Motor Insurance Loss Rate Options and Swaps

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Abstract

In an attempt to manage the risk of mismatch between the actual claims and the anticipated claim amounts in motor insurance pool, we introduce a few new concepts of motor loss rate-linked securities such as motor loss rate options and motor loss rate swaps. These hybrid derivatives can transfer the motor insurance loss rate risks to the capital markets. For the valuation of the motor loss rate-linked securities, we assume that motor insurance aggregate claims follow a compound Poisson distribution. An Esscher transform is chosen for a risk adjusted measure change. Using the Fourier transform of the risk neutral distribution of increment of loss process and its inversion, we derive integral expressions of the price of a ratchet option and a fixed-for-floating plain vanilla swap on the motor loss rate. As illustrations, some numerical examples are given under a few specified assumptions on the distribution of the discounted losses and the parameters.

Key words:

Securitization, Risk Transfer, Ratchet option, Swaps, Motor Insurance Loss Rates.

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1 Introduction

Motor loss rate risk has long been a major concern for motor insurers. Usually the amount of future claims may not be predicted completely. The motor insurers want to manage the risk of mismatch between the actual claims and the anticipated claim amounts. If the actual motor loss rates increase higher than the expected loss rates, motor insurers will have to make additional payments during the contract period and this will lead to losses on their motor insurance business. As a result, the insurers may need additional, low cost tools to manage motor loss exposure. On the contrary, if the realized losses from the motors assured decrease below to the expected, the insurers would get profits. The recent concept of insurance securitization provides additional creative options for insurers to transfer insurance risks within the insurance/reinsurance industry or to the capital market. Motor insurance companies attempt to transfer the loss rate risks to the capital market where there is greater capacity to absorb these risks compared to the reinsurance market. Insurance companies may use tradable financial securities that have little or no correlation with the original risk itself. The traditional methods of reinsurance are still dominant, but some innovative applications of securitization have been gradually introduced into the market. Cummins and Weiss (2008) provide a survey on the recent developments of various types of insurance/financial instruments. For example, AXA introduced motor insurance securitization, selling EUR 200 million of bonds in 2005. Bae et al. (2008) illustrate the motor securitization methods based on a concept similar to CDOs. Tranches of bonds are constructed on the basis of the expected loss ratio from motor insurance policy holders’ groups. And they develop motor loss rate bonds using the structure of synthetic CDOs such that the coupon payments of each tranche depend on the level of the loss rates of the underlying motor insurance pool. They show the pricing methods of the tranches and the pricing formulas where the loss distribution is modeled with a discounted compound Poisson process.

There are several motivating factors for securitization of motor insurance portfolio risks. These include the issuer gaining an alternative source of financing, a channel of risk transfer and a method of capital management which helps improve the solvency of the company. Securitization also allows the insurer to eliminate counterparty risk, which it would

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1For more detailed information and discussion see Deringer (2006), De Mey (2007), AXA Financial Protection (2005), and Towers Perrin, Tillinghast (2006).
experience from the reinsurance market, by accessing traditional asset backed securities investors. Also, it allows the insurer to access those tools that are used by banks for risk management and anticipate the expected evolution of solvency rules. These transactions will also optimize the insurer’s business and balance sheet with respect to volume, pricing and terms. Motor insurance securitization creates new investment opportunities to the investors, providing greater diversification to traditional portfolios. Such groups of investors are the insurer’s policy holders, the government, the companies and organizations that are related to the motoring industry, and the general investors who seek high yield securities.

In this paper, we show the stochastic loss rate models and introduce the new concepts of motor loss rate-linked securities such as the motor insurance loss rate options and swaps. And we describe the characteristics of these securities and derive the pricing methods using a martingale method.

We would first describe the characteristics of a motor loss rate ratchet option and swap, the stochastic motor loss rate models, then show the pricing methods and a few numerical examples, and finally conclude with discussions and suggestions on the issues of these new securities.

2 Characteristics of Motor Insurance Loss Rate Swap and Option

A motor loss rate swap is a contract to exchange cash flows in the future based on the outcome of at least one presumably random motor loss rates. The objective is for insurers to hedge motor loss rate risks by exchanging one or more future cash flows, at least one of which is random.

**Definition 1** A motor loss rate swap is an agreement between two parties to exchange payments involving at least one random motor loss rate dependent payments for a certain period of time.

In a simple case, a motor loss rate swap involves the exchange of a single fixed payment for a single random motor loss rate-dependent payment. For an illustration, suppose that at time 0, two parties enter into an agreement to exchange a pre-fixed value \( k_t \) for a random value \( S_t \).
that is dependent on the realized motor loss rate at some future time $t$. The fixed amount $k_t$ may depend on the past empirical history on loss experiences. Specifically $S_t$ is the actual loss amount realized until time $t$ so it is a random variable at time 0. It can be related to the number of accidents from time 0 to time $t$ and the realized loss amounts for each accident from a specified reference motor insurance pool. It is reasonable for the two parties to make an agreement that they would exchange only the net difference between the two payment amounts. For example, party A pays party B a value of $k_t - S_t$ if $k_t > S_t$, or B pays A a value $S_t - k_t$ if $S_t > k_t$. In this case, party A would get benefits when the actual loss amount $S_t$ realized is higher than the pre-fixed amount $k_t$ and make losses if $S_t$ turns out to be lower than $k_t$. So Party A has a long position to $S_t$, while party B has a short position to $S_t$.

Insurance companies may use motor loss rate swaps to exploit natural hedging across their motor insurance businesses.

When an insurer is particularly interested in managing extreme loss rate risks, a swap contract seems to be inappropriate. In this case the insurer has a higher chance to make a payment rather than to be paid at each settlement point. Even though the corresponding swap rate may be determined based on the historical data, both parties would like to enter into a swap contract when they expect “fair” chance of getting paid. Alternatively a motor loss rate ratchet option can be used to protect the motor insurance providers against the risks of higher motor loss rates, but without giving up the possible benefits of lower motor loss rates. In other words, a motor loss rate ratchet option is an “insurance” to the motor insurance providers. The cost of the motor loss rate option is the “insurance premium”.

**Definition 2** A motor loss rate ratchet option is a series of call options on motor loss rates where strike thresholds are reset periodically.

