Motor Insurance Loss Rate Options and Swaps

Changki Kim

(Jointly with Taehan Bae)

The University of New South Wales
Sydney – Australia

www.actuarial.unsw.edu.au
Introduction

- Motivation
- Describe the stochastic loss rate models used
- Pricing models
- Describe the characteristics of the Ratchet options, and the swaps
- Numerical examples
- Conclusions
1. Financial risk management approach.
2. Capital Market has more capacities than reinsurance market.
3. Introducing new types of risk to a capital market which is less correlated with existing risks. Make market more complete.
4. Securitization of insurance risk transfers the risks to the capital market through the creation and issuance of financial securities.
5. Life insurance securitization: Mortality Bonds (Swiss Re, 2003)
6. Auto insurance securitization: AXA, 2005
Characteristics of Motor Insurance Loss Rate Options and Swaps

**Definition 1**  A *motor loss rate swap* is an agreement between two parties to exchange payments involving at least one random motor loss rate dependent payments for a certain period of time.

**Definition 2**  A *motor loss rate ratchet option* is a series of call options on motor loss rates where strike thresholds are reset periodically.

- A motor loss rate ratchet option is a right, not an obligation, to exchange cash flows related to motor loss rates.
- On each exercise date the option holder receives the excess amount of loss above the pre-specified strike threshold.
- It is important to reset strike thresholds periodically based on the realized loss rate of the previous period.
These hybrid derivatives have several advantages over motor loss rate bonds.

They can be arranged at lower transaction costs than a bond issue.

By construction the proposed swaps and options cover multiple time periods.

Thus it is more efficient than buying a series of stop loss reinsurances.

They are more flexible and can be tailor-made. Most of the arrangements are private placement.

They do not need a liquid market.

Their flexibility and low cost provide motor insurers advantages over the traditional reinsurance treaties.
Motor Insurance Loss Rate Models

- Discounted compound Poisson model: The aggregate loss

\[ S_t = \int_{(0,t]} \int_{R^+} g(u, x) N(du, dx) = \sum_{i=1}^{N_t} g(T_i, X_i) \]

- Poisson random measure: \( N(du, dx) \), with mean measure

\[ m(du, dx) = du \nu_u(dx) \]

- Where Levy measure under the physical measure \( P \)

\[ \nu_u(dx) = \lambda_u dF_X(x) \]
Fourier transformation

Theorem 1 For any continuous bivariate function \( g \) such that
\[ \int_{(0,T]} \int_{R^+} g(u,x)N(du,dx) < \infty \]
then the following holds,

\[
E\left[ \exp\left( \int_{(0,T]} \int_{R^+} g(u,x)N(du,dx) \right) \right] = \exp\left[ \int_{(0,T]} \int_{R^+} \lambda_u \left( e^{g(u,x)} - 1 \right) dF_X(x)du \right]
\]

Corollary 2 Let us denote the distribution function of \( S_t \) by \( F_S(x,t) \). The Fourier transform of the distribution of \( S_t \) for a given \( t \) is expressed as follows.

\[
\hat{f}_S(u,t) = \int_{R^+} e^{iuX} dF_S(x,t) = E[e^{iuS_t}] = \exp\left\{ \int_{0}^{t} \lambda_s \left( \hat{f}_X(ue^{-rs}) - 1 \right) ds \right\}
\]
Measure change: Esscher Transform

- **Incomplete Market**: There are infinitely many risk neutral measures such that each choice of measure yields no arbitrage price of insurance risks.

- **Esscher Transform**
  \[
  \frac{dQ}{dP} \bigg|_{F_t} = \frac{e^{h_tS_t}}{E^P[e^{h_tS_t}]} , \text{ equivalently } dF^Q_S(x,t) = \frac{e^{h_tx}}{E^P[e^{h_tS_t}]} dF^P_S(x,t)
  \]

- **Martingale Condition**: The discounted surplus process should be a martingale under a risk neutral measure \( Q \)
  \[
  E^Q[e^{-rt}U_t \mid F_s] = e^{-rs}U_s , \quad \text{for any } 0 < s \leq t.
  \]
  where \( U_t = u_0e^{rt} + ce^{rt\alpha} - e^{rt}S_t \) and \( c = (1 + \theta)E^P[S_1] \)
Loss Distribution under Q

