Tail Conditional Expectations for Elliptical Distributions*

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Abstract

Significant changes in the insurance and financial markets are giving increasing attention to the need for developing a standard framework for risk measurement. Recently, there has been growing interest among insurance and investment experts to focus on the use of a tail conditional expectation because it shares properties that are considered desireable and applicable in a variety of situations. In particular, it satisfies requirements of a "coherent" risk measure in the spirit developed by Artzner, et al. (1999). In this paper, we derive explicit formulas for computing tail conditional expectations for elliptical distributions, a family of symmetric distributions which includes the more familiar normal and student-t distributions. We extend this investigation to multivariate elliptical distributions allowing us to model combinations of correlated risks. We are able to exploit properties of these distributions naturally permitting us to decompose the conditional expectation so that we are able to allocate contribution of individual risks to the aggregated risks. This is meaningful in practice particularly in the case of computing capital requirements for an institution who may have several lines of correlated business and is concerned of fairly allocating the total capital to these constituents.

*Keywords: elliptical distributions, tail VAR, tail conditional expectations, coherent risk measure.

1 Introduction

Consider a loss random variable X whose distribution function we shall denote by $F_X(x)$ and the tail function by $\overline{F}_X(x) = 1 - F_X(x)$. This may refer to the total claims for an insurance company or to the total loss in a portfolio of investment for an individual or institution. The *tail conditional expectation* (TCE) is defined to be

$$TCE_X(x_q) = E(X \,|X > x_q) \tag{1}$$

and is interpreted as the expected worst possible loss. Given the loss will exceed a particular value x_q, generally referred to as the q-th quantile with

$$\overline{F}_X(x_q) = 1 - q,$$

the TCE defined in (1) gives the expected loss that can potentially be experienced. This index has been initially recommended by Artzner, et al. (1999) to measure both market and non-market risks presumably for a portfolio of investments. It gives a measure of a right-tail risk, one for which actuaries are very familiar with because insurance contracts typically possess exposures subject to "low-frequency but large-losses", as pointed by Wang (1998). Furthermore, computing expectations based on conditional tail events is a very familiar process to actuaries because many insurance policies also contain deductibles below which the policyholder must incur and reinsurance contracts always involve some level of retention from the ceding insurer.

A *risk measure* ϑ is a mapping from the random variable that generally represents the risk to the set of real numbers:

$$\vartheta : X \to \mathbf{R}.$$

It is supposed to provide a value for the degree of risk or uncertainty associated with the random variable. A risk measure is said to be a *coherent risk measure* if it satisfies the following properties:

1. **Subadditivity:** For any two risks X_1 and X_2, we have

   $$\vartheta(X_1 + X_2) \leq \vartheta(X_1) + \vartheta(X_2).$$

   This property requires that combining risks will be less risky than treating the risks separately. It means that there has to be something gained from diversification.

2. **Monotonicity:** For any two risks X_1 and X_2 where $X_1 \leq X_2$ with probability 1, we have

   $$\vartheta(X_1) \leq \vartheta(X_2).$$

   This says that the value of the risk measure is greater for risks considered more risky.

3. **Positive Homogeneity:** For any risk X and any positive constant λ, we have

$$\vartheta\left(\lambda X\right)=\lambda\vartheta\left(X\right).$$

If the risk exposure of a company is proportionately increased or decreased, then its risk measure must also increase or decrease by an equal proportionate value. To illustrate, an insurer may buy a quota share reinsurance contract whereby risk X is reduced to λX. The insurer must also decrease its risk measure by the same proportion.

4. **Translation Invariance:** For any risk X and any constant α, we have

$$\vartheta\left(X+\alpha\right)=\vartheta\left(X\right)+\alpha.$$

This says that increasing (or decreasing) the risk by a constant (risk not subject to uncertainty) should accordingly increase (or decrease) the risk measure by an equal amount.

Artzner, et al. (1999) demonstrated that the tail conditional expectation satisfies all requirements for a coherent risk measure. When compared to the traditional Value-at-Risk (VaR), the tail conditional expectation provides a more conservative measure of risk for the same level of degree of confidence $(1-q)$. To see this, note that

$$VaR_X\left(1-q\right)=x_q$$

and since we can re-write formula (1) as

$$TCE_X\left(x_q\right)=x_q+E\left(X-x_q\left|X>x_q\right.\right)$$

then

$$TCE_X\left(x_q\right)\geq VaR_X\left(1-q\right)$$

because the second term is clearly non-negative. Artzner and his co-authors also showed that the Value-at-Risk does not satisfy all requirements of a coherent risk measure. In particular, it violates the sub-additivity.

Another interesting feature of the index defined in (1) is that when viewed as a function of x, for which $TCE_X\left(x\right)$ may be called the tail conditional expectation function, it completely determines the distribution for a continuous random variable X, with finite expectation. To see this, we suppose a is the smallest possible value of x and note that

$$\begin{aligned}TCE_X\left(x\right) &= \frac{1}{\overline{F}_X\left(x\right)}\int_x^\infty udF_X\left(u\right)=-\frac{1}{\overline{F}_X\left(x\right)}\int_x^\infty ud\overline{F}_X\left(u\right)\\ &= x+\frac{1}{\overline{F}_X\left(x\right)}\int_x^\infty\overline{F}_X\left(u\right)du\end{aligned}$$

which implies

$$\frac{1}{x - TCE_X(x)} = \frac{\overline{F}_X(x)}{-\int_x^\infty \overline{F}_X(u)\,du} = \frac{d}{dx}\log\left(-\int_x^\infty \overline{F}_X(u)\,du\right).$$

Thus, we have

$$\overline{F}_X(x) = \frac{TCE_X(a) - a}{TCE_X(x) - x}\exp\left\{-\int_a^x [TCE_X(u) - u]^{-1}\,du\right\}$$

which says that once the function $TCE_X(x)$ is known, the distribution of X can be uniquely determined. For example, for a positive random variable, if $TCE_X(x) = \mu + x$, for some constant μ. Then, the tail probability of X is given by

$$\overline{F}_X(x) = \frac{\mu}{\mu}\exp\left\{\int_0^x \left(-\frac{1}{\mu}\right)du\right\} = \exp\left(-\frac{x}{\mu}\right)$$

which is the tail function of an exponential with mean μ. For an exponential distribution with mean μ, we then observe that its TCE is given by $TCE_X(x_q) = \mu + x_q$.

For the familiar normal distribution $N(\mu, \sigma^2)$ with mean μ and variance σ^2, it was noticed by Panjer (2002) that

$$TCE_X(x_q) = \mu + \left[\frac{\frac{1}{\sigma}\varphi\left(\frac{x_q-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{x_q-\mu}{\sigma}\right)}\right]\sigma^2 \tag{2}$$

where $\varphi(\cdot)$ and $\Phi(\cdot)$ are respectively the density and cumulative distribution functions of a standard normal $N(0,1)$ random variable. We extend this result into the larger class of elliptical distributions for which the normal distribution belongs to. This family essentially includes symmetric distributions for which the Student-t, exponential power, and logistic distributions are other familiar examples.

In this paper, we show that for univariate elliptical distributions, tail conditional expectations have the form

$$TCE_X(x_q) = \mu + \lambda \cdot \sigma^2 \tag{3}$$

where

$$\lambda = \frac{\dfrac{1}{\sigma}f_{Z^*}\left(\frac{x_q-\mu}{\sigma}\right)}{\overline{F}_Z\left(\frac{x_q-\mu}{\sigma}\right)}\sigma_Z^2, \tag{4}$$

Z is the spherical random variable that generates the elliptical random variable X, and has variance $\sigma_Z^2 < \infty$, and $f_{Z^*}(x)$ is the density of another spherical random variable Z^* corresponding to Z. For the case of the normal distribution, $Z^* = Z$ and is therefore a standard normal random variable with $\sigma_Z^2 = 1$ and (3) coinciding with

(2). We also considered the important case when the variance of X does not exist. In general, though, we find that we can express λ in (4) as

$$\lambda = \frac{\frac{1}{\sigma}\overline{G}\left(\frac{1}{2}z_q^2\right)}{\overline{F}_Z\left(z_q\right)},$$

where $\overline{G}$ is a tail-type function involving the *cumulative generator* later defined in this paper. This generator plays an important role in developing the tail conditional expectation formulas for elliptical distributions.

The use of the tail conditional expectation to compute capital requirements for financial institutions has recently been proposed. See, for example, Wang (2002). It has the intuitive interpretation that it provides the expected amount of a loss given that a shortfall occurs. The amount of shortfall is measured by a quantile from the loss distribution. Furthermore, by the additivity property of expectation, it allows for a natural allocation of the total capital among its various constituents:

$$E\left(S\left|S > x_q\right.\right) = \sum_{k=1}^{n} E\left(X_k\left|S > x_q\right.\right),$$

where $S = X_1 + \cdots + X_n$. Thus, we see that $E\left(X_k\left|S > x_q\right.\right)$ is the contribution of the k-th risk to the aggregated risks. Panjer (2002) examined this allocation formula in the case where the risks are multivariate normal. We advance this formula in the general framework of multivariate elliptical distributions. This class of distributions is widely becoming popular in actuarial science and finance. See, for example, Embrechts, et al. (1999, 2001) and Bingham and Kiesel (2002).