A motor loss rate ratchet option is a right, not obligation, to exchange cash flows related to motor loss rates. On each exercise date the option holder receives the excess amount of loss above the pre-specified strike threshold. It is important to reset strike thresholds periodically based on the realized loss rate of the previous period. By doing this, both counterparties eager to remain in the contract even when they experience extreme loss events in some periods.
These hybrid derivatives have several advantages over motor loss rate bonds. They can be arranged at lower transaction costs than a bond issue. By construction the proposed swap and option cover multiple time periods. Thus it is more efficient than buying a series of stop loss reinsurances. They are more flexible and can be tailor-made. Most of the arrangements are private placement. They do not need a liquid market. It involves willing counterparties to exploit their comparative advantages or trade views on the development of motor loss rate over time. Their flexibility and low cost provide motor insurers advantages over the traditional reinsurance treaties.

3 Motor Insurance Loss Rate Models

The following section describes the derivation involved in the securities pricing via the pricing formulas for the stop loss premiums. For the purpose of this study, we will assume that the aggregate claims for motor insurance follow a marked point process. $^2$

Denote the Poisson process with parameter $\lambda_i$ by $N_i$. On a probability space $(\Omega, F, P)$, we assume that the aggregate loss $S_i$ follows

$$S_i = \int \int g(u, x) N(du, dx) = \sum_{i=1}^{N_i} g(T_i, X_i),$$

(1)

where $T_i$'s are jump times of the Poisson process $N_i$ and the magnitudes $X_i$'s of positive random shocks are independently and identically distributed with distribution function $F_X(x)$. The random shock $X_i$ arrived at time $t$ results in a claim measured by a continuous function $g(t, x)$ defined on $(0, T] \times \mathbb{R}_+$, which is increasing in $x$. Here we further assume that $N_i$ and $X_i$ are independent for model simplicity. Note that $N(du, dx)$ is a Poisson random measure with the mean measure

$$m(du, dx) = du \nu_\lambda(dx)$$

where $\nu_\lambda(dx) = \lambda dx F_X(x)$ is the Lévy measure.

The following theorem can be shown by using standard machinery in probability theory. See also Lemma 4.3 of Resnik (1986).

$^2$ For modeling the aggregate losses, losses will accumulated from the date of issue to maturity.
Theorem 1 For any continuous bivariate function \( g \in C((0,T] \times \mathbb{R}_+) \) such that
\[
\int_{[0,T]} \int_{\mathbb{R}_+} g(u, x) N(du, dx) < \infty,
\]
the following holds,
\[
E\left[ \exp \left( \int_{[0,T]} \int_{\mathbb{R}_+} g(u, x) N(du, dx) \right) \right] = \exp \left[ \int_{[0,T]} \int_{\mathbb{R}_+} \lambda_u \left( e^{g(u, x)} - 1 \right) dF_X(x) du \right],
\]
(2)

For model simplicity we assume that the Poisson process \( N_t \) is a time homogeneous reflecting constant rate of \( \lambda \) and the claim size distribution is a discounted random shock \( g(u, X) = e^{-ru} X \). Then the aggregate loss process (1) is also referred to be a discounted compound Poisson process. See Delbaen and Haezendonck (1987), Paulsen (1993) and Nilsen and Paulsen (1996) for more details on the distribution of a discounted compound Poisson process.

Mapping techniques such as Fourier transform and its inverse transform will be employed in calculating the market price of stop loss premium and other relevant securities.

Corollary 2 Let us denote the distribution function of \( S_t \) by \( F_S(x, t) \). The Fourier transform of the distribution of \( S_t \) for a given \( t \) is expressed as follows.
\[
\hat{F}_S(u, t) = \int_{\mathbb{R}_+} e^{iuX} dF_S(x, t) = E[e^{iuS_t}] = \exp \left[ \int_0^t \hat{\lambda}_s \left( \hat{F}_X(u e^{-rs}) - 1 \right) ds \right],
\]
(3)
where \( \hat{F}_X(u) \) is the Fourier transform of the distribution of a claim size random variable \( X \).

As a consequence of the market not being complete, immeasurable numbers of risk neutral probability measures are present, and each probability measure results in no arbitrage price of insurance risks. For such probability measure changes, the Esscher transform is one of the candidate which can be used due to its characteristics as described as follow. The Esscher transform is known as the minimal entropy martingale probability measure in a geometric Levy process model. Furthermore, Esscher transform maximizes the expected
power utility function. The attitude of market to insurance risk can be interpreted by the Esscher parameter \( h \) of the Esscher transform which can be obtained under the martingale state. The real interest rate of zero reflects constant risk adjustment parameter \( h \) (as determined by martingale condition) over time. In this study, the risk adjustment parameter will be a time invariant function due to the discounting effect. Specifically, for each maturity \( t > 0 \), we define a probability measure \( Q \) whose Radon-Nikodym derivative is

\[
\frac{dQ}{dP}_{t} = \frac{e^{hS_{t}}}{E^{P}[e^{hS_{t}}]}, \tag{4}
\]

equivalently,

\[
dF^{Q}_{S}(x,t) = \frac{e^{h_{x}}}{E^{P}[e^{hS_{t}}]}dF^{P}_{S}(x,t),
\]

provided that \( E^{P}[e^{hS_{t}}] \) exists. Note that \( h_{t} \) is non-negative deterministic function which satisfies martingale condition described below in Eq. (*). \(^4\)

By corollary 2, we have,

\[
\hat{f}^{Q}_{S}(u,t) = \int_{0}^{\infty} e^{u_{x}} \frac{e^{h_{x}}}{E^{P}[e^{hS_{t}}]} f_{S}^{P}(x,t)dx = \frac{\hat{f}^{P}_{S}(u - ih^{*},t)}{\hat{f}^{P}_{S}(-ih^{*},t)}
\]

\[
= \exp \left\{ \lambda \int_{0}^{\infty} \left\{ \hat{f}^{P}_{X}((u - ih^{*})e^{-r}) - \hat{f}^{P}_{X}(-ih^{*}e^{-r}) \right\} ds \right\}. \tag{5}
\]

Equation (5) can be represented as follows which enables us to identify the distribution of \( S_{t} \) under the changed measure \( Q \),

\[
\hat{f}^{Q}_{S}(u,t) = \exp \left\{ \lambda \int_{0}^{\infty} \left\{ \frac{\hat{f}^{P}_{X}((u - ih^{*})e^{-r})}{\hat{f}^{P}_{X}(-ih^{*}e^{-r})} - 1 \right\} ds \right\}.
\]

**Remark 1** By comparing the above with (3), for each fixed maturity \( t \), we can conclude the followings:

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\(^3\) See Gerber and Shiu (1994) or Miyahara and Fujiwara (2003) for details.

