Modelling optimal retirement planning: A simulation approach and an application to Japan

Sachi Purcal and John Piggott*

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Abstract: This paper examines lifetime personal financial planning. Using an optimising model in the tradition of Merton (1969, 1971), calibrated to stylised Japanese data, we explore the question of how individuals should determine their optimal consumption, portfolio selection, life insurance, and annuity choices, given uncertainty about investment returns and mortality. Both consumption and bequests appear as arguments in the individual’s preference function. The model explicitly recognises the existence of social security in retirement, and of loadings on insurance premiums, due to both administration costs and adverse selection, in the life insurance and annuities markets.

Three themes emerge from our results thus far. First, popular financial advice and the prescriptions of financial economics can be quantitatively assessed. An important example relates to age-phasing—the strategy of gradually reducing one’s exposure to risky assets over time. The model also suggests that the optimal level of life insurance purchase is much lower than traditional rules of thumb indicate. Second, the model provides a vehicle for assessing the impact of social security in terms of its effect on private optimisation decisions. For example, it is possible to quantify the impact of social security on expected financial accumulation and decumulation over the life cycle, and to examine its effect on life insurance and annuity purchases. Third, the model sheds light on the reasons for the thinness of voluntary life annuity markets worldwide. The relative importance of pre-existing annuitisation through social security, the role of bequests, and premium loadings are quantitatively assessed within a single optimising framework.

Further research, embracing multiple innovation processes (including a stochastic labour market), and encompassing more elaborate financial engineering strategies, is foreshadowed.
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1 Introduction

This paper examines lifetime personal financial planning—how should individuals determine their optimal consumption, portfolio selection, life insurance, and annuity choices, given uncertainty about investment returns and mortality? There is now a comprehensive literature which analyses optimal consumption and portfolio selection in a multi-period, finite-horizon setting, beginning principally with Samuelson (1969) and Merton (1969, 1971). Personal financial planning, however, also encompasses the insurance decisions of individuals, and this has not been much explored.

Governments and companies throughout the world appear to be moving in increasing numbers towards accumulation (or defined contribution) type retirement schemes in a move to make individuals responsible for their own retirement. As risk is shifted back onto individuals through the rise of defined contribution arrangements and the windback of social security, the insurance dimension of personal financial planning will become more important. An integrated quantitative approach is required, which simultaneously takes these various aspects of financial planning into account.

Merton (1971) introduced mortality to optimal consumption and investment models, incorporating a parametric survival model of mortality into his formulation. While much of the literature that has followed focuses only on optimal consumption and portfolio selection, Richard (1975) extended this model to consider additionally the optimal amount of life insurance. In doing

1 Many of its facets are mentioned in Merton (1990) and Duffie (1992).
2 See, for example, World Bank (1994) and Bodie and Papke (1992).
so, he substituted the parametric survival model of Merton (1971) with a more realistic tabular, or nonparametric, survival model along the lines of Yaari (1965).

The Richard model contains all the elements of personal financial planning. However, due to the complexity of the model, it is not at all straightforward to implement. This paper takes the Richard analysis as its starting point, and renders it operational by incorporating explicit behavioural functional forms and parameterising these using Japanese data.

Three themes emerge. First, this numerical approach allows the development of a relevant (though thus far restrictive) model of financial planning. This allows both popular financial advice and the prescriptions of financial economics to be quantitatively assessed. An important example relates to age-phasing—the strategy of gradually reducing one’s exposure to risky assets over time. There are many examples of this advice, which appears to be pervasive among financial planners. Jagannathan and Kocherlakota (1996) quote several sources for this financial advice, and offer a rule of thumb of their own—that the percentage of one's wealth in bonds should be no more than one's age.3

In addition, insurance purchases—both life insurance, which insures against early death and its impact on dependants, and annuities, which insure against outliving one’s resources—are optimally determined in the Richard model. These can be can be compared to actual purchases, and the purchases recommended by personal financial planners. In the case of life insurance, a number of authors have advocated the ‘human life value’ (henceforth HLV) of Huebner (1964).4

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3 These include The Wall Street Journal Guide to Planning Your Financial Future (Kenneth Morris, Alan Siegel, and Virginia Morris, 1995); Quinn (1991, p.489), investment columnist for Newsweek, and Malkiel (1996, p.411), all of whom advocate age-phasing strategies. Canner et al (1997) quote the commonly cited rule of thumb that states that the stock allocation should equal 100 minus an investor’s age. They note that Fidelity Investments (1994) offers a worksheet to help investors choose a portfolio allocation, which guides investors to a conservative, moderate, or aggressive portfolio based on a combination of risk preference and time horizon. Similarly, many mutual-fund companies offer “life cycle” funds in which the portfolio mix becomes more conservative as the investor ages (Vanessa O’Connell, 1995).

4 This is also the view that is taught to students of life insurance. See Black and Skipper (1987, pp 201-205).
Essentially, Huebner argues for individuals taking life insurance to the value of their future earnings, thus protecting their human capital, and protecting this asset in such a way that death leaves their family's net worth unchanged. Within the framework of the Richard model, however, such a rule of thumb can be shown to be questionable. Rather, an appropriate amount of life insurance should be based on current consumption, or a multiple thereof. In the case of annuities, the model suggests that annuitisation is under-exploited by retirees, a finding which we explore later in the paper.

Second, the model provides a vehicle for assessing existing social security arrangements, along with proposed reforms, and thus permits a novel perspective on the implications of social security. Suppose that in the absence of government intervention, individuals organise their finances optimally. Then the impact of social security can be assessed in terms of its effect on private optimisation decisions.

Finally, we use the model to explore why voluntary annuity markets are so thin, not just in Japan, but globally. There are several possible explanations for this phenomenon. Among them are the existence of a bequest motive, the loadings charged by insurance companies on private annuities, thus rendering this kind of insurance expensive, and the possibility that social security already provides annuity flows which are sufficient to meet demand. Model parameterisation permits the effects of these to be assessed.

The paper is organised as follows. We begin in section 2 with an exposition of the Richard model. Model parameterisation is also described in section 2. Section 3 reports the results of the model and analyses the findings, while section 4 concludes.
2 The Model and its Parameterization

In this section we set out the details of the Richard (1975) model which underlies this paper. Our discussion starts with a brief outline of the (rather) technical details of the model. For the reader’s convenience, we have included Table 1, which summarizes all the notation used in the model. In addition, an intuitive interpretation of the model is given in subsection 2.4, below.