For the rest of the paper, it is then organized as follows. In Section 2, we provide preliminary discussion about elliptical distributions and we state that for elliptically distributed random variables, it is closed under linear transformations. We also give examples of known multivariate distributions belonging to this class. In Section 3, we develop tail conditional expectation formulas for univariate elliptical distributions. Here we introduce the notion of a cumulative generator which plays an important role in evaluating TCE. In Section 4, we exploit the properties of elliptical distributions which allow us to derive explicit forms of the decomposition of TCE of sums of elliptical risks into individual component risks. We give concluding remarks in Section 5.

2 The Class of Elliptical Distributions

Elliptical distributions are generalizations of the multivariate normal distributions and therefore share many of its tractable properties. This class of distributions was introduced by Kelker (1970) and was widely discussed in Fang, et al. (1987). This generalization of the normal family seems to provide an attractive tool for actuarial and financial risk management because it preserves the property to be regular varying of marginal tails into the multivariate circumstance (Shmidt, 2002).

Let Ψ_n be a class of functions $\psi(t) : [0,\infty) \rightarrow \mathbf{R}$ such that function $\psi(\sum_{i=1}^{n} t_i^2)$ is an n-dimensional characteristic function (Fang, et al., 1987). It is clear that

$$\Psi_n \subset \Psi_{n-1} \cdots \subset \Psi_1.$$

Consider an n-dimensional random vector $\mathbf{X} = (X_1, X_2, ..., X_n)^T$.

Definition 1 *The random vector* $\mathbf{X}$ *has a multivariate elliptical distribution, written as* $\mathbf{X} \backsim \mathbf{E}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$, *if its characteristic function can be expressed as*

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \exp(it^T \mu)\psi\left(\tfrac{1}{2}\mathbf{t}^T \boldsymbol{\Sigma} \mathbf{t}\right) \tag{5}$$

for some column-vector $\boldsymbol{\mu}$, $n \times n$ *positive-definite matrix* $\boldsymbol{\Sigma}$, *and for some function* $\psi(t) \in \Psi_n$, *which is called the characteristic generator.*

$>$From $\mathbf{X} \backsim \mathbf{E}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$, generally speaking, it does not follow that $\mathbf{X}$ has a density $f_{\mathbf{X}}(\mathbf{x})$, but if the density exists it has the following form

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\boldsymbol{\Sigma}|}} g_n \left[\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right], \tag{6}$$

for some function $g_n(\cdot)$ called the density generator. The condition

$$\int_0^\infty x^{n/2-1} g_n(x) dx < \infty. \tag{7}$$

guarantees $g_n(x)$ to be density generator (Fang, et al. 1987, Ch 2.2). If density generator does not depend on n which may happen in many cases, we drop the subscript n and simply write g. In addition, the normalizing constant c_n can be explicitly determined by transforming into polar coordinates and the result is

$$c_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[\int_0^\infty x^{n/2-1} g_n(x) dx\right]^{-1}. \tag{8}$$

The detailed evaluation of this result is given in the appendix. One may also similarly introduce the elliptical distribution by the density generator and then write $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$.

$>$From (5) follows that $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$ and A be some $m \times n$ matrix of rank $m \leq n$ and b some m-dimensional column-vector , then

$$A\mathbf{X} + b \sim E_m\left(A\boldsymbol{\mu} + b, A\boldsymbol{\Sigma}A^T, g_m\right). \tag{9}$$

In other words, any linear combination of elliptical distributions is another elliptical distribution with the same characteristic generator ψ or from the same sequence of density generators $g_1, ...g_n$, corresponding to ψ. Therefore any marginal distribution of $\mathbf{X}$ is also elliptical with the same characteristic generator. In particular, for $k = 1, 2, ..., n$, $X_k \sim E_1\left(\mu_k, \sigma_k^2, g_1\right)$ so that its density can be written as

$$f_{X_k}(x) = \frac{c_1}{\sigma_k} g_1 \left[\frac{1}{2} \left(\frac{x - \mu_k}{\sigma_k} \right)^2 \right]. \tag{10}$$

If we define the sum $S = X_1 + X_2 + \cdots + X_n = \mathbf{e}^T \mathbf{X}$, where $\mathbf{e} = (1, ..., 1)^T$ is a column vector of ones with dimension n, then it immediately follows that

$$S \sim E_n\left(\mathbf{e}^T \boldsymbol{\mu}, \mathbf{e^T} \boldsymbol{\Sigma} \mathbf{e}, g_1\right). \tag{11}$$

Let us notice that condition (7) does not require existence of the mean and covariance of vector $\mathbf{X}$. Later we give the example of multivariate elliptical distribution with infinite mean and variance. It can be shown by a simple transformation in the integral for the mean that

$$\int_0^\infty g_1(x) dx < \infty \tag{12}$$

guarantees the existence of the mean, and then the mean vector for $\mathbf{X} \sim E_n\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n\right)$ is $E\left(\mathbf{X}\right) = \boldsymbol{\mu}$. If in addition

$$|\psi'(0)| < \infty, \tag{13}$$

the covariance matrix exists and is equal to (Cambanis, et al. 1981)

$$Cov\left(\mathbf{X}\right) = -\psi'\left(0\right) \boldsymbol{\Sigma}. \tag{14}$$

Then the characteristic generator can be chosen such that

$$\psi'\left(0\right) = -1 \tag{15}$$

so that the covariance above becomes

$$Cov\left(\mathbf{X}\right) = \boldsymbol{\Sigma}.$$

Notice that condition (13) is equivalent to the condition $\int_0^\infty \sqrt{x} g_1(x) dx < \infty$.

We now consider some important families of elliptical distributions.

2.1 Multivariate Normal Family

An elliptical vector $\mathbf{X}$ belongs to the multivariate normal family with the density generator

$$g\left(u\right) = e^{-u} \tag{16}$$

(which does not depend on n). We shall write $\mathbf{X} \sim \mathbf{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. It is easy to see that the joint density of $\mathbf{X}$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\boldsymbol{\Sigma}|}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]$$

>From (8), it immediately follows that the normalizing constant is given by $c_n = (2\pi)^{-n/2}$. It is well-known that its characterictic function is

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \exp\left(i\mathbf{t}^T\boldsymbol{\mu} - \tfrac{1}{2}\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}\right)$$

so that the characteristic generator is

$$\psi(t) = e^{-t}.$$

Notice that choosing the density generator in (16) automatically gives $\psi'(0) = -1$ and hence $\boldsymbol{\Sigma} = Cov(\mathbf{X})$.

2.2 Multivariate Student t Family

An elliptical vector $\mathbf{X}$ is said to have a multivariate Student t distribution if its density generator can be expressed as

$$g_n(u) = \left(1 + \frac{u}{k_p}\right)^{-p} \tag{17}$$

where the parameter $p > n/2$ and k_p is some constant that may depend on p. We write $\mathbf{X} \sim \mathbf{t}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}; p)$ if $\mathbf{X}$ belongs to this family. Its joint density has therefore the form

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\boldsymbol{\Sigma}|}} \left[1 + \frac{(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}{2k_p}\right]^{-p}.$$

Using (8), it can be shown that the normalizing constant is

$$c_n = \frac{\Gamma(p)}{\Gamma(p-n/2)}(2\pi k_p)^{-n/2}.$$

Here we introduced the multivariate Student t in its most general form. Similar to this form was considered in Gupta and Varga (1993) where they called this family Symmetric Multivariate Pearson Type VII distributions. Taking for example $p = (n+m)/2$ where n and m are integers, and $k_p = m$, we get the traditional form of the multivariate Student t distribution with density

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\Gamma((n+m)/2)}{(\pi m)^{n/2}\Gamma(m/2)\sqrt{|\boldsymbol{\Sigma}|}} \left[1 + \frac{(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}{m}\right]^{-(n+m)/2}. \tag{18}$$

In the univariate case where $n = 1$, Bian and Tiku (1997) and MacDonald (1996) suggested to put $k_p = (2p-3)/2$ if $p > 3/2$ to get the so-called Generalized Student t (GST) univariate distribution with density

$$f_X(x) = \frac{1}{\sigma\sqrt{2k_p}B(1/2, p-1/2)}\left[1 + \frac{(x-\mu)^2}{2k_p\sigma^2}\right]^{-p},$$

where $B(\cdot,\cdot)$ is the beta function. This parameterization leads to the important property that $Var(X) = \sigma^2$. In the case where $1/2 < p \leq 3/2$, the variance does not exist and one can put $k_p = 1/2$. In Landsman and Makov (1999) and Landsman (2002), credibility formulas were examined for this family. Figure 1 shows density functions for the Generalized Student t distributions with different parameter values of p. The values of μ and σ are respectively chosen to be 0 and 1. The smoothed curve in the figure corresponds to the case of the standard normal distribution.