\(^4\) Once maturity \( T \) is fixed, \( h_{t} \) is determined and assume to be constant over the period \( (0, T) \). For notational convenience, it is denoted by \( h^{*} \).
(i) The Poisson parameter $\lambda$ has changed to $\hat{\lambda}^Q_X = \lambda \hat{f}^p_X (-ih^* e^{-rs})$, $s \leq t$;

(ii) The distribution of claim size, $dF^Q_X(x)$, has changed to

$$dF^Q_X(x,s) = \frac{\exp \{h^* e^{-rs} x\}}{\hat{f}^p_X (-ih^* e^{-rs})} dF^p_X(x)$$
and

$$\hat{f}^Q_X(ue^{-rs}) = \frac{\hat{f}^p_X ((u-ih^*)e^{-rs})}{\hat{f}^p_X (-ih^* e^{-rs})}, \quad s \leq t.$$ Note that the Lévy measure under $Q$ is

$$\nu^Q_u (dx) = \lambda \exp \{h^* e^{-rs}\} dF^p_X(x)$$

For each $t$,

$$E^Q[S_t] = \int_0^\infty \frac{xe^{h^* x}}{E^p[e^{h^* S_t}]} dF^p_X(x,t) = E^p[S_t e^{h^* S_t}] = \frac{\lambda}{\partial h} \{\log E^p[e^{h^* S_t}]\}. \quad (6)$$

Since $E^p[e^{h^* S_t}] = \hat{f}^p_S (-ih^*,t)$, the latter would be reduced to

$$E^Q[S_t] = \lambda \int_0^t \frac{\partial}{\partial h} \hat{f}^p_X (-ih^* e^{-rs}) ds$$

$$= \lambda \int_0^t \{E^p[e^{h^* X}] - E^p[e^{h^* e^{-rs} X}]\}. \quad (7)$$

The second equality can be obtained via Corollary 2 and changing the order of integration and expectation.

We require the determination of arbitrage free price of stop loss contract with a retention level of $d$, for the motor loss rate option pricing. Mathematically represented as,

$$E^Q[(S_t - d)_+] = \int_d^\infty (x - d) dF^Q_S(x,t). \quad (8)$$

The following equation is obtained via applying Theorem 3.4 in Dufresne et al (2006):

$$E^Q[(S_t - d)_+] = \frac{E^Q[S_t]}{2} + \frac{1}{\pi} PV \int_0^\infty \text{Re} \left[ \frac{e^{-iud} (\hat{f}^Q(u,t) - 1)}{(iu)^2} \right] du \quad (9)$$

where $PV\int$ refers to Cauchy principle value integral.
Substitution of (5) and (7) into (9), result in an Esscher no arbitrage price formula of stop loss contract\(^5\).

Under the assumption of no arbitrage between insurance market and capital market, the discounted surplus process should be a martingale under a risk neutral measure \(Q\).\(^6\) Specifically we define the accumulated surplus process as follow,

\[
U_t = u_0 e^{rt} + e^{rt} C_t - e^{rt} S_t, \tag{10}
\]

where \(u_0\) is the initial surplus and \(C_t\) is time zero value of the risk adjusted aggregated premiums collected on \([0, t]\), specifically we define

\[
C_t = (1 + \theta)E^{\theta} [S_t] = (1 + \theta)\lambda E^{\theta} [X] \overline{\sigma}_\theta = C \overline{\sigma}_\theta,
\]

with a risk adjustment parameter \(\theta > 0\). We denote the continuous premium rate by \(C = (1 + \theta)\lambda E^{\theta} [X]\).

We can find an equivalent martingale measure \(Q\) that satisfies the following\(^7\),

\[
E^Q [e^{-rS} | F_s] = e^{-rs} U_s, \quad Q\text{-a.s.} \tag{*}
\]

for any \(0 \leq s \leq t\). By (7), for the maturity \(t\), we can show that \(h_t\) is the solution of the following equation which also satisfies the existence of Esscher transform (4),

\[
E^Q [S_t] = \frac{\lambda}{r h_t} \left\{ E^{\theta} [e^{h_t X}] - E^{\theta} [e^{h_t e^{r X}}] \right\} = C \overline{\sigma}_\theta. \tag{11}
\]

Using L’hopital’s rule, it can be shown that

\[
\lim_{h_t \to 0} E^Q [S_t] = \lambda E^{\theta} [X] \overline{\sigma}_\theta.
\]

Eq. (11) and the above identity imply that if the risk adjusted parameter \(\theta = 0\), or \(C = \lambda E^{\theta} [X]\), then \(h_t = 0\), which means that the market takes the risk fully.

It can be shown that the constant Esscher parameter is the solution to the equation below (similar to (11)) in the presence of zero real interest rate.

\[
E^{\theta} [X e^{hX}] = \frac{C}{\lambda}.
\]

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\(^5\) Price of a stop loss contract under the compound Poisson distribution can be calculated by using numerical method or recursion, for example, using the Panjer recursion formula. See Covens et al. (1979), Bühlmann (1984), Gerber (1982) and Panjer (1981) for reference.

\(^6\) Same idea is used in Jang and Kravavych (2004).

\(^7\) We can find a martingale measure \(Q\) or the Esscher parameter \(h_t\) directly from the market when the market becomes active and mature. While the market is new and young this approach can be an alternative.
As we can see from Eq. (5) and (11), the moment generating function of claim size distribution plays a crucial role in determining mathematical tractability of price formula. Setting aside empirical studies on the motor insurance claim size data, we illustrate a tractable (but flexible enough to use in practice) example.  

Example 1 Generalized Erlang(n) claim size distribution  
Suppose X follows a generalized Erlang(n) distribution with parameters \((n, \beta_1, \beta_2, \ldots, \beta_n)\) such that \(E^P[X] = \sum_{k=1}^{n} \beta_k\). The Fourier transform is given by \(\hat{f}_X^P(u) = \prod_{k=1}^{n} (1-i \beta_k u)^{-1}\).

To guarantee the existence of the Esscher parameter \(h_i\), which also ensures the existence of Esscher transform, we further assume that

\[
\beta_i e^{-rt} < \beta_1 < \beta_2 e^{-rt} < \beta_2 < \cdots < \beta_n e^{-rt} < \beta_n. \tag{12}
\]

This condition may hold when the interest rate \(r\) is small and the maturity \(t\) is short.

