{\min},\, a,\, b)$ *in the model M1 with corporate tax* $\tau = 15\%$





On the left hand side we have the graph that shows demand for stop-loss reinsurance: the optimal retention of stop-loss reinsurance is equal to $b^* = 93.73$ and the corresponding local maximum of return on equity is equal to $\rho(u_{\min}, 1, b^*) = 26.01\%$. On the right we have the graph $1+\rho(u_{\min},\, a,\, 0)$ on the interval $b \in [\mathrm{VaR}_\alpha[X], \infty)$ that indicates that there is no demand for reinsurance and local maximum of return on equity on this interval is equal to 22.03%. Therefore, the global maximum of return on equity in the model M1 with corporate tax 30% is equal to 26.01%.

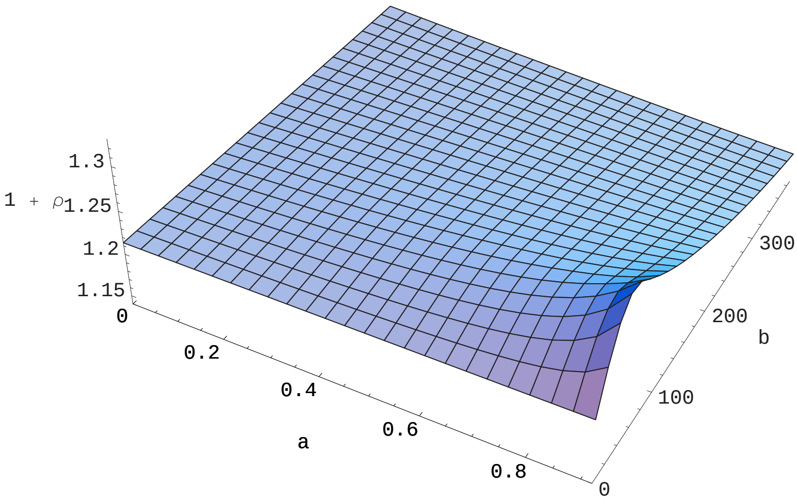


Fig. 8. *Graphical illustration of the total return* $1+\rho(a,\, b)$ *in the model M2 with corporate tax* $\tau = 30\%$

The following table [4] shows optimal reinsurance strategies in the model M1 under different levels of corporate tax.

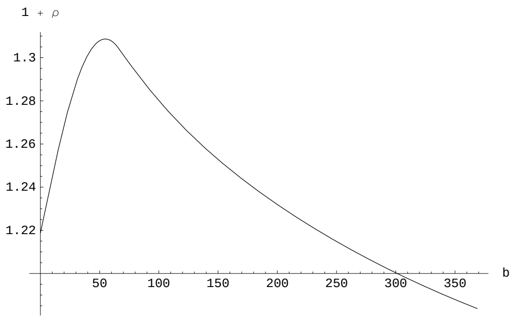
Table 3

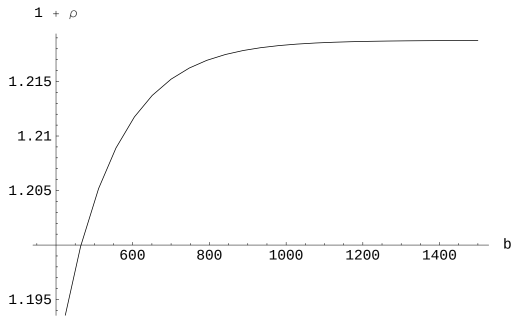
τ	Optimal reinsurance	Maximal return on equity ρ^*
15%	$b^* = \infty$ or $a^* = 0$	26.83%
20%	$b^* = \infty$ or $a^* = 0$	25.492%
25%	$b^* = 99.31$ and $a^* = 1$	26.47%
30%	$b^* = 93.73$ and $a^* = 1$	26.01%
35%	$b^* = 87.69$ and $a^* = 1$	25.302%
40%	$b^* = 82.07$ and $a^* = 1$	24.58%

It seems that there is no demand for reinsurance in the model M1 for low values of corporate tax (e.g. see the Table 3 and also the graph of the total return on equity in the model M1 with corporate tax $\tau = 15\%$ in the Figure 7). This is somewhat expected, since, as it was shown in the previous section, the model M1 does not induce demand for reinsurance in maximization of return on equity in a frictionless environment. So, including only a small amount of frictional costs (e.g. corporate tax) in the model M1 may not affect the optimal reinsurance strategy.