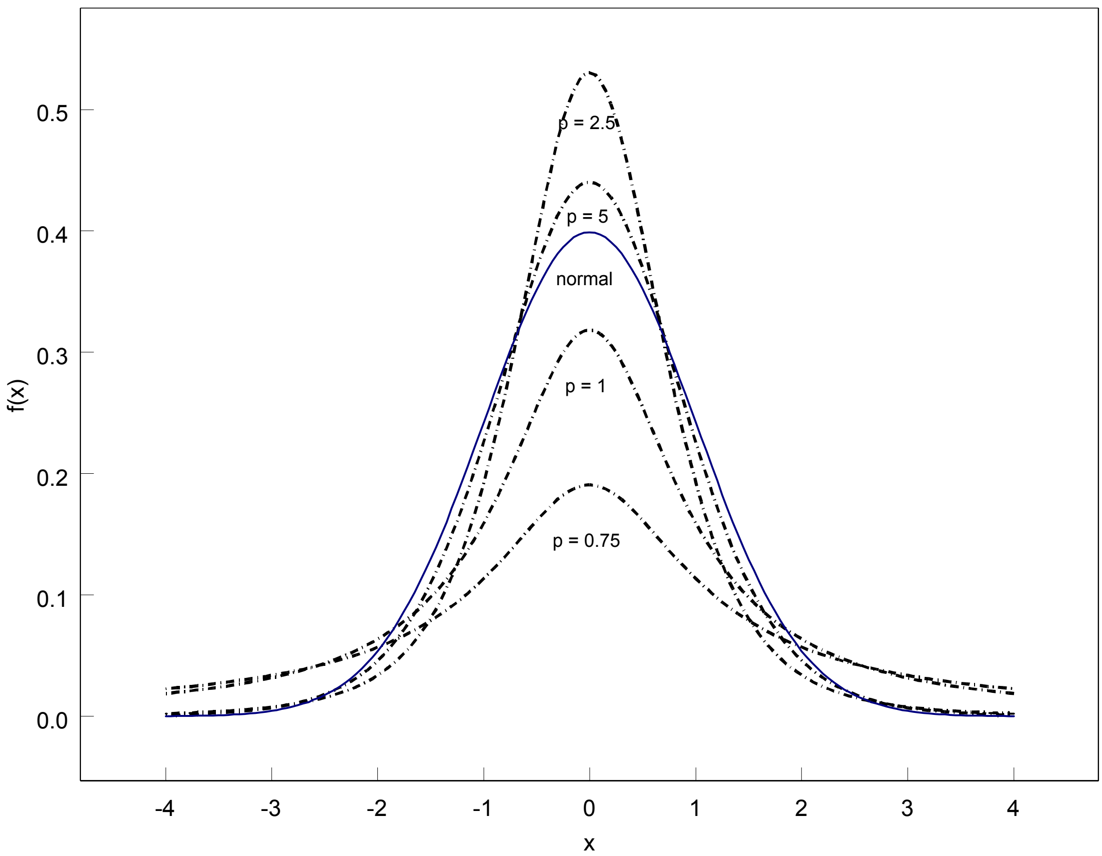

Figure 1: Density functions for the Generalized Student t distribution.

Now extending this to the multivariate case, we suggest to keep $k_p = (2p-3)/2$ if $p > 3/2$, then this multivariate GST has the advantage that

$$Cov(\mathbf{X}) = \mathbf{\Sigma}.$$

In particular, for $p = (n+m)/2$, we suggest instead of (18) to consider

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\Gamma((n+m)/2)}{[\pi(n+m-3)]^{n/2}\,\Gamma(m/2)\sqrt{|\mathbf{\Sigma}|}}\left[1+\frac{(\mathbf{x}-\boldsymbol{\mu})^T\,\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}{n+m-3}\right]^{-(n+m)/2}$$

because it also has the property that the covariance is $Cov(\mathbf{X}) = \mathbf{\Sigma}$. If $1/2 < p \leq 3/2$, the variance does not exist and we have a heavy-tailed multivariate distribution. If $1/2 < p \leq 1$, even the expectation does not exist. In the case where $p = 1$, we have the multivariate Cauchy distribution with density

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\Gamma(\frac{n+1}{2})\pi^{-(n+1)/2}}{\sqrt{|\mathbf{\Sigma}|}}\left[1+(\mathbf{x}-\boldsymbol{\mu})^T\,\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]^{-(n+1)/2}.$$

2.3 Multivariate Logistic Family

An elliptical vector $\mathbf{X}$ belongs to the family of multivariate logistic distributions if its density generator has the form

$$g(u) = \frac{e^{-u}}{(1+e^{-u})^2}.$$

Its joint density has the form

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\mathbf{\Sigma}|}}\frac{\exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\,\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]}{\left\{1+\exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\,\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]\right\}^2}$$

where the normalizing constant can be evaluated using (8) as follows

$$c_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}}\left[\int_0^\infty x^{n/2-1}\frac{e^{-x}}{(1+e^{-x})^2}dx\right]^{-1}.$$

We observe that this normalizing constant has been mistakenly printed in both Fang, et al. (1990) and Gupta and Varga (1993). Further simplification of this normalizing constant suggests that by first observing that $\frac{e^{-x}}{(1+e^{-x})^2} = \sum_{j=1}^{\infty}(-1)^{j-1}je^{-jx}$ and then re-writing it as follows:

$$
\begin{aligned}
c_n &= \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[\sum_{j=1}^{\infty} (-1)^{j-1} \int_0^{\infty} x^{n/2-1} j e^{-jx} dx \right]^{-1} \\
&= \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[\sum_{j=1}^{\infty} (-1)^{j-1} j^{1-n/2} \int_0^{\infty} y^{n/2-1} e^{-y} dy \right]^{-1} \\
&= \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[\sum_{j=1}^{\infty} (-1)^{j-1} j^{1-n/2} \Gamma(n/2) \right]^{-1} = (2\pi)^{-n/2} \left[\sum_{j=1}^{\infty} (-1)^{j-1} j^{1-n/2} \right]^{-1}.
\end{aligned}
$$

If $\mathbf{X}$ belongs to the family of multivariate logistic distributions, we shall write $\mathbf{X} \sim ML_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

2.4 Multivariate Exponential Power Family

An elliptical vector $\mathbf{X}$ is said to have a multivariate exponential power distribution if its density generator has the form

$$
g(u) = e^{-ru^s}, \text{ for } r, s > 0.
$$

The joint density of $\mathbf{X}$ can be expressed in the form

$$
f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\boldsymbol{\Sigma}|}} \exp\left\{ -\frac{r}{2} \left[(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu}) \right]^s \right\}
$$

where the normalizing constant is given by

$$
\begin{aligned}
c_n &= \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left(\int_0^{\infty} x^{n/2-1} e^{-rx^s} dx \right)^{-1} = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left(\int_0^{\infty} \frac{1}{s} y^{\frac{1}{s}(n/2-s)} e^{-ry} dy \right)^{-1} \\
&= \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left(\frac{1}{rs} r^{1-n/(2s)} \int_0^{\infty} y^{n/(2s)-1} e^{-y} dy \right)^{-1} = \frac{s\Gamma(n/2)}{(2\pi)^{n/2} \Gamma(n/(2s))} r^{n/(2s)}.
\end{aligned}
$$

When $r = s = 1$, this family of distributions clearly reduces to the multivariate normal family. When $s = 1$ alone, this family reduces to the original Kotz multivariate distribution suggested by Kotz (1975). If $s = 1/2$ and $r = \sqrt{2}$, we have the family of Double Exponential or Laplace distributions.

Figure 2 displays a comparison of the bivariate densities for some of the well-known elliptical distributions discussed in this section.