In subsection 2.2 we treat the parameterization of the model, giving details of both the numerical values and functional forms used to produce our results. Of particular importance are the assumptions relating to the bequest function and loadings.

With the parameterization of the model explained, we give a brief account of our simulated solutions to the model in subsection 2.3. Lastly, as foreshadowed above, we present an intuitive account of the workings of the model.

2.1 The Model

Richard models a multi-period utility maximizing investor with objective

$$\max E \left[ \int_T^T U(C(t),t)dt + B(Z(T),T) \right],$$

(1)

where $T$ is the investor’s uncertain time of death, and $U$, $C$, $Z$ and $B$ are the investor’s utility, consumption, legacy at death and utility from bequest. The investor is able to choose between two securities, one risky and one risk risk-free, with the price of the risky asset, $Q$, following geometric Brownian motion

$$\frac{dQ(t)}{Q(t)} = \alpha dt + \sigma dq(t),$$

(2)
where \( dq(t) \) is a Wiener increment.

### Table 1 about here

The investor’s change in wealth is given by the stochastic differential equation

\[
dW(t) = -C(t) - P(t)dt + Y(t)dt + rW(t)dt + (\alpha - r) \pi W(t)dt + \sigma \pi W(t) dq(t),
\]

(3)

where \( P(t), Y(t), W(t) \) are, respectively, the investor’s life insurance premium paid, income (assumed to be non-stochastic), and wealth at time \( t \). From equation (2), the mean return on risky investment is \( \alpha \), with standard deviation \( \sigma \), while the risk-free investment returns \( r \); the investor places a proportion \( \pi \) of wealth in the risky asset.

Richard’s model necessarily incorporates the probability of death of an investor. Let the investor’s age-at-death, \( X \), a continuous random variable, have a cumulative distribution function given by \( F(x) \) and probability density function of \( f(x) \). Consequently, \( S(x) = 1 - F(x) \) gives the probability that the investor lives to age \( x \). The function \( S(x) \) is known as the survival function. The conditional probability density function (the probability the investor dies at exact age \( x \), having survived to that age) is given by \( f(x)/S(x) \), and is known as the force of mortality by demographers and actuaries, or as the hazard rate or intensity rate by reliability theorists (Elandt-Johnson & Johnson 1980).

The investor buys instantaneous term life insurance to the amount of \( Z(t) - W(t) \). For this, a premium of \( P(t) \) is paid. If we denote the force of mortality by \( \mu(t) \), then the amount of premium paid for actuarially fair insurance will be\(^1\)

\(^1\) Richard actually considered the more general case of there being some sort of ‘loading’ to mortality. This means mortality rates are increased to above their true levels, ensuring profitability for the life insurer. For the purposes of this paper, we consider only actuarially fair mortality rates.
\[ P(t) = \mu(t)(Z(t) - W(t)) \].

The investor’s problem is to solve equation (1), subject to budget constraint (3) and initial wealth condition \( W(0) = W_0 \), by optimal choice of controls \( C, \pi \) and \( Z \). The function \( U \) is assumed to be strictly concave in \( C \) and \( B \) is assumed strictly concave in \( Z \). Equation (1) can be re-expressed as

\[
J(W, \tau) = \max_{C,Z,\pi} E_\tau \int_\tau^\omega \left( \frac{S(T)}{S(\tau)} \right) \mu(T) \left[ \int_\tau^T U(C(t),t) dt + B(Z(T),T) \right] dT,
\]

where \( \omega \) represents the limiting age of the underlying mortality table, i.e., \( X \in [0, \omega] \). Applying Fubini’s theorem, equation (5) becomes

\[
J(W, \tau) = \max_{C,Z,\pi} E_\tau \int_\tau^\omega \frac{S(T)}{S(\tau)} \left[ \mu(T) B(Z(T),T) + U(C(T),T) \right] dT
\]

after swapping the order of integration over the triangle \( T \geq t, t \geq \tau \) in \( \mathbb{R}^2 \). The Hamilton-Jacobi-Bellman equation is therefore

\[
0 = \max_{C,Z,\pi} \{ \mu(t) B(Z(t),t) + U(C(t),t) - \mu(t) J + J_t + \mu(t) \}
\]

\[
\left[ \pi \alpha W + (1 - \pi) r W + Y - C - P \right] J_w + \frac{1}{2} \sigma^2 W^2 J_{ww} \}
\]

(7)

As \( Y(t) \) in equation (7) is non-stochastic, Richard demonstrates that (7) is equivalent to an equation involving capitalized \( Y(t) \). That is, \textit{adjusted wealth} is defined as

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2 This is a novel approach to simplifying (5). For a proof that Fubini’s theorem can be used in stochastic integration see, for example, Protter (1990).
\[ \tilde{W}(t) \equiv W(t) + b(t), \] 

(8)

where \( b(t) \) is defined as the capitalized value of future income:

\[
b(t) = \int_t^\omega Y(\theta) \frac{S(\theta)}{S(t)} e^{-\rho(\theta-t)} d\theta.
\]

(9)

The standard approach (Richard 1975, Bodie, Merton & Samuelson 1992) is to remove \( Y(t) \) from (7) and substitute \( \tilde{W}(t) \) for \( W(t) \). Income is thus treated as a traded asset. As Bodie et al. (1992) rightly point out, the individual never actually ‘sells’ his or her human capital, but rather hedges in traded securities to accomplish the same thing. As markets are assumed complete, future income can be perfectly hedged.