The expectation of discounted claims under \(Q\) is

\[
E^Q[S_i] = \frac{\lambda}{rh} \left\{ \prod_{k=1}^{n} (1 - \beta_k h^*)^{-1} - \prod_{k=1}^{n} (1 - \beta_k e^{-rt} h^*)^{-1} \right\}.
\]

Note that Esscher parameter, if any, must be smaller than \(1/\beta_n\) in order to guarantee the existence of moment generating function or Esscher transform. After some algebra we can show that the Esscher parameter \(h^*\) is a solution of the following polynomial equation.  

\[
p(h) = q(h)(1 + \theta) \sum_{k=1}^{n} \beta_k \tag{13}
\]

where

\[
p(h) = \sum_{k=1}^{n} \beta_k - h(1 + e^{-rt}) \sum_{k=1}^{n} \beta_k \beta_1 + h^2 (1 + e^{-rt} + e^{-2rt}) \sum_{k=1}^{n} \beta_k \beta_1 \beta_m + \cdots + (-1)^{n-1} h^{n-1} \left( \sum_{k=0}^{n-1} e^{-rt} \right) \prod_{k=1}^{n} \beta_k
\]

\[
q(h) = \prod_{k=1}^{n} (1 - \beta_k h)(1 - \beta_k e^{-rt} h).
\]

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8 Exponential (or Gamma) distribution is often used in generalize linear models on aggregated insurance data. See Smyth and Jorgensen (2002) for example.

9 We assume that the Esscher parameter is constant once the maturity \(t\) is fixed. Thus we drop \(t\) in the expression.
Clearly, $q(h)$ has $2n$ zeros at \( \left\{ \frac{1}{\beta_1}, \frac{1}{\beta_2}e^{-\tau}, \ldots, \frac{1}{\beta_n}, \frac{1}{\beta_1}, \frac{1}{\beta_2}e^{-\tau} \right\} \). Due to the assumption (12) and the fact that

\[
p(1/\beta_j) = \frac{1}{\beta_{n-j}^{n-2}} \prod_{k=j}^{n} (\beta_j - \beta_k e^{\tau}),
\]

we can show that $p(h)$ has $n-1$ zeros and the $j$th smallest zero lies in the interval \( \left(1/\beta_{n-j+1}, 1/\beta_{n-j}\right) \). Therefore there exists only one root $h$ which is smaller than $1/\beta_n$ as illustrated in Figure 1.

\[\text{Figure 1 Finding } h^* \text{ using graphs}\]

Note that exponential distribution is a special case when $n = 1$ under which the Esscher parameter is

\[
h^* = \frac{1}{2\beta} \left\{ 1 + e^{\tau} - \sqrt{(1 + e^{\tau})^2 - 4e^{\tau} \theta/(1+\theta)} \right\}. \quad (14)
\]

It is not hard to show the following:
\[ k(u, \eta, h^*, l, t) := \lambda \int_{(l \in \mathbb{R}^+)} \hat{f}_X^p ((u(1-\eta) - ih^*) e^{-\eta s}) ds \]

\[
= \frac{\lambda}{2r} \sum_{k=1}^{\infty} \left[ 2\beta_k^{-1} \left( \arctan \left( \frac{u(1-\eta)\beta_k}{\beta_k h^* - e^{\eta t}} \right) - \arctan \left( \frac{u(1-\eta)\beta_k}{\beta_k h^* - e^{\eta t}} \right) \right) i + \beta_k \log \left( \frac{u^2 (1-\eta)^2 \beta_k^2 + (e^{\eta t} - \beta_k h^*)^2}{u^2 (1-\eta)^2 \beta_k^2 + (e^{\eta t} - \beta_k h^*)^2} \right) \right] \prod_{j \neq k} (\beta_k - \beta_j). \]

(15)

For notational simplicity we denote by

\[ A(u, \eta, l, t) = \prod_{k=1}^{\infty} \left( \frac{u^2 (1-\eta)^2 \beta_k^2 + (e^{\eta t} - \beta_k h^*)^2}{u^2 (1-\eta)^2 \beta_k^2 + (e^{\eta t} - \beta_k h^*)^2} \right)^{\frac{\lambda \beta_k}{r \prod (\beta_k - \beta_j)}} \]

and

\[ B(u, \eta, l, t) = \frac{\lambda}{r} \sum_{k=1}^{\infty} \left[ \beta_k^{-1} \left( \arctan \left( \frac{u(1-\eta)\beta_k}{\beta_k h^* - e^{\eta t}} \right) - \arctan \left( \frac{u(1-\eta)\beta_k}{\beta_k h^* - e^{\eta t}} \right) \right) \right] \prod_{j \neq k} (\beta_k - \beta_j). \]

Then the Fourier transform of the distribution of \( S_t \) can be written by

\[ \hat{f}_S^Q (u, t) = \exp \left\{ k(u, 0, h^*, 0, t) - k(0, 0, h^*, 0, t) \right\} = \frac{A(u, 0, 0, t)}{A(0, 0, 0, t)} e^{iB(u, 0, 0, t)}. \]  

(16)

Plugging the above into (9) gives a market price of stop loss contract.

4 Risk Neutral Distribution of Increment of Loss Process and a ratchet option price

In this section, we derive the Fourier transform of the risk neutral distribution of increment of loss process. By using the properties, we show the price formula of a ratchet option.

For a fixed \( \eta \geq 0 \) and \( 0 \leq l \leq t \leq T \), let us consider a general increment of the loss process,

\[ Z_{l,t}(\eta) := S_t - \eta S_l = \int_{(0,t] R^+} \int_{(0,t] R^+} e^{-\eta x} xN(du, dx) - \eta \int_{(0,t] R^+} \int_{(0,t] R^+} e^{-\eta x} xN(du, dx). \]

\[ = \int_{(0,l] R^+} \int_{(0,t] R^+} (1-\eta) e^{-\eta x} xN(du, dx) + \int_{(l,t] R^+} \int_{(0,t] R^+} e^{-\eta x} xN(du, dx). \]

(17)

As noted in the Remark 1, under \( Q \), the jump process follows an inhomogeneous Poisson process with intensity function \( \lambda_0^Q = \lambda f_X^p (-ih^* e^{-\eta s}) \) which depends on time. The
claim size distribution \(dF_Q^\epsilon(x,s) = \frac{\exp\{h^*e^{-\tau x}\}}{\hat{f}_X^P(-ih^*e^{-\tau})}dF_P^\epsilon(x)\) is also a function of time.