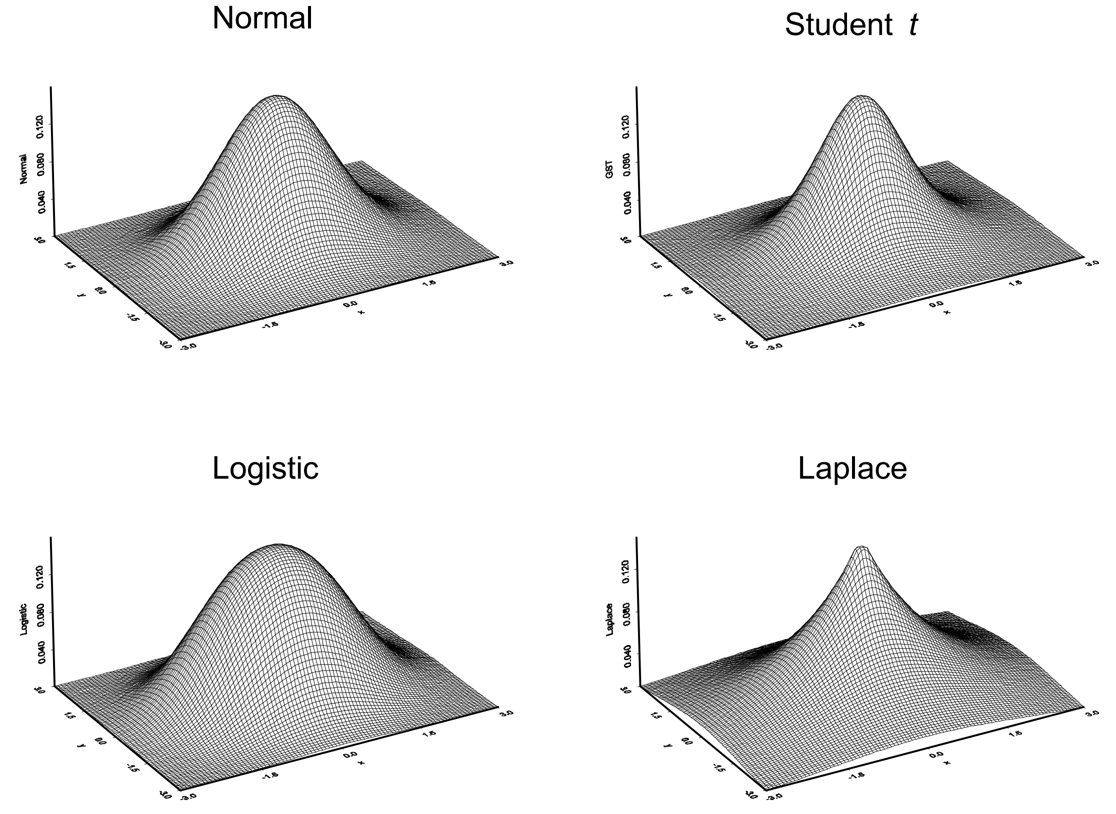

Figure 2: Comparing bivariate densities for some well-known elliptical distributions.

3 TCE Formulas for Univariate Elliptical Distributions

This section develops tail conditional expectation formulas for univariate elliptical distributions which as a matter of fact coincides with the class of symmetric distributions on the line $\mathbf{R}$. Recall that we denote by x_q the q-th quantile of the loss distribution $F_X(x)$. Because we are interested in considering the tails of symmetric distributions, we suppose that $q > 1/2$ so that clearly

$$x_q > \mu. \tag{19}$$

Now suppose $g(x)$ is a non-negative function on $[0, \infty)$ satisfying the condition that

$$\int_0^\infty x^{-1/2} g(x) dx < \infty.$$

Then (see Section 2) $g(x)$ can be a density generator of a univariate elliptical distribution of a random variable $X \sim E_1(\mu, \sigma^2, g)$ whose density can be expressed as

$$f_X(x) = \frac{c}{\sigma} g\left[\tfrac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \tag{20}$$

where c is the normalizing constant.

Note that because X has an elliptical distribution, the standardized random variable $Z = (X-\mu)/\sigma$ will have a standard elliptical (oftentimes called spherical) distribution function

$$F_Z(z) = c\int_{-\infty}^{z} g\left(\tfrac{1}{2}u^2\right) du,$$

with mean 0 and variance

$$\sigma_Z^2 = 2c\int_0^{\infty} u^2 g\left(\tfrac{1}{2}u^2\right) du = -\psi'(0),$$

if condition (13) holds. Furthermore, if the generator of the elliptical family is chosen such that condition (15) holds, then $\sigma_Z^2 = 1$.

Define the function

$$G(x) = c\int_0^{x} g(u)\, du \tag{21}$$

which we suggest to call the *cumulative generator*. This function G plays an important role in our derivation of tail conditional expectations for the class of elliptical distributions. Note that condition (12) which guarantees the existence of the expectation can equivalently be expressed as

$$G(\infty) < \infty.$$

Denote by

$$\overline{G}(x) = G(\infty) - G(x).$$

Theorem 1 *Let $X \sim E_1(\mu, \sigma^2, g)$ and G be the cumulative generator defined in (21). Under condition (12), the tail conditional expectation of X is given by*

$$TCE_X(x_q) = \mu + \lambda \cdot \sigma^2 \tag{22}$$

where λ is expressed as

$$\lambda = \frac{\frac{1}{\sigma}\overline{G}\left(\frac{1}{2}z_q^2\right)}{\overline{F}_X(x_q)} = \frac{\frac{1}{\sigma}\overline{G}\left(\frac{1}{2}z_q^2\right)}{\overline{F}_Z(z_q)} \tag{23}$$

and $z_q = (x_q-\mu)/\sigma$. Moreover, if the variance of X exists, or equivalently if (13) holds, then $\frac{1}{\sigma_Z^2}\overline{G}\left(\frac{1}{2}z^2\right)$ has the sense of a density of another spherical random variable Z^* and λ has the form

$$\lambda = \frac{\frac{1}{\sigma}f_{Z^*}(z_q)}{\overline{F}_Z(z_q)}\sigma_Z^2. \tag{24}$$

Proof. Note that

$$TCE_X(x_q) = \frac{1}{\overline{F}_X(x_q)} \int_{x_q}^{\infty} x \cdot \frac{c}{\sigma} g\left[\tfrac{1}{2}\left((x-\mu)/\sigma\right)^2\right] dx$$

and by letting $z = (x-\mu)/\sigma$, we have

$$\begin{aligned} TCE_X(x_q) &= \frac{1}{\overline{F}_X(x_q)} \int_{z_q}^{\infty} c(\mu + \sigma z)\, g\left(\tfrac{1}{2}z^2\right) dz \\ &= \frac{1}{\overline{F}_X(x_q)} \left[\mu \overline{F}_X(z_q) + c\sigma \int_{z_q}^{\infty} z g\left(\tfrac{1}{2}z^2\right) dz\right] \\ &= \mu + \lambda \cdot \sigma^2, \end{aligned}$$

where

$$\lambda = \frac{1}{\overline{F}_X(x_q)} \cdot \frac{c}{\sigma} \int_{\frac{1}{2}z_q^2}^{\infty} g(u)\, du = \frac{\frac{1}{\sigma}\bar{G}\left(\frac{1}{2}z_q^2\right)}{\overline{F}_Z(z_q)}$$

which proves the result in (23).

Now to prove (24), suppose condition (13) holds, i.e. variance of X exists and

$$\tfrac{1}{2}\sigma_Z^2 = c\int_0^{\infty} z^2 g\left(\tfrac{1}{2}z^2\right) dz = \int_0^{\infty} z dG\left(\tfrac{1}{2}z^2\right) < \infty.$$

Then, $\frac{G\left(\frac{1}{2}z^2\right)}{G(\infty)} = F_{\widetilde{Z}}(z)$ is a distribution function of some random variable $\widetilde{Z}$ with expectation given by

$$E\left(\widetilde{Z}\right) = \frac{1}{G(\infty)} \int_0^{\infty} z dG\left(\tfrac{1}{2}z^2\right) = \int_0^{\infty} \left[1 - \frac{G\left(\frac{1}{2}z^2\right)}{G(\infty)}\right] dz = \frac{1}{2}\sigma_Z^2 \frac{1}{G(\infty)} < \infty.$$

Consequently,

$$\int_0^{\infty} \overline{G}\left(\tfrac{1}{2}z^2\right) dz = \tfrac{1}{2}\sigma_Z^2$$

and $\frac{1}{\sigma_Z^2}\overline{G}\left(\frac{1}{2}z^2\right) = f_{Z^*}(z)$ is a density of some symmetric random variable Z^*, defined on $\mathbf{R}$. ■

It is clear that (22) generalizes the tail conditional expectation formula derived by Panjer (2002) for the class of normal distributions to the larger class of univariate symmetric distributions. We now illustrate Theorem 1 by considering examples for some well-known symmetric distributions which include the normal distribution. For the normal distribution, we exactly replicate the formula developed by Panjer (2002).

1. *Normal Distribution.* Let $X \sim N(\mu, \sigma^2)$ so that the function in (20) has the form $g(u) = \exp(-u)$. Therefore,

$$G(x) = c \int_0^x g(u)\, du = c \int_0^x e^{-u} du = c\left(1 - e^{-x}\right)$$

and

$$\frac{1}{\sigma}\overline{G}\left(\tfrac{1}{2}z_q^2\right) = \frac{c}{\sigma}\exp\left(-\tfrac{1}{2}z_q^2\right) = \frac{c}{\sigma}\sqrt{2\pi}\varphi(z_q) = \frac{1}{\sigma}\varphi(z_q)$$

where it is well-known that the normalizing constant is $c = \left(\sqrt{2\pi}\right)^{-1}$. Thus for the normal distribution, we find $\sigma_Z^2 = 1$ and

$$\lambda = \frac{\frac{1}{\sigma}\varphi(z_q)}{1 - \Phi(z_q)}, \tag{25}$$

where $\varphi(\cdot)$ and $\Phi(\cdot)$ denote respectively the density and distribution functions of a standard normal distribution. Notice that Z^* in Theorem 1 is simply the standard normal variable Z.