In the model M2 we observe demand for change-loss reinsurance (see Figure 8). The following two figures represent the graph $1+\rho(a,\, b)$ on intervals $b \in [0, \mathrm{VaR}_\alpha[X]]$ and $b \in [\mathrm{VaR}_\alpha[X], \infty)$.

[4] The figures in the table are high from practical point of view. This table serves as an illustration example only.





On the left hand side we can see that there is demand for change-loss reinsurance, i.e. $b^* = 58.41$, $a^* = 0.92$ and corresponding local maximum for return on equity is equal to $\rho(a^*, b^*) = 31.02\%$. On the right we see that it is optimal not to buy reinsurance for $b \in [\mathrm{VaR}_\alpha[X], \infty)$ and local maximum of return on equity on this interval is equal to 22.03%. Therefore, the global maximum of return on equity in the model M2 with corporate tax 30% is equal to 31.02%. Note that the demand for reinsurance in the model M2 with corporate tax is higher than demand in the same model without corporate tax.

Comparing two models M1 and M2 with corporate tax $\tau = 30\%$ we conclude that both models induce demand on reinsurance, however the maximal return on equity in the model M2 is higher than analogous value in the model M1. The latter can be explained in terms of insurer capitalization. In the model M1 an insurer is more capitalized, since this model does not allow insurer to reduce risk capital after taking reinsurance. On the other hand the model M2 does allow insurer to reduce risk capital after taking reinsurance and thus insurer is less capitalized. Holding extra risk capital reduces the maximum value of return on equity.

4 Demand for reinsurance in shareholder's value creation: one-period model under the presence of costs of financial distress

In this section the model M1 is reconsidered by admitting the possibility of costs of financial distress. The main assumption underlying our approach of incorporation of additional costs of financial distress into the model M1 is the distinction between firm's economic states *"default"* (or *"financial distress"*) and *"insolvency"* (or *"bankruptcy"*). The notion that default and insolvency are different states has been introduced in the finance literature (e.g. see Jarrow and Purnanandam (2004 [9]) and references therein). We consider three economic states of the insurance company at the end of the period: *no-default*, *default* and *insolvency*. The default is defined as a low net cash-flow state of the insurance company in which the insurer incurs additional deadweight losses without being insolvent. Insolvency, on the other hand, occurs on the terminal date (i.e. at the end of the period) if the terminal value of assets is less than the insurer underwriting liability (risky debt). It is assumed that if the net cash-flow reaches some exogenously predetermined boundary (*default barrier*) K, the insurer incurs deadweight losses $1 - w$ $(0 < w < 1)$ of its asset values. Financial distress can be costly due to both direct costs, such as legal fees (third party costs) and lost value from distressed sales ("fire sale" losses), and indirect costs, mainly loss of reputation and franchise value. These costs along with default barrier are exogenously defined within this paper. Some empirical studies (e.g. Opler and Titman (1994 [12]), Andrade and Kaplan (1998 [1])) of financial companies have revealed that financial distress results in costs of around $10 - 20\%$ of market value of assets. These costs are likely to be higher in the insurance industry due to the credit-sensitive nature of policyholders.

In a single period model defined on $[0,T]$ the shareholders receive liquidating dividends (insurer terminal wealth) at the end of the period. Due to equity's limited liability, the terminal payoff to the shareholders E_T is zero if the terminal asset value A_T is below the value of insurance liability L_T (insolvency state). In the event of no-default (i.e. $A_T - L_T > K$) the shareholders get a liquidating dividend of $A_T - L$. In the event of default (i.e. $A_T - L_T \leq K$), where financial distress is experienced, the shareholders receive liquidating dividends of $wA_T - L_T$ if insurer is still solvent (i.e. $wA_T - L_T > 0$) and they receive nothing if it is insolvent (i.e. $wA_T - L_T \leq 0$).