Fortunately, the distribution still has an independent increments property. That is, the first and the second terms in (17) are non-overlapping and thus independent under both measures \(P\) and \(Q\). The property provides a following expression for the Fourier transform of \(Z_{t,t}(\eta)\) under \(Q\).

**Corollary 3** For a fixed \(\eta \geq 0\) and \(0 \leq t \leq T\), the Fourier transform of risk neutral distribution of the general increment process \(Z_{t,t}(\eta)\) is expressed as following:

\[
\hat{f}_{Z_{t,t}}^Q(u,\eta) = E^Q\left[e^{iu(S_{t} - \eta S_{t})}\right]
\]

\[
= E^Q\left[\exp\left\{iu\left(\int_{(0,t]}(1-\eta)e^{-\tau x}N(ds, dx) + \int_{(t,T]}e^{-\tau x}N(ds, dx)\right)\right\}\right]\]

\[
= E^Q\left[\exp\left\{\int_{(0,t]}iu(1-\eta)e^{-\tau x}N(ds, dx)\right\}\right]E^Q\left[\exp\left\{\int_{(t,T]}iu e^{-\tau x}N(ds, dx)\right\}\right]
\]

\[
= \exp\left\{\lambda\left(\int_{(0,t]}\hat{f}_X^P((u(1-\eta) - ih^*)e^{-\tau x})ds + \int_{(t,T]}\hat{f}_X^P((u - ih^*)e^{-\tau x})ds - \int_{(0,t]}\hat{f}_X^P(-ih^*e^{-\tau})ds\right)\right\}.
\]

(18)

The expectation of \(Z_{t,t}(\eta)\) is

\[
E^Q[Z_{t,t}(\eta)] = \frac{\lambda}{rh^*}\left(1-\eta\right)E^P[e^{h^*X}] - E^P[e^{h^*e^{-\tau}X}] + \eta E^P[e^{h^*e^{-\tau}X}]\right\}.
\]

(19)

By substituting the Fourier transform (18) and the expectation (19) into the stop-loss price formula (9), we can obtain the following:

\[
E^Q[(S_t - \eta S_t),_s] = E^Q[(Z_{t,t}(\eta)),_s] = \frac{E^Q[Z_{t,t}(\eta)]}{2} + \frac{1}{\pi} PV \int_{0}^{\infty} \frac{\left(\hat{f}_{Z_{t,t}}^Q(u,\eta) - 1\right)}{(iu)^2} du.
\]

(20)

**Example continued: Generalized Erlang(n) claim size distribution**

The expectation of \(Z_{t,t}(\eta)\) is

\[
E^Q[Z_{t,t}(\eta)] = \frac{\lambda}{rh^*}\left((1-\eta)\prod_{k=1}^{n}(1 - \beta_k e^{-\tau h^*})^{-1} - \prod_{k=1}^{n}(1 - \beta_k e^{-\tau h^*})^{-1} + \eta \prod_{k=1}^{n}(1 - \beta_k e^{-\tau h^*})^{-1}\right).
\]

Recalling Eq. (15), the Fourier transform (18) can be written as
\[
\hat{f}^0_{z_i}(u, \eta) = \exp\left\{k(u, \eta, h^*, 0, t) + k(u, 0, h^*, l, t) - k(0, \eta, h^*, 0, t)\right\}
\]
\[
= \frac{A(u, \eta, 0, l) \cdot A(u, 0, l, t)}{A(0, \eta, 0, t)} \exp\left[\frac{(B(u, \eta, 0, l) + B(u, 0, l, t))\eta}{\theta}\right].
\]
By substituting these formulae into the price formula (20), we can obtain an integral expression of stop loss contract where the threshold depends on the loss history. ■

**Ratchet option on motor loss rate**

The actual loss ratio is defined as the actual aggregate loss divided by the total gross premium over a period of time \([0, t]\). In practice, the fixed loss rate is essentially determined by the historical claims data and is usually accomplished by simulating the future loss that can be assumed to be retained by the insurance companies.

We denote gross aggregate premium collected on \([0, t]\) by

\[
G_t = (1 + \theta')\mathbb{E}^\theta[S_t] = (1 + \alpha)(1 + \theta)\lambda \mathbb{E}^\theta[X] \bar{a}_{t,\eta} = G\bar{a}_{t,\eta},
\]

where \(\theta\) is a risk adjustment parameter, and \(\alpha\) is a security loading factor (expense rate).

Let us denote the actual cumulative loss rate by \(q_t\),

\[
q_t = \frac{S_t}{\bar{a}_{t,\eta}} = \frac{L_t}{\bar{s}_{t,\eta}},
\]

where \(L_t = S_t e^{\pi t}\) is the cumulative losses until time \(t\), and \(\bar{a}_{t,\eta}\) and \(\bar{s}_{t,\eta}\) are the present value and the accumulated value of continuously paying annuities with unit of annual payment, respectively.

For each settlement point, the threshold of the ratchet option is defined as follows:

\[
\hat{q}_t(\pi) = \begin{cases} 
(1 + \pi)E^\theta[q_t] = (1 + \pi) \frac{E^\theta[S_t]}{\bar{a}_{t,\eta}}, & i = 1, \\
(1 + \pi)q_{t,i} = (1 + \pi) \frac{S_{t,i}}{\bar{a}_{t,i}}, & i > 1.
\end{cases}
\]

Note that the threshold evolves over time and depends on the actual loss of the previous settlement point. The parameter \(\pi (> -1)\) determines the level of subsequent strike thresholds and must be specified upfront. If an insurer would like to hedge extreme losses above previous period’s realized loss, large \(\pi\) is preferred.
At each settlement date, the protection seller (issuer) will pay the premium collected times excessive loss rate beyond the prefixed threshold. Thus, a no arbitrage price of the n-year ratchet option is given as following:

\[
V(\pi; 0) = E^Q \left[ \sum_{i=1}^{n} G_{i}^{\pi} \left( q_{i} - \hat{q}_{i}^{\pi} \right) \right]
\]

\[
= E^Q \left[ G_{i}^{\pi} \left( q_{i} - (1 + \pi) E^Q[ q_{i} ] \right) \right] + E^Q \left[ \sum_{i=2}^{n} G_{i-1}^{\pi} \left( q_{i} - (1 + \pi) q_{i-1} \right) \right]
\]

\[
= E^Q \left[ \left( S_{i} - (1 + \pi) E^Q[ S_{i} ] \right) \right] + \sum_{i=2}^{n} E^Q \left[ \left( S_{i} - (1 + \pi) \delta(t_{i-1}, t_{i}) S_{i-1} \right) \right], \tag{24}
\]

where \( \delta(t_{i-1}, t_{i}) := \frac{1 - e^{-\pi t_{i}}}{1 - e^{-\pi t_{i-1}}} \).