- By corollary 2, we have

\[
\hat{f}_S^Q(u, t) = \int_0^\infty e^{iu x} \frac{e^{h_i x}}{E^P[e^{h_i S}]} f^P_S(x, t) dx = \frac{\hat{f}_S^P(u - ih_t, t)}{\hat{f}_S^P(-ih_t, t)}
\]

\[
= \exp \left\{ \lambda \int_0^t \left\{ \hat{f}_X^P((u - ih_t)e^{-rs}) - \hat{f}_X^P(-ih_te^{-rs}) \right\} ds \right\}
\]

\[
= \exp \left\{ \int_0^t \lambda \hat{f}_X^P(-ih_te^{-rs}) \left\{ \frac{\hat{f}_X^P((u - ih_t)e^{-rs})}{\hat{f}_X^P(-ih_te^{-rs})} - 1 \right\} ds \right\}
\]

- Thus under Q,
  - The Poisson parameter \( \lambda \) has changed to \( \lambda_s^Q = \lambda \hat{f}_X^P(-ih_te^{-rs}) \).
  - The distribution of claim size \( dF^P_X(x) \) has changed to

\[
dF^Q_X(x) = \frac{\exp\{ h_te^{-rs} x \}}{\hat{f}_X^P(-ih_te^{-rs})} dF^P_X(x)
\]
Market value of Stop Loss Premium

- Theorem 3.4 in Dufresne, Garrido and Morales (2006)

\[
E^Q[(S_t - d)_+] = \frac{E^Q[S_t]}{2} + \frac{1}{\pi} \text{PV} \int_0^\infty \text{Re} \left[ \frac{e^{-iut}(\hat{f}_S^Q(u,t) - 1)}{(iu)^2} \right] du
\]

- Example: Suppose \( X \) follows Generalized Erlang\((n)\) claim size distribution, then we have

\[
E^Q[S_t] = \frac{\lambda}{rh^*} \left\{ \prod_{k=1}^n (1 - \beta_k h^*)^{-1} - \prod_{k=1}^n (1 - \beta_k e^{-\gamma h^*})^{-1} \right\},
\]
Example (continued):

Suppose $X$ follows Generalized Erlang($n$) claim size distribution. For notational simplicity we denote by

$$A(u, \eta, l, t) = \prod_{k=1}^{n} \left\{ \frac{u^2 (1-\eta)^2 \beta_k^2 + (e^{rl} - \beta_k h^*)^2}{u^2 (1-\eta)^2 \beta_k^2 + (e^{rl} - \beta_k h^*)^2} \right\}^{\frac{\lambda \beta_k}{2r \prod_{j \neq k} (\beta_k - \beta_j)}},$$

$$B(u, \eta, l, t) = \frac{\lambda}{r} \sum_{k=1}^{n} \left[ \beta_k^{n-1} \left\{ \arctan \left( \frac{u (1 - \eta) \beta_k}{\beta_k h^* - e^{rl}} \right) - \arctan \left( \frac{u (1 - \eta) \beta_k}{\beta_k h^* - e^{rl}} \right) \right\} \prod_{j \neq k} (\beta_k - \beta_j) \right]$$

Then the Fourier transform of the distribution of $S_t$ can be written by

$$\hat{f}^O_S(u, t) = \frac{A(u, 0, 0, t)}{A(0, 0, 0, t)} e^{iB(u, 0, 0, t)}$$

Using this we calculate the market price of stop loss contract.
Risk Neutral Distribution of Increment of Loss Process

- let us consider a general increment process

\[ Z_{t_1,t} (\eta) := S_t - \eta S_{t_1} \]

\[ = \int_{(0,t]} \int_{R^+} e^{-ru} xN(du, dx) - \eta \int_{(0,t]} \int_{R^+} e^{-ru} xN(du, dx) \]

\[ = \int_{(0,t]} \int_{R^+} (1 - \eta) e^{-ru} xN(du, dx) + \int_{(l,t]} \int_{R^+} e^{-ru} xN(du, dx) \]

- Corollary 3 The Fourier transform of risk neutral distribution of the general increment process is expressed as follows:

\[ \hat{f}^Q_{Z_{t_1,t}} (u, \eta) = E^Q \left[ e^{iu(\eta S_t - \eta S_{t_1})} \right] \]

\[ = \exp \left\{ \lambda \left( \int_{(0,t]} \hat{f}^P_X (u (1 - \eta) - ih^*) e^{-rs} ) ds \right. \right. \]

\[ \left. \left. + \int_{(l,t]} \hat{f}^P_X (u - ih^*) e^{-rs} ) ds - \int_{(0,t]} \hat{f}^P_X (ih^* e^{-rs} ) ds \right\} \]
Risk Neutral Distribution of Increment of Loss Process

The expectation can be easily obtained as follows.