Richard provides an algebraic solution to the above model for CRRA utility. He demonstrates that when

\[
U(C(t),t) = h(t) \frac{C^\gamma(t)}{\gamma}, \gamma < 1, h > 0, C > 0
\]

(10)

\[
B(Z(t),t) = m(t) \frac{Z^\gamma(t)}{\gamma}, \gamma < 1, h > 0, Z > 0
\]

(11)

the optimal controls are given by

\[
C^*(W,t) = \left( \frac{h(t)}{a(t)} \right)^{\frac{1}{\gamma}} \left[ W + b(t) \right],
\]

(12)
\[ Z^*(W,t) \equiv W + \frac{P^*(W,t)}{\mu(t)} = \left( \frac{m(t)}{a(t)} \right)^{1/(1-\lambda)} [W + b(t)], \]

(13)

and

\[ \pi^*(W,t)W = \frac{\alpha-r}{(1-\lambda)\sigma^2} [W + b(t)] = \tilde{\pi}^*W \]

(14)

where

\[ \hat{a}(t) = \left\{ \int_{\sigma}^{\alpha} k(\theta) S(\theta) \exp \left[ \frac{\gamma}{1-\gamma} \left( \frac{(\alpha-r)^2}{2(1-\gamma)\sigma^2} + r \right) \right] d\theta \right\}^{1-\gamma} \]

(15)

and 4

\[ k(t) = \left\{ h^{\lambda/(1-\gamma)}(t) + \left[ \frac{1}{\mu(t)} \right]^{\gamma/(1-\gamma)} \left[ \mu(t)m(t)^{\lambda/(1-\gamma)} \right] \right\}. \]

(16)

The solutions are linear in adjusted wealth, a familiar result for HARA (hyperbolic absolute risk aversion) utility functions (Merton 1971). The solution for \( \tilde{\pi}^* \) indicates investment in the risky asset should be a constant fraction of adjusted wealth. This is an example of the well-known result that optimal investment behaviour over the life cycle, for utility functions that display constant relative risk aversion, is “myopic”, with individuals always investing a constant proportion of wealth in the risky asset and ignoring the future distribution of asset returns.

1 Note that equation (13) simplifies equation (32) of Richard, due to the consideration of actuarially fair insurance.

4 Equation (39) of Richard, which gives the formula for \( k(t) \), contains a typographical error. Equation (16) is our corrected version—for the case of actuarially fair life insurance.
2.2 Parameterization of the model

A critical step in producing numerical results for the model is the appropriate parameterization of the model. Although Richard has partially parameterized his model, by presenting a solution to the model for isoelastic utility and utility of bequest, key parts of the model remain unspecified. We start by considering the discount functions, \( h(t) \) and \( m(t) \).

2.2.1 Bequest Function

A plausible choice of \( h(t) \) is \( e^{-\rho t} \), where \( \rho \) is the rate of the investor’s time preference. The choice of \( m(t) \) is not clear, however. If \( m(t) \) is set equal to \( h(t) \) consideration of equations (12) and (13) readily yields that such a model would produce optimal consumption and bequest amounts that are identical. This result isn’t particularly appealing. Prevailing social norms seem to indicate a reasonable value to leave a surviving spouse would be an amount sufficient to provide \( \frac{2}{3} \) of the deceased’s current income for life. This notion has in fact been enshrined in pension benefits regulations in Canada, for example, where the surviving spouse of a deceased pensioner is provided with between 50% and 66\( \frac{2}{3} \% \) pension continuation.\(^5\)

Thus, we may provide an optimal legacy of

\[
Z^*(t) = \frac{\gamma}{2} C^*(t) \int_{\theta}^{\phi(\theta)} S(\theta) e^{-r(\theta-i)} d\theta
\]

and implies \( m(t) = e^{-\rho t} \phi^{k^{-T}}(t) \), from equations (12) and (13) above. While this would seem a highly appropriate choice for \( m(t) \), the inclusion of this functional form into the Richard model is not at all

\(^5\) See, for example, s.45(2) of Ontario’s Pensions Benefits Act, 1987.
straightforward, due to the interaction of survival probabilities of both the consumer and beneficiary.\textsuperscript{6}

Instead, we consider a simpler case. Let us assume that the investor wants to leave a certain annuity to his surviving spouse, which pays $\frac{3}{2} C^*(t)$ from the date of death to the limiting age of the mortality table. Thus, we wish to set

$$Z^*(t) = \frac{3}{2} C^*(t) \int e^{-r(t-\theta)} d\theta$$

implying that

$$m(t) = e^{-rt} \left( \frac{3}{2} \int e^{-r(t-\theta)} d\theta \right)^{-\gamma}.$$  

This is a more generous bequest function than the ideal, and this overprovision should be borne in mind when considering the results below.

\section*{2.2.2 Loadings}

Up to now, our consideration of the Richard model has been from the perspective of actuarially fair insurance and annuity provision. It is well known that insurers add loadings to their products to cover administrative costs and make provision against unforeseen fluctuations. Here we adjust the model to take these factors into account.

While the Richard model does consider the case of loadings to mortality rates, we do not adopt this approach. Rather, we consider the modification of premia in the presence of loadings. This is the approach to treating expenses commonly adopted by actuaries (Bowers et. al. 1997).

From the consumer’s perspective, an insurance premium consists of the actuarially fair premium, plus loadings. We thus model expenses in this way:

$$P(t) = \mu(t)(Z(t) - W(t)) + \Gamma(t)$$

\textsuperscript{6}These complications have been noted by other authors (Borch 1990, pp.257–260).
where $\Gamma(t)$ is the loading amount applied to the premium at time $t$. This adjustment to the Richard model entails minimum change to its solution; the optimum solution given by equations (12) and (14) remain unchanged, while we modify equation (13) to

$$Z^*(t) = W(t) + \frac{P^*(t) + \Gamma(t)}{\mu(t)}$$

the rest of the equation remaining unchanged.

Note that in the model with loadings, it is only the insurance and annuity side of the market which have loadings added to them. The capitalization of the individual’s income is still done at the (unloaded) actuarially fair rates.

### 2.2.3 Numerical Values

The economic and financial data we use to parameterize the model are summarized in Table 2. In Table 3 we set out the parameters we adopt for the model. The values we adopt reflect real values of asset accumulation, hence $\alpha$, the rate of return on the risky asset, is chosen as the real rate of return on the Nikkei: $(1.064/1.039) - 1 \approx 0.025$. The safe rate, $r$, is similarly chosen. We adopt Japanese male population mortality given by the Ministry of Health and Welfare (1995) Japanese Life Table, number 18, excluding the effects of the Kobe earthquake. Values of $\gamma$ of $-0.5$ and $-4$ reflect an individual who is somewhat risk averse and quite risk averse, respectively. We set the individual's yearly earnings to 12 times the average monthly cash earnings of regular employees for calendar year 1999: $12 \times ¥ 364,638 = ¥ 4,375,656$.\(^7\)

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\(^7\)These values are from the Japanese Institute of Labour webpage, [http://www.jil.go.jp/estatis/e0301.htm](http://www.jil.go.jp/estatis/e0301.htm).
We assume this amount remains constant over the employee’s working life.