Recalling (20) and letting \( \eta = (1 + \pi) \delta(t_{i-1}, t_{i}) \) give an integral expression of each summand in the ratchet option price formula (24).

5 Pricing Motor Loss Rate Swaps

Plain Vanilla Motor Loss Rate Swaps

First we consider a fixed-for-floating plain vanilla motor loss rate swap settled in arrears as a simple example. Even though we have a continuous time loss rate model we only consider a finite collection of discrete future dates, \( \{ T_j, j = 0, 1, \ldots, n \} \) with \( T_0 = 0 \). The dates \( T_0, \ldots, T_{n-1} \) are known as reset dates, and the dates \( T_1, \ldots, T_n \) are known as settlement dates. The payments are made on the settlement dates and the number of payments \( n \) is called the length of a swap. The first date \( T_0 \) is referred to as the start date of a swap and we assume it is today for simplicity. The period \( [T_{j-1}, T_j] \) is called the \( j \)-th accrual period. We assume that Party A agrees to pay Party B a fixed amount of losses derived from a pre-agreed fixed loss ratio denoted by \( \hat{q}_{T_j} \) at each settlement date \( T_j, j = 1, \ldots, n \). In return, Party B agrees to pay Party A a floating amount of losses realized until each settlement date \( T_j, j = 1, \ldots, n \). The two parties usually need to pay the net amount, the difference between the two mutual obligations. So Party B should pay when the actual loss ratio exceeds the predetermined fixed loss ratio, \( q_{T_j} > (1+s) \hat{q}_{T_j} \), where \( s \) is a real number to be determined. That is, \( q_{T_j} > (1+s) \hat{q}_{T_j} \) implies that \( L_{T_j} > (1+s) \hat{q}_{T_j} G \bar{s}_{T_j} \) and Party B should pay \( L_{T_j} - (1+s) \hat{q}_{T_j} G \bar{s}_{T_j} \). If \( q_{T_j} \leq (1+s) \hat{q}_{T_j} \), then Party A should pay \( (1+s) \hat{q}_{T_j} G \bar{s}_{T_j} \) at time \( T_j \).
We consider the value of a motor loss rate swap as a function of a real number $s$ at time $0 = T_0$,

$$
MS(s) = E^Q \left[ \sum_{j=1}^{n} e^{-\gamma_{T_j}} (L_{T_j} - (1 + s) \hat{q}_{T_j} G\bar{a}_{T_j}) \right] 
$$

$$
= E^Q \left[ \sum_{j=1}^{n} (S_{T_j} - (1 + s) \hat{q}_{T_j} G\bar{a}_{T_j}) \right].
$$

(25)

Since we know that a swap value is zero at initiation, we naturally define the spread of a motor loss rate swap.

**Definition 3** *The spread of a motor loss rate swap is the value of $s$ that makes the value of a motor loss rate swap zero, i.e., the value of $s$ for which $MS(s) = 0$.*

Using Definition 3, we obtain an explicit formula for a motor loss rate swap spread,

$$
s = \frac{\sum_{j=1}^{n} E^Q[S_{T_j}]}{\sum_{j=1}^{n} \hat{q}_{T_j} G\bar{a}_{T_j}} - 1 = \frac{1}{1 + \alpha} \frac{\sum_{j=1}^{n} \bar{a}_{T_j}}{\sum_{j=1}^{n} \hat{q}_{T_j} \bar{a}_{T_j}} - 1.
$$

(26)

When we only consider an one-period swap, i.e. $n=1$, the price is

$$
s = \frac{1}{1 + \alpha} \frac{1}{\hat{q}_{T_1}} - 1.
$$

If the expected loss rate $\hat{q}_{T_1}$ is well produced enough to predict the real loss rate for the next period then the swap prices should be zero and the expense rate should be

$$
\alpha = \frac{1}{\hat{q}_{T_1}} - 1.
$$

Given the value of $s$ determined above, we can calculate the market price of the swap at any time $t \geq T_0$. The following corollary can be shown relying on the fact that the loss distribution $S_t$ still has independent increments property after the measure change as we can see in the remark 1 in the previous section.

**Corollary 4** Under Esscher transformed measure $Q$, the Fourier transform of the conditional distribution $S_t$ given $S_s = y$ has the following expression:

$$
\hat{f}_S^Q(u,t; y,l) = E^Q \left[ e^{iuS_t} \mid S_s = y \right] = E^Q \left[ \exp \left\{ iuy + \int_{l}^{u} \int \left. iux e^{-\gamma s} N(ds, dx) \right\} S_s = y \right]
$$
\[
= \exp \left\{ i u y + \lambda \int_0^T \left[ \hat{f}_x^p ((u - i h_t)e^{-r_s}) - \hat{f}_x^p (-i h_t e^{-r_s}) \right] ds \right\}. \tag{27}
\]

Then we can easily show that the conditional expectation can be written as follows.

\[
E^Q[S_t | S_t = y] = \frac{1}{i} \frac{d}{du} \hat{f}_x^Q (u, t; y, t) \bigg|_{u=0} = y + \frac{\lambda}{r h_t} \left[ E^p \left[ e^{b e^{-r_s} X} \right] - E^p \left[ e^{h e^{-r_s} X} \right] \right]. \tag{28}
\]

By using the above conditional expectation under the measure \( Q \), the market price of the swap at a time in the j-th accrual period, \( t \in (T_{j-1}, T_j) \), can be expressed as follows:

\[
V(t) = E^Q \left[ e^\alpha \sum_{i=j}^n (S_{t_i} - (1 + s) \hat{q}_{t_i} G \alpha \tau_{t_i}) | S_t \right] = e^\alpha \left\{ \sum_{i=j}^n \sum_{i=1}^n E^Q \left[ S_{t_i} \right] - \frac{\sum_{i=1}^n \hat{q}_{t_i} \alpha \tau_{t_i} \sum_{i=1}^n E^Q \left[ S_{t_i} \right]}{\sum_{i=1}^n \hat{q}_{t_i} \alpha \tau_{t_i}} \right\} \tag{29}
\]

\[
= e^\alpha \left( n-j+1)S_t + \sum_{i=j}^n \frac{\lambda}{r h_t} \left[ E^p \left[ e^{b e^{-r_s} X} \right] - E^p \left[ e^{h e^{-r_s} X} \right] \right] - C \sum_{i=1}^n \frac{\sum_{i=1}^n \hat{q}_{t_i} \alpha \tau_{t_i}}{\sum_{i=1}^n \hat{q}_{t_i} \alpha \tau_{t_i}} \right). \tag{29}
\]

### 6 Numerical Examples \(^{10}\)

In this section we consider some numerical examples under some specified assumptions on the distribution of the discounted losses and the parameters. The results may vary when there are changes in the assumptions. For illustrative purpose, we assume that the losses follow a discounted compound Poisson process with generalized Erlang (2) claim size distribution. Specifically we assume that the Poisson parameter \( \lambda = 12, \beta_1 = 5, \beta_2 = 15 \), and maturity \( T = 5 \).