2. *Generalized Student-t Distribution.* Let X belong to the univariate Generalized Student t family with density generator expressed as in (17) so that

$$G(x) = c_p \int_0^x g(u)\, du = c_p \int_0^x \left(1 + \frac{u}{k_p}\right)^{-p} du = c_p \frac{k_p}{p-1}\left[1 - \left(1 + \frac{x}{k_p}\right)^{1-p}\right]$$

provided $p > 1$. Here we denote the normalizing constant by c_p with the subscript p to emphasize that it depends on the parameter p. Recall from Section 2.2 that c_p can be expressed as

$$c_p = \frac{\Gamma(p)}{\sqrt{2k_p}\Gamma(1/2)\Gamma(p-1/2)} = \frac{\Gamma(p)}{\sqrt{2\pi k_p}\Gamma(p-1/2)}. \tag{26}$$

Note that the case where $p = 1$ gives the Cauchy distribution for which the mean does not exist and therefore its TCE also does not exist. Now considering the case only where $p > 1$, we get

$$\begin{aligned} \frac{1}{\sigma}\overline{G}\left(\tfrac{1}{2}z_q^2\right) &= \frac{c_p}{\sigma}\frac{k_p}{p-1}\left(1 + \frac{z_q^2}{2k_p}\right)^{-p+1} \\ &= \frac{1}{\sigma}\frac{c_p}{c_{p-1}}\frac{k_p}{(p-1)} \cdot f_Z\left(\sqrt{\frac{k_{p-1}}{k_p}}\, z_q;\ p-1\right), \end{aligned} \tag{27}$$

where $f_Z(\cdot; p)$ denotes the density of a standardized GST with parameter p, and $k_{p-1} = 1/2$, $c_{p-1} = 1/\sqrt{2k_{p-1}} = 1$, when $0 < p - 1 \le 1/2$. Recall from section 2.2 again that for the GST family, we have

$$k_p = \begin{cases} \frac{2p-3}{2}, & \text{if } p > 3/2 \\ \frac{1}{2}, & \text{if } 1/2 < p \le 3/2 \end{cases}. \tag{28}$$

For the case $p > 3/2$, the variance of X exists and GST was constructed such that $Var\,(X) = \sigma^2$, that is, $\sigma_Z^2 = 1$ (see Section 2.2). From (26), it follows that

$$\frac{c_p}{c_{p-1}} = \frac{\Gamma\,(p)\,\Gamma\,(p-3/2)}{\Gamma\,(p-1/2)\,\Gamma\,(p-1)}\sqrt{\frac{k_{p-1}}{k_p}} = \frac{(p-1)}{(p-3/2)}\sqrt{\frac{k_{p-1}}{k_p}}, \tag{29}$$

and then from (27), (29), and (28),

$$\frac{1}{\sigma}\overline{G}\left(\tfrac{1}{2}z_q^2\right) = \frac{1}{\sigma}\cdot\sqrt{\frac{k_{p-1}}{k_p}}\, f_Z\left(\sqrt{\frac{k_{p-1}}{k_p}}\,z_q;\ p-1\right). \tag{30}$$

Moreover, when $p > 5/2$, $p-1 > 3/2$, so that we can re-express (30) as follows:

$$\frac{1}{\sigma}\overline{G}\left(\tfrac{1}{2}z_q^2\right) = \frac{1}{\sigma}\cdot\sqrt{\frac{2p-5}{2p-3}}\, f_Z\left(\sqrt{\frac{2p-5}{2p-3}}z_q; p-1\right).$$

Thus, we have

$$\lambda = \frac{\frac{1}{\sigma}\sqrt{\frac{2p-5}{2p-3}}\cdot f_Z\left(\sqrt{\frac{2p-5}{2p-3}}z_q; p-1\right)}{\overline{F}_Z\,(z_q; p)} \tag{31}$$

and Z^* is simply a scaled standardized GST with parameter $p-1$. Notice that (see, for example, Landsman and Makov, 1999) when $p \to \infty$, the GST distribution tends to the Normal distribution. It is clear from (31) that λ will tend to that of the normal distribution in (25).

For $3/2 < p \leq 5/2$, $1/2 < p-1 \leq 3/2$, and taking into account (28), we have

$$\frac{k_{p-1}}{k_p} = \frac{1}{2p-3},$$

and

$$\lambda = \frac{\frac{1}{\sigma}\sqrt{\frac{1}{2p-3}}\cdot f_Z\left(\sqrt{\frac{1}{2p-3}}z_q;\ p-1\right)}{\overline{F}_Z\,(z_q; p)}.$$

Now considering the case where $1 < p \leq 3/2$, we have $0 < p-1 \leq 1/2$, $\dfrac{k_{p-1}}{k_p} = 1$ and therefore

$$\lambda = \frac{\frac{1}{\sigma}f_Z\,(z_q; p-1)}{\overline{F}_Z\,(z_q; p)}.$$

Notice here that in this case, $f_Z\,(z_q; p-1)$ preserves the form of the density for GST, but it is not a density function because $\int_{-\infty}^{\infty} f_Z\,(x; p-1)\,dx$ diverges. In Figure 3, we provide a graph relating λ and the parameter p, for $p > 1$ and $q = 0.95$, for the GST distribution. The dotted line in the figure is the limiting case ($p \to \infty$) which is exactly that of the Normal distribution.

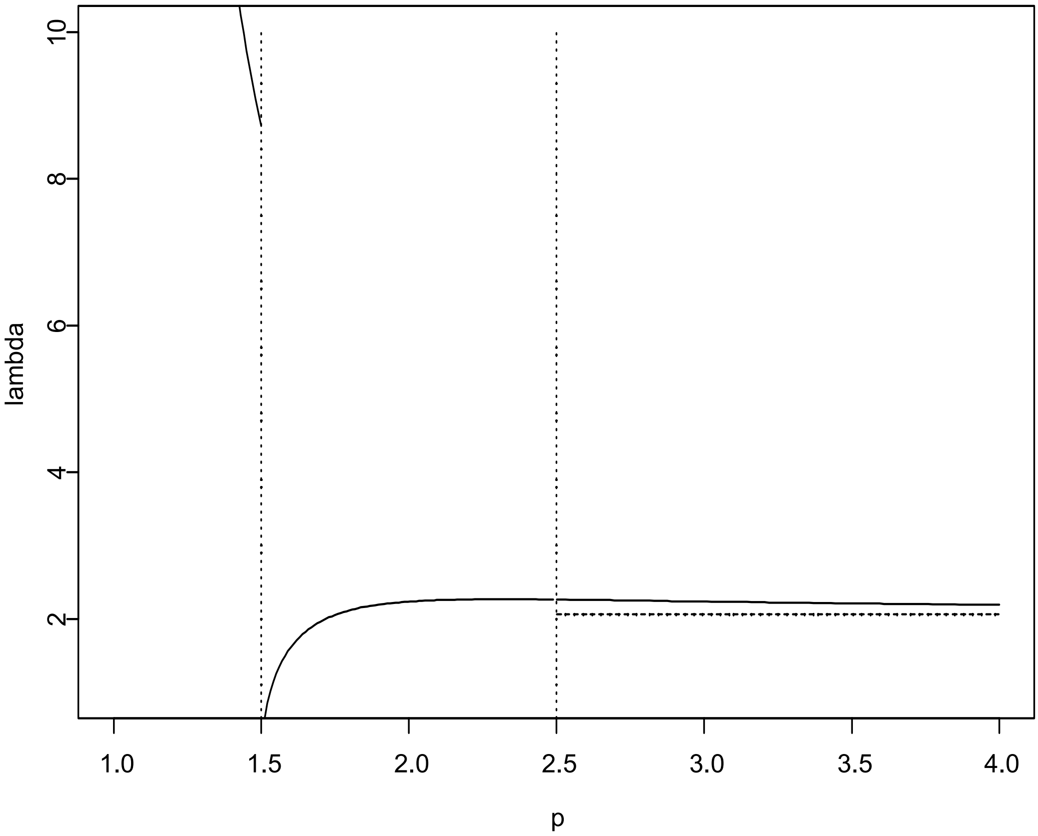

Figure 3: The relationship between λ and the parameter p for the GST distribution.