### 2.3 Simulation

We determine the expected values of the state and control variables over an investor’s lifetime by simulation. Drawing on a standard result in mathematical finance, it can be shown that a valid approximation to equation (2) is given by

\[
\frac{\Delta Q(t)}{Q} = \alpha \Delta t + \sigma \sqrt{\Delta t} \varepsilon_t,
\]

where \( \varepsilon_t \sim N(0,1) \) and independent, and we measure time units in fortnights (\( \Delta t = 1/26 \) of a year), so that for an individual aged 30, we have 2,080 periods as they age from 30 to 110, the limiting age of Japanese Life Table, number 18, for males.

Using the approximation to \( dq(t) \) of \( \sqrt{\Delta t} \varepsilon_t \), we can also simulate the path of the state variable, \( W(t) \) in equation (3) over an individual’s lifetime. The investor’s optimal behaviour is given by equations (12) to (14), which may be evaluated analytically given simulated values of \( W(t) \) and an initial value, \( W(0) \). For the results we present below, we calculate expected values of the state and control variables at each of the 2,080 discrete each points in the individual’s lifetime. These expected values are based on 10,000 simulations of the individual’s lifetime.

### 2.4 Intuitive Interpretation

We conclude this section with an intuitive interpretation of the Richard model.

The investor’s objective in this model is to maximize his expected lifetime discounted utility (including the utility from bequest). This naturally leads to a stochastic control problem with...
a risky state variable, wealth, and several controls—the investor will attempt to achieve his objective through optimal choice of consumption, investment in risky assets and bequest.

In our model, we assume the investor will start at age 30 with initial wealth equal to one year of salary. The investor’s stock of wealth will then be reduced by consumption and expenditures on life insurance; it will be augmented by the receipt of income and annuity payments, as well as returns from his investments in risky and safe assets. This process is summarized in equation (3).

The investor’s purchase of life insurance is determined by both his optimal desired legacy, \( Z(t) \), and the current level of this financial wealth, \( W(t) \). Should the investor’s desired legacy exceed this wealth on hand, he necessarily buys life insurance to provide for this shortfall, \( Z(t) - W(t) \), on his death. In continuous time, the insurance premium he would pay for this benefit is \( \mu(t)(Z(t) - W(t)) \), which is given by equation (4) above.

Should the investor have no bequest motive, or should his financial wealth grow to exceed his desired bequest, then this excess will be annuitized. In this respect an annuity is the negative of life insurance.\(^8\) Indeed, it is represented in this model as a negative premium, contributing to the investor’s stock of wealth, rather than depleting it. That this approach to annuitization makes sense can be seen by considering the following: we can imagine the investor handing over the (excess) capital sum \( W(t) - Z(t) \) to the insurance company, to be invested strictly according to his preferences, in return for an annuity payment. In essence, the annuity payments are the bets the insurer pays the investor for his excess cash—the insurer is betting on the investor’s death.

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\(^{8}\) This symmetry has been observed by earlier authors. See, for example, Fischer (1973). In the case of life insurance, the insured makes a payment, in return for which the insurance company will pay a benefit contingent on death. In the case of a life annuity, the annuitant receives a payment, in return for which the insurance company will receive a benefit if the annuitant dies.
At the end of the contract \((t+dt\) in our continuous time model), the insurer can keep \(W(t) - Z(t)\) plus returns if the investor is dead; should the investor survive, \(W(t) - Z(t)\) plus returns will be returned to the investor. The process closely resembles that of a variable term life annuity. Note that if the investor had a bequest motive, the model indicates this bequest will be paid; the investor annuitizes only when \(W(t) > Z(t)\).

In the model we assume markets are complete, and the investor receives a deterministic wage. With respect to investment decisions, then, the investor considers his total wealth—consisting both his wealth on hand and the capitalized value of future income—represented by equations (8) and (9) above.

For isoelastic utility, and bequest and if geometric Brownian motion governs the stochastic process associated with risky asset returns, Richard has determined the values of the optimal controls, is given by equations (12)–(14) above. Using the parameterization of the model we have set out above, one can show that the optimal consumption of an individual at instant \(t\) is equivalent to the annuity payment he would receive if his total wealth at that instant were used to purchase both (i) a level life annuity, and (ii) a level reversionary annuity, paying at two-thirds the level of the life annuity, with certainty from the date of death to age \(\omega\), the limiting age of the mortality table. Thus, his optimal consumption decision is made such that he considers his total wealth as having to provide both for his lifetime consumption and an insurance benefit on his death. Richard’s model also shows that investor’s invest a constant fraction of total wealth in the risky asset.

These conclusions are similar to the simpler results of Merton (1969), where, in a model with certain lifetime and no bequest or insurance considerations, the optimal solutions for a model with isoelastic utility and geometric Brownian motion underlying the risky asset, optimal
consumption at a point in time was determined by annuitizing wealth at that moment over the consumer’s remaining certain life. The optimal investment in risky was also a constant fraction of wealth.

3 Results

In this section we present and interpret the results generated by the model so far. It is convenient to start with some simple observations. First, in an entirely deterministic world, standard preference maximisation, in which individuals are assumed to maximise a time-separable lifetime power utility function, generates a level consumption stream. When uncertain lifetimes are introduced, this result continues to hold, provided that actuarially fair annuities are available. In the absence of bequests, individuals will annuitise completely, and there will, of course, be no demand for life insurance. Introducing a bequest motive does not alter the level consumption result, provided that actuarially fair life insurance contracts are available. That is, the introduction of uncertain lifetimes makes no difference to economic behaviour, and in particular, individual preference for level consumption, provided this uncertainty can be insured against at actuarially fair prices.