Therefore, the equity value at time $t = 0$ is equal to

$$\begin{aligned} E_0 &= \mathbb{E}_Q\left[(A_T - L_T)(1 - \mathbf{1}_{\{\mathrm{Def}\}}) + (wA_T - L_T)\mathbf{1}_{\{\mathrm{Def-Solv}\}}\right] \\ &= \mathbb{E}_Q\left[(A_T - L_T)\mathbf{1}_{\{A_T - L_T > K\}} + (wA_T - L_T)\mathbf{1}_{\{A_T - L_T \leq K\}\cap\{wA_T - L_T > 0\}}\right], \end{aligned} \tag{20}$$

where Q [5] is an equivalent martingale measure, that exists under the assumption that the market for assets and liabilities is arbitrage free, but incomplete (as it is typical in insurance).

The equity value in (20) can be rewritten in the following way

$$\begin{aligned} E_0 &= \mathbb{E}_Q\left[(A_T - L_T)\mathbf{1}_{\{A_T - L_T > K\}} + (wA_T - L_T)\mathbf{1}_{\{A_T - L_T \leq K\}\cap\{wA_T - L_T > 0\}}\right] \\ &= \mathbb{E}_Q\left[(A_T - L_T) - (A_T - L_T)\mathbf{1}_{\{A_T - L_T \leq K\}}\right. \\ &\quad \left. + (wA_T - L_T)\mathbf{1}_{\{A_T - L_T \leq K\}} - (wA_T - L_T)\mathbf{1}_{\{A_T - L_T \leq K\}\cap\{wA_T - L_T < 0\}}\right] \\ &= \mathbb{E}_Q[A_T - L_T] - \mathbb{E}_Q\left[(1-w)A_T\mathbf{1}_{\{A_T - L_T \leq K\}}\right] \\ &\quad + \mathbb{E}_Q\left[(L_T - wA_T)\mathbf{1}_{\{A_T - L_T \leq K\}\cap\{wA_T - L_T < 0\}}\right] \end{aligned} \tag{21}$$

We can see in (21) that the equity value has three components. The first term $\mathbb{E}_Q[(A_T - L_T)]$ represents the net asset value of the firm. The second term $\mathbb{E}_Q\left[(1-w)A_T\mathbf{1}_{\{A_T - L_T \leq K\}}\right]$ represents the deadweight losses caused by financial distress. The shareholders of a financially distressed but solvent insurance company bear financial distress costs and thus the terminal equity value is reduced by this amount. These costs create a risk-reducing incentive. The third term represents the savings to shareholders of a levered (by insurance risky debt) insurance company due to the limited liability in the event of insolvency. Existence of this positive term in (21) induces the risk-enhancing incentives of the shareholders. By increasing the upside risk of net liability $L_T - A_T$, the shareholders can make themselves better off by increasing the call option value. But, at the same time the expected losses in the event of financial distress also increases since increasing upside risk of net insurance liability immediately implies increasing of downside risk of the net asset value of the company. Therefore, the optimal level of integrated investment-underwriting risk is determined by the trade-off between this two incentives.

An insurer can use two possibilities to control investment risk:
1) it can construct an investment portfolio with minimal volatility; or
2) the volatility can be fixed and an insurer can reduce risk by buying derivative contracts such as options.

[5] For the sake of simplicity the risk-free interest rate is set to be 0. This is can be done due to the Numeraire Invariance Theorem (see Duffie (2001 [4]))