3. *Logistic Distribution.* As earlier described, for this class of distribution, the density generator has the form $g(u) = \frac{e^{-u}}{(1+e^{-u})^2}$. Therefore,

$$G(x) = c\int_0^x \frac{e^{-u}}{(1+e^{-u})^2}du = c\left[\left(1+e^{-x}\right)^{-1} - 1/2\right]$$

where it can be verified that the normalizing constant $c = 1/2$. Thus,

$$\begin{aligned}
\frac{1}{\sigma}\overline{G}\left(\tfrac{1}{2}z_q^2\right) &= \frac{1}{2\sigma}\left[1-\left(1+e^{-\frac{1}{2}z_q^2}\right)^{-1}\right] = \frac{1}{2\sigma}\frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z_q^2}}{\frac{1}{\sqrt{2\pi}}+\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z_q^2}} \\
&= \frac{1}{2}\frac{\frac{1}{\sigma}\varphi\left(z_q\right)}{\left(\sqrt{2\pi}\right)^{-1}+\varphi\left(z_q\right)},
\end{aligned}$$

where $\varphi(\cdot)$ is the density of a standard normal distribution. Therefore, for a logistic random variable, we have the expression for λ:

$$\lambda = \left[\frac{1}{2} \frac{1}{\left(\sqrt{2\pi}\right)^{-1} + \varphi(z_q)} \right] \frac{\frac{1}{\sigma}\varphi(z_q)}{\overline{F}_Z(z_q)}$$

which resembles that for a normal distribution but with a correction factor.

4. *Exponential Power Distribution.* For exponential power distribution with density generator of the form $g(u) = \exp(-ru^s)$ for some $r, s > 0$, we have

$$\begin{aligned} G(x) &= c \int_0^x e^{-ru^s} du = c \left(sr^{1/s}\right)^{-1} \int_0^{rx^s} w^{1/s-1} e^{-w} dw \\ &= c \left(sr^{1/s}\right)^{-1} \Gamma(rx^s; 1/s), \end{aligned}$$

where

$$\Gamma(z; 1/s) = \int_0^z w^{1/s-1} e^{-w} dw \tag{32}$$

denotes the incomplete Gamma function. One can determine the normalizing constant to be

$$c = \frac{sr^{1/(2s)}}{\sqrt{2}\Gamma(1/(2s))} \tag{33}$$

by a straightforward integration of the density function. In effect, we have

$$\frac{1}{\sigma}\overline{G}\left(\tfrac{1}{2}z_q^2\right) = \left[\sqrt{2}\Gamma(1/(2s))\,\sigma\right]^{-1} \left\{\Gamma(1/s) - \Gamma\left[r\left(\tfrac{1}{2}z_q^2\right)^s; 1/s\right]\right\}$$

and

$$\lambda = \frac{1}{\overline{F}_Z(z_q)} \frac{1}{\sqrt{2}\Gamma(1/(2s))\,\sigma} \left\{\Gamma(1/s) - \Gamma\left[r\left(\tfrac{1}{2}z_q^2\right)^s; 1/s\right]\right\}. \tag{34}$$

It is clear that when $s = 1$ and $r = 1$, the density generator for the exponential power reduces to that of a normal distribution. From (33), it follows that $c = \left(\sqrt{2\pi}\right)^{-1}$, and from (34), it follows that

$$\begin{aligned} \lambda &= \frac{1}{1-\Phi(z_q)} \left(\sqrt{2\pi}\right)^{-1} \left[1 - \Gamma\left(\tfrac{1}{2}z_q^2; 1\right)\right] \\ &= \frac{1}{1-\Phi(z_q)} \left(\sqrt{2\pi}\right)^{-1} \left[1 - \left(1 - e^{-\frac{1}{2}z_q^2}\right)\right] = \frac{\frac{1}{\sigma}\varphi(z_q)}{1-\Phi(z_q)} \end{aligned}$$

which is exactly that of a normal. The Laplace or Double Exponential distribution is another special case belonging to the exponential power family. In this case, $s = 1/2$ and $r = \sqrt{2}$. From (34), it follows that

$$\begin{aligned}
\lambda &= \frac{1}{\overline{F}_Z(z_q)} \frac{1}{2\sigma} \left[\Gamma(2) - \Gamma(|z_q|; 2)\right] \\
&= \frac{1}{\overline{F}_Z(z_q)} \frac{1}{2\sigma} \left(1 - \int_0^{|z_q|} we^{-w} dw\right) \\
&= \frac{1}{\overline{F}_Z(z_q)} \frac{1}{2\sigma} e^{-|z_q|} (1 + |z_q|) \\
&= 2\frac{1}{\overline{F}_Z(z_q)} \frac{1}{\sigma} f_{Z^*}(z_q),
\end{aligned}$$

where $f_{Z^*}(z) = \frac{1}{2} f_Z(z)(1+|z|) = \frac{1}{4} e^{-|z|}(1+|z|)$ is density of the new random variable Z^*, and $\sigma_Z^2 = 2$ is a variance of standard Double Exponential distribution that well confirms with (24).

4 TCE and Multivariate Elliptical Distributions

Let $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ be a multivariate elliptical vector, i.e. $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \mathbf{\Sigma}, g_n)$. Denote the (i, j) element of $\mathbf{\Sigma}$ by σ_{ij} so that $\mathbf{\Sigma} = \|\sigma_{ij}\|_{i,j=1}^n$. Moreover, let

$$F_Z(z) = c_1 \int_0^z g_1\left(\tfrac{1}{2}x^2\right) dx$$

be the standard one-dimensional distribution function corresponding to this elliptical family and

$$G(x) = c_1 \int_0^x g_1(u)\, du \tag{35}$$

be its cumulative generator. From Theorem 1 and (10), we observe immediately that the formula for computing tail conditional expectations for each component of the vector $\mathbf{X}$ can be expressed as

$$TCE_{X_k}(x_q) = \mu_k + \lambda_k \cdot \sigma_k^2$$

where

$$\lambda_k = \frac{\frac{1}{\sigma_k}\overline{G}\left(\frac{1}{2}z_{k,q}^2\right)}{\overline{F}_Z(z_{k,q})} \text{ and } z_{k,q} = \frac{x_q - \mu_k}{\sigma_k},$$

or

$$\lambda_k = \frac{\frac{1}{\sigma_k} f_{Z^*}(z_q)}{\overline{F}_Z(z_q)} \sigma_Z^2 \ ,$$

if $\sigma_Z^2 < \infty$.

4.1 Sums of Elliptical Risks

Suppose $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$ and $\mathbf{e} = (1, 1, ..., 1)^T$ is the vector of ones with dimension n. Define

$$S = X_1 + \cdots + X_n = \sum_{k=1}^{n} X_k = \mathbf{e}^T \mathbf{X} \tag{36}$$

which is the sum of elliptical risks. We now state a theorem for finding the TCE for this sum.

Theorem 2 *The tail conditional expectation of S can be expressed as*

$$TCE_S(x_q) = \mu_S + \lambda_S \cdot \sigma_S^2 \tag{37}$$

where $\mu_S = \mathbf{e}^T \boldsymbol{\mu} = \sum_{k=1}^{n} \mu_k$, $\sigma_S^2 = \mathbf{e}^T \boldsymbol{\Sigma} \mathbf{e} = \sum_{i,j=1}^{n} \sigma_{ij}$ *and*

$$\lambda_S = \frac{\frac{1}{\sigma_S} \overline{G}\left(\frac{1}{2} z_{S,q}^2\right)}{\overline{F}_Z(z_{S,q})} \tag{38}$$

with $z_{S,q} = \dfrac{\mu_S - x_q}{\sigma_S}$. *If the covariance matrix of* $\mathbf{X}$ *exists,* λ_S *can be represented by (24).*

Proof. It follows immediately from (11) that $S \sim E_n\left(\mathbf{e}^T \boldsymbol{\mu}, \mathbf{e}^{\mathbf{T}} \boldsymbol{\Sigma} \mathbf{e}, g_1\right)$ and the result follows using Theorem 1. ■

4.2 Portfolio Risk Decomposition with TCE

When uncertainty is due to different sources, it is often natural to ask how to decompose the total level of uncertainty to these sources. Frees (1998) suggested methods for quantifying the degree of importance of various sources of uncertainty for insurance systems. In particular, he showed the effectiveness of the use of coefficient of determination in such decomposition and applied it in situations involving risk exchanges and risk pooling.

For our purposes, suppose that the total loss or claim is expressed as in (36) where one can think of each X_k as the claim arising from a particular line of business or product line in the case of insurance, or the loss resulting from a financial instrument or a portfolio of investments. As it was noticed by Panjer (2002), from the additivity of expectation, the tail conditional expectation allows for a natural decomposition of the total loss:

$$TCE_S(x_q) = \sum_{k=1}^{n} E(X_k | S > x_q). \tag{39}$$

Note that this is not in general equivalent to the sum of the tail conditional expectations of the individual components. This is because

$$TCE_{X_k}(x_q) \neq E(X_k | S > x_q).$$

Instead we denote this as

$$TCE_{X_k|S}(x_q) = E(X_k | S > x_q),$$

the contribution to the total risk attributable to risk k. It can be interpreted as follows: that in the case of a disaster as measured by an amount at least as large as the quantile of the total loss distirbution, this refers to the average amount that would be due to the presence of risk k. Panjer (2002) obtained important results for this decomposition in the case where the risks have a multivariate normal distribution. In this paper, we extend his result for essentially more general elliptical multivariate class for which the multivariate normal family belongs to.