When a risky asset is introduced into this framework, individuals maximise expected utility through some exposure to the risky asset. That is, in this setting, most individuals will prefer not to insure completely against investment risk. The result is that the expected consumption stream will not be level in general. Under the standard assumption that the rate of time preference is equal to the safe rate of return, the expected consumption path will rise through time. An individual who is extremely risk averse, and invests only in safe assets, will enjoy a certain and level consumption stream—the intuition of the previous paragraph continues to hold.
Overall, in the optimal solution to the financial planning problem, the proportion of total wealth invested in the risky asset will be constant, a result first derived by Merton (1969). This holds not only through the accumulation phase of the life cycle, but through the retirement phase as well. This feature of optimal financial planning has profound implications for the nature of retirement securities which are marketed. Recall that in the present model, human capital is assumed a safe asset. It therefore follows that early in the life cycle, when the value of human capital is high, the proportion of financial wealth exposed to the risky asset is very high. As human capital declines with age, more financial wealth is allocated to the safe asset, so as to maintain overall constancy of exposure to the risky asset.

This age-phasing result has been analysed most thoroughly by Bodie et al (1992). They provide two explanations for this pattern of lifetime financial investment. The first is based upon the Merton derivation alluded to above. The second is that typically, households have much more flexibility earlier in the life cycle, and are therefore better placed to adapt to a negative shock than they would be later in the life cycle. Only the first of these is relevant here, since there is no choice variable reflecting flexibility in our model. The standard control variable used in formal analysis of this problem is labour supply, which we are assuming fixed.

Bodie et al also point out that in practice, the purchase of owner-occupied housing provides a ready mechanism for debt-financed exposure to a risky asset. An exposure to risky assets equal to, roughly, 1000% of net financial wealth corresponds to a 90% mortgage, which is a fairly standard financial position among young home buyers.
Model results depicting expected time paths of variables of interest have been generated through numerical simulation in which the model is solved many times. Here, we typically take the average of 10,000 simulations.  

3.1 Financial planning results.

Figure 1 presents expected lifetime consumption for the 70 year period between the ages of 30 and 100 years. These accord with the above intuition—highly risk averse consumers make choices which generate a nearly level, but fairly low, expected consumption stream, while those who are less risk averse start at the same consumption point, but make choices which generate a gradually increasing consumption stream. Corresponding time profiles of expected wealth are depicted in Figure 2. Note that both expected financial and total wealth follow higher trajectories, the less risk averse the consumer.

![Figure 1 about here](image1)

![Figure 2 about here](image2)

The exposure of the consumer’s portfolio to the risky asset is very high at younger ages. Figure 3 reveals that at the start of the life cycle, for the less risk averse consumer, it exceeds 1,000%, and even for the more risk averse, exposure is several times the value of net financial wealth. After retirement, exposure is 33% for the less risk averse consumer, and 10% for the more risk averse.

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9 With a maximum effective length of life of 80 years, and discrete time periods spanning two weeks, more than
We next report results on insurance. Figure 4 contains the time profiles of both intended bequests, and sums assured, for households with the two reference risk aversion parameters. The intended bequest at any point is the sum of the sum assured (the life insurance payout at that point of time in the event of death) and accumulated wealth. The central specification of bequests in the model is the amount necessary to purchase a level and certain income stream, equal to two-thirds of consumption at that point in time, for a period of years equal to 110 less the age of the life insurance purchaser. Intended bequests thus gradually fall with age.

Figure 4 illustrates the symmetry of life insurance and annuities, described in subsection 2.4 above, and neatly exploited by the Richard model. The sum assured becomes negative as the consumer enters his 60s; a negative sum assured represents the capital sum released to the annuity provider and upon which annuity payments are based. After the age of 65, when the individual is forced to retire, the excess wealth gradually diminishes. The consumer is able to gradually run down his wealth, holding enough in reserve to meet his (diminishing) bequest target, and in each period, annuitising wealth not earmarked for either bequests or current consumption.

Figure 5 reports the patterns of life insurance premia and annuity payments. Life insurance premia begin at a low level, because the risk of mortality is small at young ages. The mortality effect more than offsets the large required death-contingent benefit. With time, however, increasing

20,000,000 separate solutions are obtained for each model specification.
mortality forces the premium up, even though the required sum assured is declining. As wealth accumulation approaches the bequest target, the requirement of life insurance diminishes, and with it the premium paid.

**Figure 5 about here**

Analogous with Figure 4, the insurance premium paid transforms into an annuity payout for the individual in his early 60s. At 65, there is a kink in the payout at retirement, but it then increases over time, along with mortality, even though the annuity premium gradually diminishes. At the age of 90, on average annuity payouts finance one third of all consumption expenditures, with the rest financed from wealth decumulation.

There are two surprises from this analysis. First, the life insurance requirement based on the HLV rule of thumb is much greater than the requirements suggested by the Richard model—see Table 4. At the age of 30, the HLV guidelines suggest a sum assured requirement some 40% greater than the present analysis. This ratio increases with age: at the age of 50, for example, HLV suggests a figure some 70% higher than that implied by our analysis.

**Table 4 about here**

Second, our analysis suggests that annuitisation would be far more widespread if actuarially fair variable life annuities were available. Table 5 indicates that the contribution of annuity income to consumption would be considerable, with about 13% of consumption funded from annuity income at age 70, rising to about 30% at age 90. This contrasts sharply with observed voluntary life annuity purchases, which in Japan, and in most other countries, are negligible.
The thinness of voluntary annuity markets has puzzled analysts for many years. Among possible explanations is the presence of social security, which provides a form of mandatory annuitisation for retired workers. The importance of social security in modifying personal financial choices has been underestimated by the personal finance industry worldwide. It is to this issue that we turn next.

3.2 The impact of social security

We introduce social security into the model by making use of the following contrivance. Since social security entitlements depend on employment, and since employment in this model is fixed by assumption, a social security scheme is equivalent to a modification of the human capital depreciation schedule. Wage income is adjusted to take account of contingent social security entitlements, such that the present value of total wealth at the beginning of the modelled life cycle is unchanged. This allows appropriately calibrated comparison of optimal financial choices with and without social security.

Japan’s social security system was established in the 1940s, with different groups of workers being covered by different programs. In 1985 it was restructured to resemble something close to the current system, although further reforms were enacted in 1994 and 1999. Until recently, benefit levels have progressively increased since its introduction.