\[
E^Q [Z_{l,t}(\eta)]
= \frac{\lambda}{r h_t} \left\{ (1 - \eta) E^P [e^{h^*X}] - E^P [e^{h^*e^{-rt}X}] + \eta E^P [e^{h^*e^{-rt}X}] \right\}
\]

The stop-loss price can be obtained as following:

\[
E^Q \left[ (S_t - \eta S_t)_+ \right] = E^Q \left[ (Z_{l,t}(\eta))_+ \right]
= \frac{E^Q [Z_{l,t}(\eta)]}{2} + \frac{1}{\pi} \text{PV} \int_0^\infty \text{Re} \left[ \frac{\hat{f}^Q_{Z_{l,t}} (u, \eta) - 1}{(iu)^2} \right] \]
Example: Generalized Erlang(n) claim size distribution

- The expectation of the general increment process is

\[
E^Q[Z_{t,t}(\eta)] = \frac{\lambda}{r h^*} \left\{ (1 - \eta) \prod_{k=1}^{n} (1 - \beta_k h^*)^{-1} - \prod_{k=1}^{n} (1 - \beta_k e^{-rt} h^*)^{-1} + \eta \prod_{k=1}^{n} (1 - \beta_k e^{-rl} h^*)^{-1} \right\}
\]

- The Fourier transform of the general increment process is

\[
\hat{f}^Q_{Z_{t,t}}(u, \eta) = \frac{A(u, \eta, 0, l) \cdot A(u, 0, l, t)}{A(0, \eta, 0, t)} \exp\left\{B(u, \eta, 0, l) + B(u, 0, l, t)i\right\}
\]

- Using the above formulas, we can obtain the price of stop loss contract.
The actual loss ratio is defined as

\[
\text{(The actual aggregate loss)} / \text{(the total gross premium)}, \text{ over a period of time}
\]

- In practice, the fixed loss rate is essentially determined by the historical claims data.

- It is usually accomplished by simulating the future losses that can be assumed to be retained by the insurance companies.

- We denote gross aggregate premium collected on \([0,t]\) by

\[
G_t = (1 + \theta^*) E^P [S_t] = (1 + \alpha) C_t = (1 + \alpha)(1 + \theta) \lambda E^P [X] \bar{a}_{\tilde{t}} = G \bar{a}_{\tilde{t}}
\]

where \(\theta\) is a risk adjustment parameter, and \(\alpha\) is a loading factor (expense rate).
Ratchet option on motor loss rate

Let us denote the actual cumulative loss rate by $q_t$, 

$$q_t = \frac{S_t}{G\overline{a}_t} = \frac{L_t}{G\overline{s}_t},$$

where $L_t = S_t e^{rt}$ is the cumulative losses until time $t$.

For each settlement point, the threshold of the Ratchet option is defined as follows:

$$\tilde{q}_{t_i} = \begin{cases} 
(1 + \pi) E^Q [q_{t_i}] = (1 + \pi) \frac{E^Q [S_{t_i}]}{G\overline{a}_{t_i}}, & i = 1, \\
(1 + \pi) q_{t_{i-1}} = (1 + \pi) \frac{S_{t_{i-1}}}{G\overline{a}_{t_{i-1}}}, & i > 1.
\end{cases}$$

Note that the threshold evolves over time and depend on the actual loss of the previous settlement point.
Then no arbitrage price of the Ratchet option is given as following:

\[
V(\pi;0) = E^Q\left[ \sum_{i=1}^{n} G\bar{a}_{t_i} (q_{t_i} - \tilde{q}_{t_i})_+ \right]
\]

\[
= E^Q\left[ (S_{t_1} - (1 + \pi)E^Q[S_{t_1}]_+) + \sum_{i=2}^{n} E^Q\left[ (S_{t_i} - (1 + \pi)\delta(t_{i-1}, t_i)S_{t_{i-1}})_+ \right] \right]
\]

where \( \delta(t_{i-1}, t_i) = \frac{1 - e^{-rt_i}}{1 - e^{-rt_{i-1}}} \)
Ratchet option on motor loss rate

(continued)

- For each summand in the sum, by letting \( \eta = (1 + \pi)\delta(t_{i-1}, t_i) \), we can obtain an integral expression of the price of the Ratchet option on the motor loss rate.
Plain Vanilla Motor Loss Rate Swaps

- First we consider a fixed-for-floating plain vanilla motor loss rate swap settled in arrears as a simple example.

- Even though we have a continuous time loss rate model we only consider a finite collection of discrete future dates, \( \{T_j, j = 0, 1, \ldots, n\} \) with \( T_0 = 0 \).