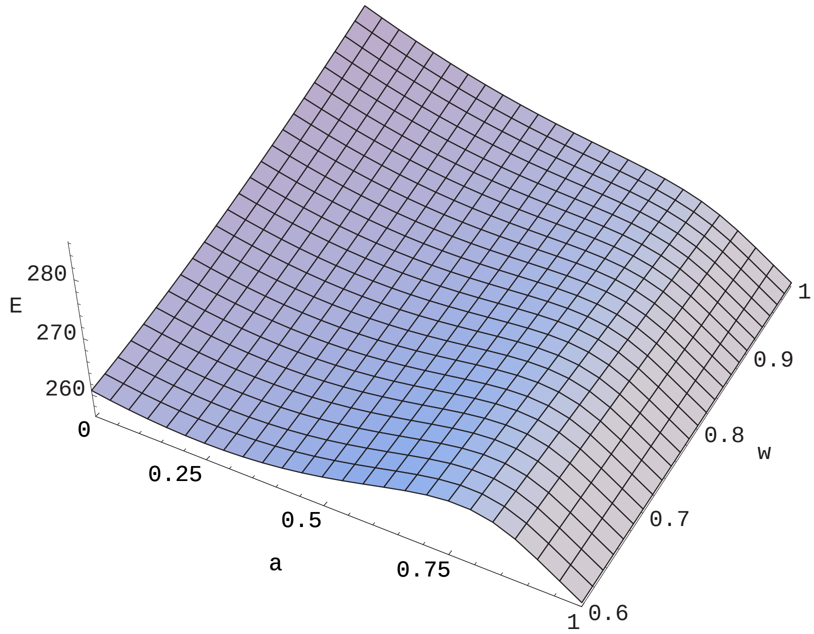
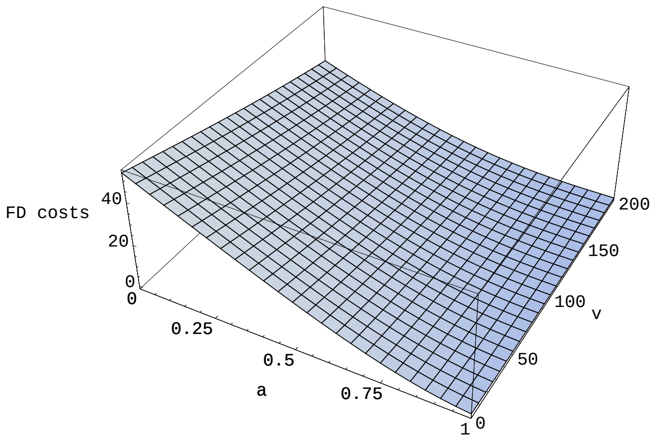


Fig. 9. *Graphical illustration of the equity value E_0 as a function of parameter a of reinsurance and parameter w financial distress costs in the model M1 with costs of financial distress (with parameter of the default barrier $k = 0.7$)*

Underwriting risk can be reduced by traditional *cession* (risk transfer), i.e. through purchase of a reinsurance contract in the reinsurance market. It is worth noticing that to some extent the reinsurance contract resembles derivative contracts in finance. Using both, an agent reduces risk (volatility of investment portfolio or underwriting risk, which is traditionally measured by the probability of default (insolvency)). However such alteration of risk is costly, and thus it reduces the net asset value of the insurance company (first term in (21)). In this section we consider some illustrations of the equity value of the revised model M1 with costs of financial distress. As in the previous section we assume that at the beginning of the period an insurer has insurance premium income P to cover contingent underwriting loss X and it invests this amount of premiums along with required minimal value of risk capital in the capital market. We assume that investment return $R = 1 + i$ is deterministic. The default barrier K is exogenously defined and is assumed to be proportionate to the market value of assets, i.e. $K = K(A) = (1-k)A$. We choose proportional reinsurance to control underwriting risk. Using the same numerical parameters of distribution of insurance loss as in the previous sections and additionally assuming that $k = 0.7$, we can see that if the deadweight losses $1 - w$ of financial distress are small (i.e the equity value of current model is close to the analogous value in the model M1), then there is no demand for reinsurance. However if the deadweight losses increase (i.e. w decreases) and such that $1 - w > 0.263$, then it is optimal to buy reinsurance to maximize equity value E_0 (see Figure 9).