The replacement rate for a full time worker on average earnings for 40 years is about 55%, if he retires at 65. Our calculations suggest that the replacement rate after 30 years is 33.3%, again
assuming retirement at 65. At this point, life expectancy is 17.1 years.\textsuperscript{10} We have stylised this in our model by using as our base case a 50% replacement rate after 35 years continuous employment. Since no taxes, other than an implicit social security tax, is levied, this replacement rate should be thought of as a net-of-income tax rate.

Our social security paradigm comprises a life annuity for the retired worker equal to a given proportion of his wage. Our base case sets this at 50%, but we model a 20% wage income replacement as well for some simulations. There is no provision for survivor insurance within this stylised social security program—bequests are unaffected by the introduction of the policy. Implicitly, as an implication of the model calibration to equivalence initial wealth in alternative model specifications, a payroll tax which raises revenue on an actuarially fair basis is levied on wage income.

Introducing social security into the model in this way is equivalent to mandating a safe saving flow. The household compensates for this by saving less privately, and by investing a higher proportion of personal financial wealth in risky assets. In the specifications considered here, it is possible for the consumer to offset the impact of social security completely.

The proportion of expected consumption financed by social security varies with the degree of risk aversion of the household. With relatively low risk aversion ($\gamma = -0.5$), social security payments finance about 60% of retirement consumption; with high risk aversion ($\gamma = -4$) social security dominates. This is because expected private wealth accumulation is inversely related to the degree of risk aversion.

In addition, life insurance and annuity purchases are affected by the presence of social security. Figure 6 shows that as social security becomes more important, the household will buy

\textsuperscript{10} Ministry of Health and Welfare, Japan.
more life insurance (because social security as specified does not carry a survivor’s benefit) and less annuities, and will postpone annuity purchase. For the case where social security is set at 50% of pre-retirement income, life annuities are not purchased at all.

The proportion of wealth invested in risky increases to compensate for the mandated safe investment embodied in social security. For $\gamma = -0.5$, the proportion of financial wealth invested in risky assets at age 40 rises from 135% in the no-social security case to 165% for the 50% case. At age 50, the increase is more dramatic—from 67% to 86%.

For a household on average earnings, the pattern of private wealth accumulations and insurance holdings revealed here is more likely to approximate the optimal pattern than the estimates reported for the no-social-security case, since most developed economies have well developed social security systems offering generous retirement benefits. An implication of the model is that for such a typical household, expected financial accumulations at retirement are nearly 25% less in the presence of social security. This suggests a much lower private saving rate in the presence of social security.

3.3 Annuity markets

The third area of interest which we develop from our model results focuses on the life annuity market. Voluntary annuity markets are thin everywhere, for reasons which are not clear. We begin this discussion with a brief discussion of the literature on this question.

Possible explanations for low annuity demand are a bequest motive, the desire to hold precautionary balances to cope with uninsurable events, over-annuitisation through publicly
provided social security, and insurance company loadings, linked either to adverse selection or administrative fees. What does seem clear is that life annuity prices are high, relative to population life expectancy and alternative investment returns. In an early widely cited paper which computes estimated loadings, Friedman and Warshawsky (1990) report that a typical male aged 65 in the US in the 1980s would have enjoyed a premium of 4.21 per cent per annum, had he purchased a government bond rather than a life annuity.

High annuity prices are frequently attributed to adverse selection. Mitchell et al. (1999) report load factors on actuarially fair quotes (the difference between the premium and the expected pension benefit) of 18% in the US for 1995. They attribute about half of the load factor to adverse selection. They also report a significant increase in the effect of adverse selection with age and a significantly smaller effect of adverse selection on annuity prices for women.

Of the alternative explanations for low annuity demand, economic analysis has focussed mainly on the bequest motive. While a desire to leave bequests would no doubt discourage voluntary annuity purchase, observation suggests that even those who might be supposed not to have a strong bequest motive (for example, the elderly with no children) do not purchase annuities. Hurd (1990) considers the interaction between private annuities and bequests in some detail, and tentatively concludes that the bequest motive is not necessary to explain the lack of demand for individual annuities. Further, many people who have the possibility of securing an individual life annuity in developed countries own their own homes, and their bequest motives may be satisfied through the transfer of this asset.

This assessment would appear to lead back to adverse selection as a major reason for the low demand for individual annuities. Hurd, however, also reports that in at least two experimental programs, elderly people in subsidised housing programs were offered the choice between a lump
sum and an actuarially fair annuity. Almost all took the lump sum, even though many had no children. One plausible explanation for this is the desire to hold precautionary balances to cope with uninsurable events. A second possible explanation is that the annuity promised a flat payment path, and it may be that very elderly individuals prefer lower consumption in return for higher consumption possibilities earlier in their retirement. Finally, the participants in these programs had relatively low incomes, and may have been “overannuitised” through social security.

The optimising model we have developed is able to shed new light on the question of which factors are the most important in determining annuity demand. We are able to manipulate bequest targets, social security, and administrative loadings in the specification of the model, and generate the optimal time paths for life insurance premiums and annuity payments under various combinations of these parameters. This allows comparison of optimal behaviour towards annuity purchase under alternative specifications to be compared, so that the relative importance of alternative explanations for low annuity demand can be weighed.

Figure 7 depicts time paths for five such combinations. When all three parameters of interest are set to zero—that is, there is no bequest motive, no social security, and the annuity quotes are actuarially fair—there is a gradual increase in annuity purchase throughout life. For the specifications we have chosen, the introduction of bequests reduces annuity demand the most. Social security comes next, followed by administrative loadings. When all these factors are present together, annuity demand is zero.

Figure 7 about here

In Figure 8, we explore the impact of bequests further, this time for our benchmark case in which 50% social security is present. (Loadings are kept at zero.) Because of the presence of social
security, individuals reduce their voluntary annuity purchases late in life, and begin to draw down their wealth at a rate faster than full annuitisation permits. As the bequest target rises, annuity demand diminishes, to a quite low level in the presence of bequests guaranteeing a 1/3 consumption stream to survivors, and eliminating annuity purchase completely with a 2/3 annuity purchase.