To develop the formula for decomposition, first, we need the following two lemmas.

Lemma 1 *Let* $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$. *Then for* $1 \leq k \leq n$, *the vector* $\mathbf{X}_{k,S} = (X_k, S)^T$ *has an elliptical distribution with the same generator, i.e.,* $\mathbf{X}_{k,S} \sim E_2(\boldsymbol{\mu}_{k,S}, \boldsymbol{\Sigma}_{k,S}, g_2)$, *where* $\boldsymbol{\mu}_{k,S} = \left(\mu_k, \sum_{j=1}^n \mu_j\right)^T$,

$$\boldsymbol{\Sigma}_{k,S} = \begin{pmatrix} \sigma_k^2 & \sigma_{kS} \\ \sigma_{kS} & \sigma_S^2 \end{pmatrix},$$

and $\sigma_k^2 = \sigma_{kk}, \sigma_{kS} = \sum_{j=1}^n \sigma_{kj}, \sigma_S^2 = \sum_{i,j=1}^n \sigma_{ij}$.

Proof. Define the matrix A as

$$A = \begin{pmatrix} 0 & 0 & .... & 1 & .... & 0 & 0 \\ 1 & 1 & .... & 1 & .... & 1 & 1 \end{pmatrix}$$

which consists of 0's in the first row, except the k-th column which has a value of 1, and all of 1's in the second row. Thus, it is clear that

$$A\mathbf{X} = (X_k, S)^T = \mathbf{X}_{k,S}.$$

It follows from (9) that

$$A\mathbf{X} \sim E_2\left(A\boldsymbol{\mu}, A\boldsymbol{\Sigma}A^T, g_2\right)$$

where its mean vector is

$$\boldsymbol{\mu}_{k,S} = A\boldsymbol{\mu} = \left(\mu_k, \sum_{j=1}^n \mu_j\right)^T$$

and its variance-covariance structure is

$$\mathbf{\Sigma}_{k,S} = A\mathbf{\Sigma}A^T = \begin{bmatrix} \sigma_k^2 & \sum_{j=1}^n \sigma_{kj} \\ \sum_{j=1}^n \sigma_{kj} & \sigma_S^2 \end{bmatrix}.$$

Thus, we see that $\mathbf{X}_{k,S} \sim E_2\left(\boldsymbol{\mu}_{k,S}, \mathbf{\Sigma}_{k,S}, g_2\right)$. ■

Lemma 2 *Let* $\mathbf{Y} = (Y_1, Y_2)^T \sim E_2(\boldsymbol{\mu}, \mathbf{\Sigma}, g_2)$ *such that condition (12) holds. Then*

$$\begin{aligned} TCE_{Y_1|Y_2}(y_q) &= E(Y_1 | Y_2 > y_q) \\ &= \mu_1 + \lambda_2 \cdot \sigma_1\sigma_2\rho_{12} \end{aligned}$$

where

$$\lambda_2 = \frac{\frac{1}{\sigma_2}\overline{G}\left(\frac{1}{2}z_{2,q}^2\right)}{\overline{F}_Z(z_{2,q})}$$

and $\rho_{12} = \dfrac{\sigma_{12}}{\sigma_1\sigma_2}$, $\sigma_1 = \sqrt{\sigma_{11}}$, $\sigma_2 = \sqrt{\sigma_{22}}$, *and* $z_{2,q} = \dfrac{y_q - \mu_2}{\sigma_2}$.

Proof. First note that by definition and from (6), we have

$$\begin{aligned} & E(Y_1 | Y_2 > y_q) \\ = & \frac{1}{\overline{F}_{Y_2}(y_q)} \int_{-\infty}^{\infty}\int_{y_q}^{\infty} y_1 f_{\mathbf{Y}}(y_1, y_2)\, dy_2 dy_1 \\ = & \frac{1}{\overline{F}_Z(z_{2,q})} \int_{-\infty}^{\infty}\int_{y_q}^{\infty} y_1 \frac{c_2}{\sqrt{|\mathbf{\Sigma}|}} g_2\left[\tfrac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^T \mathbf{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})\right] dy_2 dy_1 \qquad (40) \\ = & \frac{1}{\overline{F}_Z(z_{2,q})} \times I, \end{aligned}$$

where I is the double integral in (40). In the bivariate case, we have

$$|\mathbf{\Sigma}| = \begin{vmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{vmatrix} = \left(1 - \rho_{12}^2\right)\sigma_1^2\sigma_2^2$$

and

$$\begin{aligned} & (\mathbf{y}-\boldsymbol{\mu})^T \mathbf{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu}) \\ = & \frac{1}{(1-\rho_{12}^2)}\left[\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2 - 2\rho_{12}\left(\frac{y_1-\mu_1}{\sigma_1}\right)\left(\frac{y_2-\mu_2}{\sigma_2}\right) + \left(\frac{y_2-\mu_2}{\sigma_2}\right)^2\right] \\ = & \frac{1}{(1-\rho_{12}^2)}\left\{\left[\left(\frac{y_1-\mu_1}{\sigma_1}\right) - \rho_{12}\left(\frac{y_2-\mu_2}{\sigma_2}\right)\right]^2 + \left(1-\rho_{12}^2\right)\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2\right\}. \end{aligned}$$

Using the transformations $z_1 = \dfrac{y_1 - \mu_1}{\sigma_1}$ and $z_2 = \dfrac{y_2 - \mu_2}{\sigma_2}$, and the property that the marginal distributions of multivariate elliptical distribution are again elliptical distributions with the same generator, we have

$$\begin{aligned} I &= \frac{c_2}{\sqrt{1-\rho_{12}^2}} \int_{z_{2,q}}^{\infty} \int_{-\infty}^{\infty} (\mu_1 + \sigma_1 z_1)\, g_2 \left[\tfrac{1}{2} \frac{(z_1 - \rho_{12} z_2)^2}{(1-\rho_{12}^2)} + \tfrac{1}{2} z_2^2 \right] dz_1 dz_2 \qquad (41) \\ &= \mu_1 \overline{F}_Z (z_{2,q}) + \sigma_1 I', \end{aligned}$$

where

$$I' = c_2 \int_{z_{2,q}}^{\infty} \int_{-\infty}^{\infty} c_2 \frac{z_1}{\sqrt{1-\rho_{12}^2}} g_2 \left[\frac{1}{2} \frac{(z_1 - \rho_{12} z_2)^2}{(1-\rho_{12}^2)} + \frac{1}{2} z_2^2 \right] dz_1 dz_2$$

is the double integral in the second term of the above equation. After transformation $z' = \dfrac{z_1 - \rho_{12} z_2}{\sqrt{1-\rho_{12}^2}}$ we get

$$I' = \sqrt{1-\rho_{12}^2} \int_{z_{2,q}}^{\infty} \int_{-\infty}^{\infty} c_2 \left(z' + \frac{\rho_{12} z_2}{\sqrt{1-\rho_{12}^2}} \right) g_2 \left[\tfrac{1}{2} \left(z'^2 + z_2^2 \right) \right] dz' dz_2. \qquad (42)$$

By noticing that the integral of odd function

$$\int_{-\infty}^{\infty} z' c_2 g_2 \left[\tfrac{1}{2} \left(z'^2 + z_2^2 \right) \right] dz' = 0,$$

and again using the property of the marginal elliptical distribution, giving

$$\int_{-\infty}^{\infty} c_2 g_2 \left[\tfrac{1}{2} \left(z'^2 + z_2^2 \right) \right] dz' = c_1 g_1 \left(\tfrac{1}{2} z_2^2 \right),$$

we have in (42)

$$\begin{aligned} I' &= \int_{z_{2,q}}^{\infty} \rho_{12} z_2 c_1 g_1 \left(\tfrac{1}{2} z_2^2 \right) dz_2 = \rho_{12} \int_{\frac{1}{2} z_{2,q}^2}^{\infty} c_1 g_1 (u)\, du \\ &= \rho_{12} \sigma_2 \frac{1}{\sigma_2} \bar{G} \left(\tfrac{1}{2} z_{2,q}^2 \right) \qquad (43) \end{aligned}$$

and the result in the theorem then immediately follows from (40), (41) and (43). ■

Using these two lemmas, we obtain the following result.

Theorem 3 *Let* $\mathbf{X} = (X_1, X_2, ..., X_n)^T \sim E_n (\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$ *such that condition (12) holds and let* $S = X_1 + \cdots + X_n$*. Then the contribution of risk* $X_k, 1 \le k \le n$, *to the total tail conditional expectation can be expressed as*

$$TCE_{X_k|S} (x_q) = \mu_k + \lambda_S \cdot \sigma_k \sigma_S \rho_{k,S}, \text{ for } k = 1, 2, ..., n \qquad (44)$$

where $\rho_{k,S} = \dfrac{\sigma_{k,S}}{\sigma_k \sigma_S}$, *and* λ_S *is the same as in Theorem 2.*

Proof. The result immediately follows from Lemma 2 by simply putting $\mathbf{Y} = (X_k, S)^T$ and using Lemma 1. ■

Let us observe that at the same time that matrix $\mathbf{\Sigma}$ coincides with the covariance matrix up to a constant, see (14), the formally defined

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}}$$

as the ratio of elements of matrix $\mathbf{\Sigma}$, is really a correlation coefficient between X_i and X_j. The same can be said about $\rho_{k,S}$.