- The dates \( T_0, \ldots, T_{n-1} \) are known as reset dates, and the dates \( T_1, \ldots, T_n \) are known as settlement dates.

- The payments are made on the settlement dates and the number of payments \( n \) is called the length of a swap.

- The first date \( T_0 \) is referred to as the start date of a swap and we assume it is today for simplicity.
Plain Vanilla Motor Loss Rate Swaps

(continued)

- The period \([T_{j-1}, T_j]\) is called the j-th accrual period.

- We assume that Party A agrees to pay Party B a fixed amount of losses derived from a pre-agreed fixed loss ratio at each settlement date \(T_j, j = 1, \ldots, n\).

- In return, Party B agrees to pay Party A a floating amount of losses realized until each settlement date \(T_j, j = 1, \ldots, n\).

- The two parties usually need to pay the net amount, the difference between the two mutual obligations.
Plain Vanilla Motor Loss Rate Swaps

- We consider the value of a motor loss rate swap as a function of a real number $s$ at time 0,

$$
MS( s ) = E^Q \left[ \sum_{j=1}^{n} e^{-r_{T_j}} (L_{T_j} - (1 + s) \hat{q}_{T_j} Gs_{T_j}) \right] \\
= E^Q \left[ \sum_{j=1}^{n} (S_{T_j} - (1 + s) \hat{q}_{T_j} G\bar{a}_{T_j}) \right]
$$
Plain Vanilla Motor Loss Rate Swaps

- Since we know that a swap value is zero at initiation we define the price of a motor loss rate swap.

- **Definition 3** The price of a motor loss rate swap is the value of \( s \) that makes the value of a motor loss rate swap zero, i.e., the value of \( s \) for which \( MS( s ) = 0 \).
From Definition 3, we obtain an explicit formula for a motor loss rate swap price,

\[ s = \frac{\sum_{j=1}^{n} E^Q[ S_{T_j} ]}{\sum_{j=1}^{n} \hat{q}_{T_j} G \overline{a}_{T_j}} - 1 = \frac{1}{1 + \alpha} \frac{\sum_{j=1}^{n} \overline{a}_{T_j}}{\sum_{j=1}^{n} \hat{q}_{T_j} \overline{a}_{T_j}} - 1 \]

When we only consider an one-period swap, i.e. \( n=1 \), the price is

\[ s = \frac{1}{1 + \alpha} \frac{1}{\hat{q}_{T_j}} - 1 \]
By using the conditional expectation under the measure $Q$, the market price of the swap can be easily obtained as follows for the time in the $j$-th accrual period, $t \in (T_{j-1}, T_j)$

$$V(t) = E^Q \left[ e^{rt} \sum_{i=j}^n (S_{T_i} - (1 + \pi) \hat{q}_{T_i} G \overline{\alpha}_{T_i}) \bigg| S_t \right] = e^{rt} \left\{ \sum_{i=j}^n E^Q \left[ S_{T_i} \big| S_t \right] - \frac{\sum_{i=j}^n \hat{q}_{T_i} \overline{\alpha}_{T_i}}{\sum_{i=1}^n \hat{q}_{T_i} \overline{\alpha}_{T_i}} \sum_{i=1}^n E^Q \left[ S_{T_i} \right] \right\}$$

$$= e^{rt} \left\{ (n - j + 1)S_t + \sum_{i=j}^n \frac{\lambda}{r h^*} \left\{ E^P \left[ e^{h^* e^{rt} X} \right] - E^P \left[ e^{h^* e^{-rt} X} \right] \right\} - C \sum_{i=1}^n \overline{\alpha}_{T_i} \right\}$$
For illustrative purpose, we assume that the losses follow a discounted compound Poisson process with generalized Erlang (2) claim size distribution.

Specifically we assume that the Poisson parameter $\lambda = 12$, maturity $T = 5$, and $\beta_1 = 5, \beta_2 = 15$. 
# Table 1 Premium rate and Esscher parameters