These results are probably sensitive to the stylised social security plan in the model, which has no survivor benefits. A social security scheme offering generous survivor benefit may well blunt the importance of the bequest specification as a determinant of annuity demand. It would in principle be possible to specify a social security scheme offering such generous survivor benefits that the household response would be increased annuity purchase, rather than increased life insurance purchase.

Figure 9 depicts the impact of loadings on expected consumption. As we suggested earlier, the expected consumption stream, with actuarially fair life insurance and annuity markets, will be level, or will gradually escalate with exposure to investment risk. When life and longevity insurance contracts are not actuarially fair, however, households will tend to self-insure in some degree. This effect is seen in Figure 9. Expected consumption (contingent on survival) peaks during the household’s 90s, and declines thereafter.

Table 6 reports the value of annuity payouts from age 65, capitalised to age 65. These values reflect the effects already demonstrated in the time-profiles. With no bequest, no social
security, and no loading, the value of annuity purchase stands at ¥29.2 million. This is an underestimate of the total value of annuity purchases, since under this specification, a household will annuitise throughout working life.

Table 6 about here

Interestingly, Table 6 suggests that loadings themselves are not a major deterrent to annuity purchase. A household with no social security entitlements and no bequest motive will buy two thirds of the actuarially fair annuity value, even with a 30% loading of the absolute value of the final premium.

4 Conclusion

In this paper, we have examined how households might optimally allocate their resources between different kinds of assets for investment purposes, between life insurance and longevity insurance, and between saving and consumption, in a framework in which both investment returns and mortality are uncertain. Households faced a variety of policy and market specifications, including varying social security payouts and a range of administrative loadings on life insurance and annuity purchases. Household preferences were varied across a range of risk aversion and bequest parameters.

Three sets of results are emphasised. First, this numerical approach allows the development of a relevant (though thus far restrictive) model of financial planning. This allows both popular financial advice and the prescriptions of financial economics to be quantitatively assessed. The model generates the age-phasing result of Bodie et al (1992), among other findings. In addition, life insurance demand, generated by a desire to purchase insurance against early death and its impact on
dependants, is shown to be considerably less than that advocated by guides such as the Human Life Value rule.

Second, the model provides a vehicle for assessing existing social security arrangements, along with proposed reforms, and thus permits a novel perspective on the macroeconomic implications of social security. Private financial accumulations are reduced by about 25% at the age of retirement as a result of a 50% of wage social security promise.

Finally, we use the model to explore why voluntary annuity markets are so thin, not just in Japan, but globally. There are several possible explanations for this phenomenon. Among them are the existence of a bequest motive, the loadings charged by insurance companies on private annuities, thus rendering this kind of insurance expensive, and the possibility that social security already provides annuity flows which are sufficient to meet demand. Results suggest that the bequest motive is the strongest single deterrent to annuity purchase, followed by social security. Administrative loadings, even when set as high as 30%, do not on their own lead to dramatic reductions in annuity purchases.

The major simplifying assumptions in the model as developed thus far revolve around the specification of labour. It is assumed that human capital is a safe asset, and that lifetime labour supply is fixed. These are clearly unrealistic assumptions—wages are uncertain, involuntary spells of unemployment occur, and the choice between labour and leisure is very flexible over the life cycle, particularly with respect to secondary labour force participation and choice of retirement age. We plan to focus on this issue in future research.
References


CEIC Database, DX Data, Sydney, 2000.


Table 1: Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Equation</th>
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<tr>
<td>$U$</td>
<td>Utility</td>
<td>(1)</td>
</tr>
<tr>
<td>$B$</td>
<td>Utility from bequest</td>
<td>(1)</td>
</tr>
<tr>
<td>$C$</td>
<td>Consumption</td>
<td>(1)</td>
</tr>
<tr>
<td>$Z$</td>
<td>Legacy at death</td>
<td>(1)</td>
</tr>
<tr>
<td>$Q$</td>
<td>Price of risky asset</td>
<td>(2)</td>
</tr>
<tr>
<td>$P$</td>
<td>Premium</td>
<td>(3)</td>
</tr>
<tr>
<td>$Y$</td>
<td>Income</td>
<td>(3)</td>
</tr>
<tr>
<td>$W$</td>
<td>Financial wealth</td>
<td>(3)</td>
</tr>
<tr>
<td>$\bar{W}$</td>
<td>Adjusted wealth</td>
<td>(8)</td>
</tr>
<tr>
<td>$S$</td>
<td>Survival function</td>
<td>(5)</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Limiting age of mortality table</td>
<td>(5)</td>
</tr>
<tr>
<td>$b$</td>
<td>Capitalised value of future income</td>
<td>(9)</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Hazard rate</td>
<td>(4)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Mean return of risky asset</td>
<td>(2)</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Standard deviation of risky asset return</td>
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</tr>
<tr>
<td>$r$</td>
<td>Risk-free asset return</td>
<td>(3)</td>
</tr>
<tr>
<td>$dq$</td>
<td>Weiner increment</td>
<td>(2)</td>
</tr>
<tr>
<td>$J$</td>
<td>Bellman function, or derived utility</td>
<td>(5)</td>
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<tr>
<td>$\gamma$</td>
<td>Risk aversion parameter</td>
<td>(10)</td>
</tr>
<tr>
<td>$h$</td>
<td>Discount function for utility</td>
<td>(10)</td>
</tr>
<tr>
<td>$m$</td>
<td>Discount function for bequest</td>
<td>(11)</td>
</tr>
<tr>
<td>$\pi$</td>
<td>Proportion of financial wealth invested in risky asset</td>
<td>(3)</td>
</tr>
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<td>$\tilde{\pi}$</td>
<td>Proportion of adjusted wealth invested in risky asset</td>
<td>(14)</td>
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<tr>
<td>$\rho$</td>
<td>Rate of time preference</td>
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<tr>
<th></th>
<th>Wages(^1)</th>
<th>Prices(^2)</th>
<th>Nikkei(^3)</th>
<th>Bill Rate(^4)</th>
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<td>5.9%</td>
<td>3.9%</td>
<td>6.4%</td>
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<td>Volatility</td>
<td>2.2%</td>
<td>2.5%</td>
<td>19.9%</td>
<td>0.9%</td>
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<tr>
<td>Range</td>
<td>(-1.9%, 29.1%)</td>
<td>(-1.1%, 24.7%)</td>
<td>(-41.1%, 99.4%)</td>
<td>(0.5%, 12.2%)</td>
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Sources:
Table 3: Parameters used in numerical simulation of the model.