Notice that if we take the sum of $TCE_{X_k|S}(x_q)$ in (44), we have

$$\begin{aligned} \sum_{k=1}^{n} TCE_{X_k|S}(x_q) &= \sum_{k=1}^{n} \mu_k + \lambda_S \sum_{k=1}^{n} \sigma_k \sigma_S \rho_{k,S} \\ &= \mu_S + \lambda_S \sum_{k=1}^{n} \sigma_{k,S} \\ &= \mu_S + \lambda_S \cdot \sigma_S^2, \end{aligned}$$

because from Lemma 1 we get that

$$\sum_{k=1}^{n} \sigma_{k,S} = \sum_{k=1}^{n} \sum_{j=1}^{n} \sigma_{kj} = \sigma_S^2$$

which gives the result for the TCE of a sum of elliptical risks, as given in (37). It was demonstrated in Panjer (2002) that in the case of a multivariate normal random vector i.e. $\mathbf{X} \sim \mathbf{N}_n(\boldsymbol{\mu}, \mathbf{\Sigma})$, we have

$$E(X_k | S > x_q) = \mu_k + \left[\frac{\frac{1}{\sigma_S} \varphi\left(\frac{x_q - \mu}{\sigma_S}\right)}{1 - \Phi\left(\frac{x_q - \mu}{\sigma_S}\right)} \right] \sigma_k^2 \left(1 + \rho_{k,-k} \frac{\sigma_{-k}}{\sigma_k}\right), \tag{45}$$

where they have used the negative subscript $-k$ to refer to the sum of all the risks excluding the kth risk, that is, $S_{-k} = S - X_k$. Therefore, according to this notation, we have

$$\begin{aligned} \rho_{k,-k} \frac{\sigma_{-k}}{\sigma_k} &= \frac{\sigma_{k,-k}}{\sigma_k \sigma_{-k}} \frac{\sigma_{-k}}{\sigma_k} = \frac{\sigma_{k,-k}}{\sigma_k^2} \\ &= \frac{Cov(X_k, S - X_k)}{\sigma_k^2} = \frac{\sigma_{k,S}}{\sigma_k^2} - 1. \end{aligned}$$

Thus, the formula in (45) becomes

$$E(X_k | S > x_q) = \mu_k + \left[\frac{\frac{1}{\sigma_S} \varphi\left(\frac{x_q - \mu}{\sigma_S}\right)}{1 - \Phi\left(\frac{x_q - \mu}{\sigma_S}\right)} \right] \sigma_k \sigma_S \rho_{k,S}$$

that (44) gives in the case of multivariate normal, and consequently (44) generalizes (45) for the class of elliptical distributions.

5 Conclusion

In this paper, we have developed an appealing way to characterize the tail conditional expectations for elliptical distributions. In the univariate case, the class of elliptical distributions consists of the class of symmetric distributions which include familiar distributions like normal and Student t. This class can easily be extended into the multivariate framework by simply characterizing them either in terms of the characteristic generator or the density generator. This paper studied this class of multidimensional distributions rather extensively to allow the reader to understand them more thoroughly particularly many of the properties of the multivariate normal is shared by this larger class. Thus, someone wishing to use multivariate elliptical distributions in their practical work may find this paper self-contained. Furthermore, this paper defines the *cumulative generator* resulting from the definition of the density generator, and uses this generator quite extensively to generate formulas for tail conditional expectations. We also know that tail conditional expectations naturally permits a decomposition of this expectation into individual components consisting of the individual risks making up the multivariate random vector. We extended TCE formulas developed for the univariate case into the case where there are several risks which when taken together behaves like an elliptical random vector. We further extended the results into the case where we then decompose the TCE into individual components making up the sum of the risks. We are able to verify, using the results developed in this paper, the formulas that were investigated and developed by Panjer (2002) in the case of the multivariate normal distribution.

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Appendix. In this appendix, we prove (8), that is the normalizing constant in the density of a multivariate elliptical random variable can be expressed as

$$c_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[\int_0^\infty x^{n/2-1} g_n(x) dx \right]^{-1}.$$

We prove this by transformation from the rectangular to polar coordinates in several dimensions. The authors show this because this is uncommon knowledge to actuaries and that this procedure is not readily available in several calculus textbooks. The polar transformation considered in what follows has been suggested by Anderson (1984). The transformation from rectangular to polar coordinates in several dimension is the following:

$$\begin{aligned}
x_1 &= r \sin\theta_1 \\
x_2 &= r \cos\theta_1 \sin\theta_2 \\
x_3 &= r \cos\theta_1 \cos\theta_2 \sin\theta_3 \\
&\cdots \\
&\cdots \\
x_{n-1} &= r \cos\theta_1 \cos\theta_2 \cdots \cos\theta_{n-2} \sin\theta_{n-1} \\
x_n &= r \cos\theta_1 \cos\theta_2 \cdots \cos\theta_{n-2} \cos\theta_{n-1}
\end{aligned}$$

where $-\pi/2 < \theta_k \leq \pi/2$ for $k = 1, 2, ..., n-2$, and $-\pi < \theta_{n-1} \leq \pi$. It can be shown that

$$\mathbf{x}^T\mathbf{x} = \sum_{k=1}^{n} x_k^2 = r^2$$

and that the Jacobian of the transformation is

$$|J| = \left| \frac{\partial (x_1, ..., x_n)}{\partial (\theta_1, ..., \theta_{n-1}, r)} \right| = r^{n-1} \cos^{n-2}\theta_1 \cos^{n-3}\theta_2 \cdots \cos\theta_{n-2}.$$

Thus, for the density in (6) to be valid, it must integrate to 1. Without loss of generality, we consider the case where $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = I_n$ (the identity matrix). Therefore,

$$\begin{aligned}
& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} c_n g_n\left(\frac{1}{2}\mathbf{x}^T\mathbf{x}\right) d\mathbf{x} \\
= & \ c_n \int_{-\pi/2}^{\pi/2}\cdots\int_{-\pi}^{\pi}\int_{0}^{\infty} r^{n-1}\cos^{n-2}\theta_1\cos^{n-3}\theta_2\cdots\cos\theta_{n-2} g_n\left(\frac{1}{2}r^2\right) d\theta_1\cdots d\theta_{n-1}dr \\
= & \ c_n \cdot \prod_{k=1}^{n-2}\int_{-\pi/2}^{\pi/2}\cos\theta_k^{n-(k+1)}d\theta_k \cdot \int_{-\pi}^{\pi} d\theta_{n-1} \cdot \int_{0}^{\infty} r^{n-1} g_n\left(\frac{1}{2}r^2\right) dr.
\end{aligned}$$

By letting $u = \cos^2\theta_k$ so that $du = 2\cos\theta_k \sin\theta_k d\theta_k$ and recognizing we get a beta function, it can be shown that

$$\int_{-\pi/2}^{\pi/2}\cos\theta_k^{n-k-1}d\theta_k = \frac{\Gamma\left[\frac{1}{2}(n-k)\right]\Gamma\left(\frac{1}{2}\right)}{\Gamma\left[\frac{1}{2}(n-k+1)\right]} = \frac{\Gamma\left[\frac{1}{2}(n-k)\right]\sqrt{\pi}}{\Gamma\left[\frac{1}{2}(n-k+1)\right]}.$$

Furthermore, we have

$$\begin{aligned}
\int_0^{\infty} r^{n-1} g_n\left(\tfrac{1}{2}r^2\right) dr &= \int_0^{\infty}\left[(2x)^{1/2}\right]^{n-2} g_n(x)\, dx \\
&= 2^{n/2-1}\int_0^{\infty} x^{n/2-1} g_n(x)\, dx.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
c_n &= \left\{\prod_{k=1}^{n-2}\frac{\Gamma\left[\frac{1}{2}(n-k)\right]\sqrt{\pi}}{\Gamma\left[\frac{1}{2}(n-k+1)\right]}\cdot 2\pi\cdot 2^{n/2-1}\int_0^{\infty} x^{n/2-1} g_n(x)\, dx\right\}^{-1} \\
&= \left[\frac{\Gamma(1)\,\pi^{n/2-1}}{\Gamma(n/2)}\cdot 2\pi\cdot 2^{n/2-1}\int_0^{\infty} x^{n/2-1} g_n(x)\, dx\right]^{-1} \\
&= \left[\frac{(2\pi)^{n/2}}{\Gamma(n/2)}\int_0^{\infty} x^{n/2-1} g_n(x)\, dx\right]^{-1}
\end{aligned}$$

and the desired result immediately follows.