<table>
<thead>
<tr>
<th>θ</th>
<th>r</th>
<th>C</th>
<th>h</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.01</td>
<td>264</td>
<td>0.002967</td>
</tr>
<tr>
<td></td>
<td>0.03</td>
<td></td>
<td>0.003108</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td></td>
<td>0.003248</td>
</tr>
<tr>
<td></td>
<td>0.07</td>
<td></td>
<td>0.003386</td>
</tr>
<tr>
<td>0.2</td>
<td>0.01</td>
<td>288</td>
<td>0.005534</td>
</tr>
<tr>
<td></td>
<td>0.03</td>
<td></td>
<td>0.005802</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td></td>
<td>0.006066</td>
</tr>
<tr>
<td></td>
<td>0.07</td>
<td></td>
<td>0.006326</td>
</tr>
<tr>
<td>0.3</td>
<td>0.01</td>
<td>312</td>
<td>0.007828</td>
</tr>
<tr>
<td></td>
<td>0.03</td>
<td></td>
<td>0.008207</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td></td>
<td>0.008581</td>
</tr>
<tr>
<td></td>
<td>0.07</td>
<td></td>
<td>0.008946</td>
</tr>
</tbody>
</table>
Figure 2 Evolution of loss process
We can see from Figure 2 that the distributions of the discounted losses under the Esscher transform are translated to the right and have slightly heavier left tail and lighter right tail.

This implies that the Esscher transform puts more weight on smaller extreme values.
### Table 2 Time zero price of 5-year ratchet options

<table>
<thead>
<tr>
<th>Ω</th>
<th>Phi</th>
<th>$V$</th>
<th>Ω</th>
<th>Phi</th>
<th>$V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td></td>
<td>183.65</td>
<td>0.0</td>
<td></td>
<td>201.59</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0</td>
<td>80.27</td>
<td>0.1</td>
<td>0.0</td>
<td>83.81</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0</td>
<td>40.29</td>
<td>0.2</td>
<td>0.0</td>
<td>40.62</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0</td>
<td>22.84</td>
<td>0.3</td>
<td>0.0</td>
<td>22.36</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0</td>
<td>14.02</td>
<td>0.4</td>
<td>0.0</td>
<td>13.34</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0</td>
<td>9.11</td>
<td>0.5</td>
<td>0.0</td>
<td>8.42</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0</td>
<td>6.18</td>
<td>0.6</td>
<td>0.0</td>
<td>5.56</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0</td>
<td>4.35</td>
<td>0.7</td>
<td>0.0</td>
<td>3.81</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0</td>
<td>3.15</td>
<td>0.8</td>
<td>0.0</td>
<td>2.70</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>192.85</td>
<td>0.1</td>
<td>0.1</td>
<td>209.84</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1</td>
<td>82.14</td>
<td>0.2</td>
<td>0.1</td>
<td>85.27</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1</td>
<td>40.49</td>
<td>0.3</td>
<td>0.1</td>
<td>40.69</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1</td>
<td>22.61</td>
<td>0.4</td>
<td>0.1</td>
<td>22.09</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1</td>
<td>13.68</td>
<td>0.5</td>
<td>0.1</td>
<td>13.00</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1</td>
<td>8.76</td>
<td>0.6</td>
<td>0.1</td>
<td>8.10</td>
</tr>
<tr>
<td>0.7</td>
<td>0.1</td>
<td>5.86</td>
<td>0.7</td>
<td>0.1</td>
<td>5.28</td>
</tr>
<tr>
<td>0.8</td>
<td>0.1</td>
<td>4.07</td>
<td>0.8</td>
<td>0.1</td>
<td>3.57</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>2.91</td>
<td>0.2</td>
<td>0.2</td>
<td>2.49</td>
</tr>
</tbody>
</table>
Figure 4  Market prices of 5-year ratchet options according to \( \Phi \)
**Table 4  Swap spreads**

<table>
<thead>
<tr>
<th>Threshold</th>
<th>10%tile</th>
<th>20%tile</th>
<th>30%tile</th>
<th>40%tile</th>
<th>50%tile</th>
<th>60%tile</th>
<th>70%tile</th>
<th>80%tile</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>0.616</td>
<td>0.458</td>
<td>0.358</td>
<td>0.280</td>
<td>0.213</td>
<td>0.152</td>
<td>0.090</td>
<td>0.025</td>
</tr>
</tbody>
</table>
Conclusions

- As hedging methods for motor loss rate risks, we suggest the use of hybrid derivates for motor insurance loss rate risk transfer.

- We consider a few motor insurance linked derivatives such as motor insurance loss rate options and swaps which can be traded over the counter in a capital market.

- They are designed not only to provide the insurer with innovative new hedging methods for its loss rate risks but also to give more investment choices to the potential investors in the financial market.
References

References

Thank You!
Questions?

Contact information:

Dr. Changki Kim  
Actuarial Studies  
Australian School of Business  
The University of New South Wales  
Sydney NSW 2052 Australia  

Tel: +61 2 9385 2647, +61 406 918 163  
Fax: +61 2 9385 1883  
Email: c.kim@unsw.edu.au