<table>
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<tr>
<th>Parameter</th>
<th>Value</th>
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<td>0.025</td>
</tr>
<tr>
<td>$r$</td>
<td>0.005</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.005</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.2</td>
</tr>
<tr>
<td>Mortality</td>
<td>JLT18 (male)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>(-0.5, -4)</td>
</tr>
<tr>
<td>$\omega$</td>
<td>110</td>
</tr>
<tr>
<td>$Y (=W(0))$</td>
<td>¥ 4 375 686</td>
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Table 4: Optimal Bequests and Sums Assured (γ varying) versus HLV Sums Assured

<table>
<thead>
<tr>
<th>Age</th>
<th>γ = -0.5</th>
<th></th>
<th>γ = -4</th>
<th></th>
<th>HLV Sum Assured</th>
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</thead>
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<tr>
<td></td>
<td>Bequest</td>
<td>Sum Assured</td>
<td>Bequest</td>
<td>Sum Assured</td>
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</tr>
<tr>
<td>30</td>
<td>105.37</td>
<td>101.00</td>
<td>105.48</td>
<td>101.10</td>
<td>140.53</td>
</tr>
<tr>
<td>40</td>
<td>100.41</td>
<td>68.67</td>
<td>95.99</td>
<td>70.08</td>
<td>102.85</td>
</tr>
<tr>
<td>50</td>
<td>94.11</td>
<td>35.83</td>
<td>85.68</td>
<td>38.11</td>
<td>63.24</td>
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<tr>
<td>60</td>
<td>85.48</td>
<td>2.25</td>
<td>74.32</td>
<td>4.76</td>
<td>21.61</td>
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<tr>
<td>70</td>
<td>74.44</td>
<td>-12.95</td>
<td>61.86</td>
<td>-10.72</td>
<td>#N/A</td>
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(All amounts in ¥m. Negative sums assured indicate capital sums going toward annuity purchase.)
Table 5: Capital used for annuity purchase together with resulting annuity flow
(γ varying)

<table>
<thead>
<tr>
<th>Age</th>
<th>Capital for Annuity Purchase (¥m)</th>
<th>Capital for Annuity Payment (¥m p.a.)</th>
<th>Capital for Annuity Purchase (¥m)</th>
<th>Capital for Annuity Payment (¥m p.a.)</th>
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<td>60</td>
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<td>#N/A</td>
<td>#N/A</td>
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<tr>
<td>70</td>
<td>12.95</td>
<td>0.33</td>
<td>10.72</td>
<td>0.27</td>
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<tr>
<td>80</td>
<td>7.69</td>
<td>0.56</td>
<td>6.09</td>
<td>0.44</td>
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<tr>
<td>90</td>
<td>4.16</td>
<td>0.89</td>
<td>3.15</td>
<td>0.67</td>
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<td>100</td>
<td>2.42</td>
<td>1.12</td>
<td>1.75</td>
<td>0.81</td>
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Table 6: Expected Present Value of Annuity Payments, from age 65, Expense Loading and Bequest Motive Varying

Zero Bequest

<table>
<thead>
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<th>0%</th>
<th>20%</th>
<th>50%</th>
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<tr>
<td>Zero</td>
<td>29.23</td>
<td>23.53</td>
<td>16.22</td>
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<tr>
<td>Loading 15%</td>
<td>23.23</td>
<td>18.69</td>
<td>12.77</td>
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<tr>
<td>Loading 30%</td>
<td>19.27</td>
<td>15.50</td>
<td>10.41</td>
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Bequest: 1/3 C* Annuity

<table>
<thead>
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<th>Social Security Replacement Rate</th>
<th>0%</th>
<th>20%</th>
<th>50%</th>
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</thead>
<tbody>
<tr>
<td>Zero</td>
<td>16.48</td>
<td>10.81</td>
<td>3.55</td>
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<tr>
<td>Loading 15%</td>
<td>13.87</td>
<td>9.08</td>
<td>2.45</td>
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<tr>
<td>Loading 30%</td>
<td>11.95</td>
<td>7.79</td>
<td>1.40</td>
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Bequest: 2/3 C* Annuity

<table>
<thead>
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<th>50%</th>
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<tbody>
<tr>
<td>Zero</td>
<td>7.14</td>
<td>1.45</td>
<td>0.00</td>
</tr>
<tr>
<td>Loading 15%</td>
<td>6.12</td>
<td>1.01</td>
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<tr>
<td>Loading 30%</td>
<td>5.34</td>
<td>0.57</td>
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(Loadings are expressed as a percentage of the final premium)
Figure 1: Expected Lifetime Consumption, $\gamma$ Varying

![Graph showing expected lifetime consumption with $\gamma$ varying from -4 to 0.5.](japan_half_4_graphs.xls)
Figure 2: Expected Wealth Components, $\gamma$ Varying
(In legend—$W(t)+b(t)$: total wealth; $W(t)$: financial wealth; $b(t)$: human capital)
Figure 3: Expected Investment in Risky—as a Proportion of Financial Wealth, $W(t)$
Figure 4: Expected Bequests and Sums Assured
(B and SA, respectively, in legend)
Figure 5: Expected Insurance Premia Paid (+ values) and Annuity Payments Received (− values)
Figure 6: Expected Insurance Premia Paid (+ values) and Annuity Payments Received (– values), Social Security Replacement Rates Varying

vary_bequest_vary_ss_replacement.xls
Figure 7: Expected Insurance Premia Paid (+ values) and Annuity Payments Received (− values)—Varying Social Security Replacement, Bequest Motives, and Expense Loadings

(Legend—S: social security replacement rate, B: bequest annuity, L: premium loading)
Figure 8: Expected Insurance Premia Paid (+ values) and Annuity Payments Received (– values), Varying by Bequest (50% social security replacement)

(In legend—B: bequest annuity)
Figure 9: Expected Lifetime Consumption, Loadings Varying
(In legend—L: loading to actuarially fair premium, as a percentage of absolute value of final premium)