This is an expected result, since in the presence of low costs of financial distress the current model of equity valuation resembles the model M1 defined in the frictionless environment. And as it has been shown the model M1 does not induce demand for reinsurance. Yet another method of mitigation of the expected financial distress is to use extra risk capital. However, creating additional equity capital may be more costly than purchasing reinsurance. In this case reinsurance can create an additional layer of "synthetic equity" capital to reduce the expected costs of financial distress by reducing the probability of default event, in which an insurer encounters financial distress. As an illustration we provide the graph (see figure below) of trade-off between reinsurance (i.e. quota share $a \in (0, 1)$) and excess of risk capital $v \geq 0$. This graph shows us that for every fixed level of expected financial distress (FD) costs the more reinsurance an insurer purchases the less additional layer of equity capital it needs.



5 Conclusion

In this article, we have investigated the demand for change-loss reinsurance in two single-period models of shareholders value creation. In both models the gross insurer premium is determined using an expected value premium principle that does not reflect the effect of insolvency on policy payoff. In the first model the required minimal risk capital is predetermined at the beginning of the period without taking into account possible purchase of reinsurance. In the second model an insurer is allowed to reduce its risk capital to the level under which the minimum solvency requirements are satisfied. We showed that there is no demand for reinsurance in the first, more conservative model without frictional costs. However, under the presence of frictional costs, such as corporate tax and financial distress costs, this model induces demand for reinsurance.

At the same time it was shown that the second model induces demand for reinsurance. In the frictionless environment this model has an optimal trade-off between the required minimal level of the risk capital and purchase of reinsurance. There is also demand for reinsurance in the second model under the presence of corporate tax and costs of financial distress.

The demand for reinsurance in the second model under the absence of frictional costs is likely due to the assumption of actuarial premium principle, according to which the premium is not adjusted with respect to the value of insolvency exchange option.

References

[1] Andrade, G. and Kaplan, S. How costly is financial (not economic) distress? Evidence

from highly levereged transactions that became distressed, *Journal of Finance*, Vol.53, pp.1443-1493, 1998.

[2] Brealey, R. and Myers, S. *Principle of corporate finance*, McGraw-Hill, 2003.

[3] Culp, C. *The ART of Risk Management*, John Wiley and Sons, 2002.

[4] Duffie, D. *Dynamic asset pricing theory*, Princeton University Press, 2001.

[5] Daykin, C., Pentikänen, T. and Pesonen, M. *Practical risk theory for actuaries*, Chapmen & Hall, New York, 1996.

[6] Garven, J. On the application of finance theory to the insurance firm, *Journal of Financial Services Research*, Vol.1, pp.57-76, 1987.

[7] Gerber, H. *An introduction to mathematical risk theory*, S.S. Huebner Foundation, Philadelphia, PA,1980.

[8] Hürlimann, W. Solvabilité et Réassurance, *Bulletin of the Swiss Association of Actuaries*, pp. 229-249, 1993.

[9] Jarrow, R. and Purnanandam, A. Capital structure and the present value of firm's investment opportunities: a reduced form credit risk perespective, *Working Paper, Johnson Graduate School of Management, Cornell University, NY*, January 15, 2004.

[10] Myers, S. and Cohn, R. Insurance rate regulation and the capital asset pricing model, In D. Cummins and S.Harrington, editors, *Fair Rate of Return in Property-Liability Insurance*, Kluwer Academic Publishers, Norwell, MA, 1987.

[11] Nakada, P., Shan, H., Koyluoglu, H. and Collingon, O. P&C RAROC: A Catalyst for Improved Capital Management in the Property and Casualty Insurance Industry, *The Journal of Risk Finance*, Fall 1999.

[12] Opler, T. and Titman, S. Financial distress and coporate performance, *Journal of Finance*, Vol.49, pp.1015-1040, 1994.

[13] SOA Economic Capital Calculation and Allocation Subgroup, Specialty Guide on Economic Capital, Technical Report, June 2003.

[14] Taksar, M. Optimal risk and divident distribution control models for an insurance company, *Math.Meth.Oper.Res.*, Vol.1, pp. 1-42, 2000.

[15] Venter, G. Premium calculation implications of reinsurance without arbitrage, *ASTIN Bulletin*, Vol.21, No.2, pp. 223-256, 1991.