\(^{10}\) Numerical integrations are implemented using R-function \textit{integrate}. 100,000 simulations of \( S_t \) under \( Q \) are conducted based on Remark 1. Specifically, for each simulation, risk neutral arrival times of inhomogeneous Poisson process with rate function \( \hat{\lambda}_t \) are simulated by thinning algorithm and for each arrival time, a risk neutral claim amount is generated by the density \( f_x^Q (x, t) \) using rejection algorithm. See Ross (2002) for details on these algorithms.

\(^{11}\) The choice of loss frequency and claim size parameters is more or less arbitrary. We roughly use the 2007 US private passenger insurance losses data given by Insurance Information Institute (2009) available at
Table 1 summaries the risk adjusted premium rate $C = (1 + \theta) \lambda E^\theta [X]$ and the Esscher parameter $h^\star = h_\theta$ for several different choices of the risk adjustment parameter $\theta$ and the interest rate $r$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$r$</th>
<th>$C$</th>
<th>$h^\star$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.01</td>
<td>0.002967</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.03</td>
<td>0.003108</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.05</td>
<td>0.003248</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.07</td>
<td>0.003386</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.01</td>
<td>0.005534</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.03</td>
<td>0.005802</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.05</td>
<td>0.006066</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.07</td>
<td>0.006326</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.01</td>
<td>0.007828</td>
<td></td>
</tr>
<tr>
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<td></td>
</tr>
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<td>0.2</td>
<td>0.05</td>
<td>0.008581</td>
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<td>0.2</td>
<td>0.07</td>
<td>0.008946</td>
<td></td>
</tr>
<tr>
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<td>0.01</td>
<td>0.010539</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.03</td>
<td>0.010918</td>
<td></td>
</tr>
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<td>0.3</td>
<td>0.05</td>
<td>0.011292</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.07</td>
<td>0.011666</td>
<td></td>
</tr>
</tbody>
</table>

We can see from the Table 1 that the Esscher parameter increases in both the loading factor and the interest rate. The risk adjustment parameter and interest rate are the two main input components which determine the riskiness of underlying loss process. The higher value of Esscher parameter results in a shift of the loss distribution to the right.

Figure 2 shows the evolution of the discounted loss distribution $S_t$ over time and compares two densities $f^p_S(x,t)$ and $f^Q_S(x,t)$ when $\theta = 0.2$ and $r = 0.03$.

http://www.iii.org/media/facts/statsbyissue/auto/ . The statistics indicate that the frequency of liability claims is about 5 per 100 vehicles and the average claim severity is about $15,000 when we add bodily injury and property damage. For collision and comprehensive coverage, the frequency is about 7 per 100 vehicles and the average per claim severity is roughly $5,000. We simply use Poisson parameter $\lambda = 12 (= 5+7)$, $\beta_1 = 5$ and $\beta_2 = 15$ for illustrative purpose.
We can see from Figure 2 and 3 that the distributions of the discounted losses under the Esscher transform are translated to the right and have slightly heavier left tail and lighter right tail. This implies that the Esscher transform puts more weight on smaller extreme values. The second plot of Figure 3 compares the distributions of $S(1)$ and $S(5)$ under $P$. Again we can see that the left tail of $S(5)$ is heavier and right tail becomes thinner than those of $S(1)$. This is because of the effect of discounting. In other words, the discounted values of
claims that happen in the fifth year contribute less to the total aggregated loss than claims which arrive in the first year.

Table 2 gives the time zero prices of 5-year ratchet options for different choices of $\theta$ by several coverage levels $\pi$. We assume that the interest rate $r$ is 0.03.

<table>
<thead>
<tr>
<th>$\Theta$</th>
<th>$\pi$</th>
<th>$V(\pi; \theta)$</th>
<th>$\Theta$</th>
<th>$\pi$</th>
<th>$V(\pi; \theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>183.65</td>
<td>0.0</td>
<td>0.0</td>
<td>201.59</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>80.27</td>
<td></td>
<td>0.1</td>
<td>83.81</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
<td>40.29</td>
<td></td>
<td>0.2</td>
<td>40.62</td>
</tr>
<tr>
<td></td>
<td>0.3</td>
<td>22.84</td>
<td>0.3</td>
<td>0.3</td>
<td>22.36</td>
</tr>
<tr>
<td>0.1</td>
<td>0.4</td>
<td>14.02</td>
<td>0.4</td>
<td>0.4</td>
<td>13.34</td>
</tr>
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<td></td>
<td>0.5</td>
<td>9.11</td>
<td>0.5</td>
<td>0.5</td>
<td>8.42</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>6.18</td>
<td>0.6</td>
<td>0.6</td>
<td>5.56</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>4.35</td>
<td>0.7</td>
<td>0.7</td>
<td>3.81</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>3.15</td>
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<td>0.8</td>
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<tr>
<td>0.2</td>
<td>0.0</td>
<td>192.85</td>
<td>0.0</td>
<td>0.0</td>
<td>209.84</td>
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<tr>
<td></td>
<td>0.1</td>
<td>82.14</td>
<td>0.1</td>
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</tr>
<tr>
<td></td>
<td>0.2</td>
<td>40.49</td>
<td>0.2</td>
<td>0.2</td>
<td>40.69</td>
</tr>
<tr>
<td></td>
<td>0.3</td>
<td>22.61</td>
<td>0.3</td>
<td>0.3</td>
<td>22.09</td>
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<tr>
<td></td>
<td>0.4</td>
<td>13.68</td>
<td>0.4</td>
<td>0.4</td>
<td>13.00</td>
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<td>0.5</td>
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<td>0.5</td>
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<td>0.7</td>
<td>4.07</td>
<td>0.7</td>
<td>0.7</td>
<td>3.57</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>2.91</td>
<td>0.8</td>
<td>0.8</td>
<td>2.49</td>
</tr>
</tbody>
</table>

High risk adjustment parameter makes the option price large for the cases of moderate coverage levels ($\pi$ = 0.0, 0.1 or 0.2). On the contrary, the prices become smaller for extreme coverage cases. This is because of the fact that the left tail of the Esscher transformed distribution becomes heavier as the risk adjustment parameter gets larger.

Figure 4 shows that the option prices decrease rapidly as the coverage rate $\pi$ increases when $\theta = 0.2$ and $r = 0.03$. 
Swap spread depends on the trigger $\hat{q}_t$, which determines the level of insurance coverage of the swap contract. In this example we use percentiles of loss ratio process $q_t$ under the physical measure $P$.

**Table 3** Evolution of percentiles of loss ratio under $P$ ($\theta = 0.2$, $r = 0.03$, $\alpha = 0.1$)

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>yr1</th>
<th>yr2</th>
<th>yr3</th>
<th>yr4</th>
<th>yr5</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>0.42</td>
<td>0.51</td>
<td>0.56</td>
<td>0.58</td>
<td>0.60</td>
</tr>
<tr>
<td>20%</td>
<td>0.52</td>
<td>0.59</td>
<td>0.62</td>
<td>0.64</td>
<td>0.65</td>
</tr>
<tr>
<td>30%</td>
<td>0.59</td>
<td>0.65</td>
<td>0.67</td>
<td>0.68</td>
<td>0.69</td>
</tr>
<tr>
<td>40%</td>
<td>0.67</td>
<td>0.70</td>
<td>0.71</td>
<td>0.72</td>
<td>0.72</td>
</tr>
<tr>
<td>50%</td>
<td>0.73</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
</tr>
<tr>
<td>60%</td>
<td>0.81</td>
<td>0.80</td>
<td>0.79</td>
<td>0.79</td>
<td>0.78</td>
</tr>
<tr>
<td>70%</td>
<td>0.88</td>
<td>0.85</td>
<td>0.84</td>
<td>0.83</td>
<td>0.82</td>
</tr>
<tr>
<td>80%</td>
<td>0.98</td>
<td>0.92</td>
<td>0.89</td>
<td>0.87</td>
<td>0.86</td>
</tr>
</tbody>
</table>

Table 3 provides the percentiles of loss ratio process at the end of each year. We can see that the lower percentiles become larger and upper percentiles get smaller as time goes by. This is again because of the effect of discounting.

**Table 4** Swap spreads: $\theta = 0.2$, $r = 0.03$, $\alpha = 0.1$
Table 4 gives the 5-year swap spreads for each percentile based threshold \( \hat{q} \). Suppose, for example, that we choose the 80th percentile for trigger \( \hat{q} = 0.98 \). The protection seller should pay \( 1.025 \times 0.98 \times s \) to the protection buyer if the actual loss ratio is bigger than \( 1.025 \times 0.98 \). The protection buyer has to pay the amount when the opposite case happens.

As one can see from the spread formula (26), the fair spread has an inverse relationship with expense rate \( \alpha \) and trigger \( \hat{q} \). In general counterparties can agree to use arbitrary thresholds. For some cases, the spread can even be negative when the expense rate is chosen much higher than the ideal level which makes the spread equal to zero, i.e., \( \sum_{j=1}^{n} \alpha_{T_{j}} / \sum_{j=1}^{n} \hat{q}_{j} \alpha_{T_{j}} - 1 \).

When the P-percentile based thresholds are used as illustrated in this example, the effect of increase in expense rate is cancelled out due to the decrease in the threshold at the same rate. Thus the swap spreads remains unchanged regardless of the choice of expense rate. This suggests that P-percentile is a sensible choice for trigger \( \hat{q} \).

**7 Conclusions**

When an insurer decides to sell motor insurance contracts covering the losses from motor accidents it should consider appropriate models to estimate how much the expected amount of claims each year would be, based on its collected data. However, the insurer can be exposed to the risk of the actual loss rate being higher than expected. As a hedging method, we suggest the use of hybrid derivatives for motor insurance loss rate risk transfer. We consider a few motor insurance linked derivatives such as motor insurance loss rate options and swaps which can be traded over the counter in a capital market. They are designed not only to

<table>
<thead>
<tr>
<th>( \hat{q} )</th>
<th>10%tile</th>
<th>20%tile</th>
<th>30%tile</th>
<th>40%tile</th>
<th>50%tile</th>
<th>60%tile</th>
<th>70%tile</th>
<th>80%tile</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>0.616</td>
<td>0.458</td>
<td>0.358</td>
<td>0.280</td>
<td>0.213</td>
<td>0.152</td>
<td>0.090</td>
<td>0.025</td>
</tr>
</tbody>
</table>
provide the insurer with innovative new hedging methods for its loss rate risks but also to
give more investment choices to the potential investors in the financial market.
The insurer may want to exchange random cash flows in the future based on the outcome of
motor loss rates with pre-fixed values. In this case the insurance company would use motor
loss rate swaps. When an insurer is particularly interested in managing extreme loss rate risks,
a swap contract would be inappropriate. Alternative method considered is a motor loss rate
ratchet option which can be used to protect the motor insurance providers against the risks of
higher motor loss rates, without giving up the possible benefits of lower motor loss rates.
The pricing formulas of a ratchet option and swaps on motor insurance loss rates are given
under a few assumptions on aggregate loss process. We choose the Esscher transform to link
between insurance market and capital market. The risk neutral pricing formulas for the
ratchet option and swaps are obtained by using Fourier inversion and are expressed in integral
forms.
We show numerical examples using numerical integration and simulation methods to
illustrate the derivative prices and their characteristics when the claim amount distribution
follows a generalized Erlang distribution.

References

AXA Financial Protection, 2005. AXA launches the first securitization of a motor insurance
portfolio. Press Release, AXA Financial Protection, November 3, 2005. Available at:


premium, Astin Bulletin 10, 318-324.

and securitized risk transfer solutions. Available at: SSRN:
http://ssrn.com/abstract=1260399


Dufresne, D., Garrido, J., Morales, M., 2006. Fourier inversion formulas in option pricing
and insurance, Technical Report No. 06/06 December 2006. Concordia University.


