Securitization of Motor Insurance Loss Rate Risks

Taehan Bae† and Changki Kim*

Abstract

We try to transfer the loss rate risks in motor insurance to the capital market. We use the tranche technique to hedge the motor insurance risks. As an example, we focus on AXA and their securitization of French motor insurance in 2005. Though this application is new, this transaction is based on a concept similar to CDO. Tranches of bonds are constructed on the basis of the expected loss ratio from motor insurance policy holders' groups. As a consequence we develop motor loss rate bonds using the structure of synthetic CDOs. The coupon payments of each tranche depend on the level of the loss rates of the underlying motor insurance pool. We show an integral formula for a risk adjusted price of loss tranche contract where loss distribution is modelled with discounted compound Poisson process. Esscher transform is chosen for a risk adjusted measure and Fourier inversion method is used to calculate the price of the motor claim rate securities. The pricing methods of the tranches are illustrated, and possible suggestions to improve the pricing method or the design of these new securities follow.

Key words:
Securitization, Risk Transfer, Motor Insurance Loss Rates, CDOs.

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1 Introduction

- For motor insurance providers, future claims cannot be completely predicted. This risk, the mismatch of actual claims from those anticipated, is a significant one and must be managed.

- In 2005, AXA pioneered this strategy, selling EUR 200 million of bonds, as securitization of their motor insurance portfolio. Since the issue of the innovative motor insurance securities from AXA, motor securitization has been receiving attention considerably

- Securitization transfers risks to the capital markets, where there is greater capacity to absorb these risks compared to the reinsurance market.

- Motor insurance securitization also creates new investment opportunities, providing greater diversification to the traditional assets normally offered. Investors are given the freedom to choose among tranches of bonds with different risk ratings.

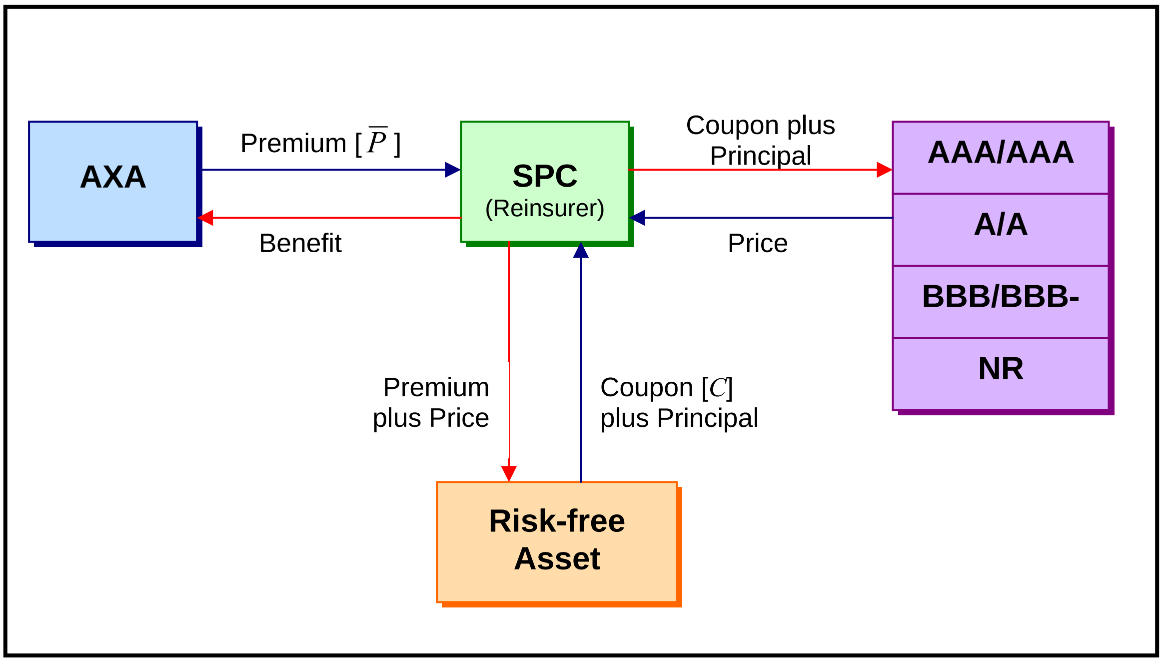
- We consider the structure of a synthetic collateralized debt obligation (CDO) for the securitization of motor insurance loss rates.

- We derive the pricing formulas for the securitization of motor insurance loss rate risks using CDO tranche pricing methods under a risk adjusted measure followed by numerical examples and discussions.

2 Characteristics of AXA Motor Insurance Securitization

- Prior to the discussion of the general motor insurance securitization we first consider the AXA motor securitization to illustrative the essential characteristics of this revolutionary system.

Figure 1: Simplified Structure of Overall Transaction



- AXA takes reinsurance of its individual motor insurance portfolio through a Special Purpose Company (SPC). AXA pays a premium to the SPC for the reinsurance service. The SPC issues several bonds in three tranches rated by AAA/AAA, A/A, BBB/BBB- and non-rated (NR) tranche. The price is transferred to investors as coupon payments plus the principal.

- The note holders are required to pay a price to the SPC in exchange for holding the bonds. The SPC invests the premium income that it receives from AXA and the proceeds that it receives from the note holders into a risk-free asset. From the risk-free asset, the SPC receives a coupon payment plus the principal. There is then some benefit that is received by AXA from the SPC. This contingent payment is measured by the difference between the real loss experienced and the expected loss where no benefit is received by AXA in the case this figure is negative.

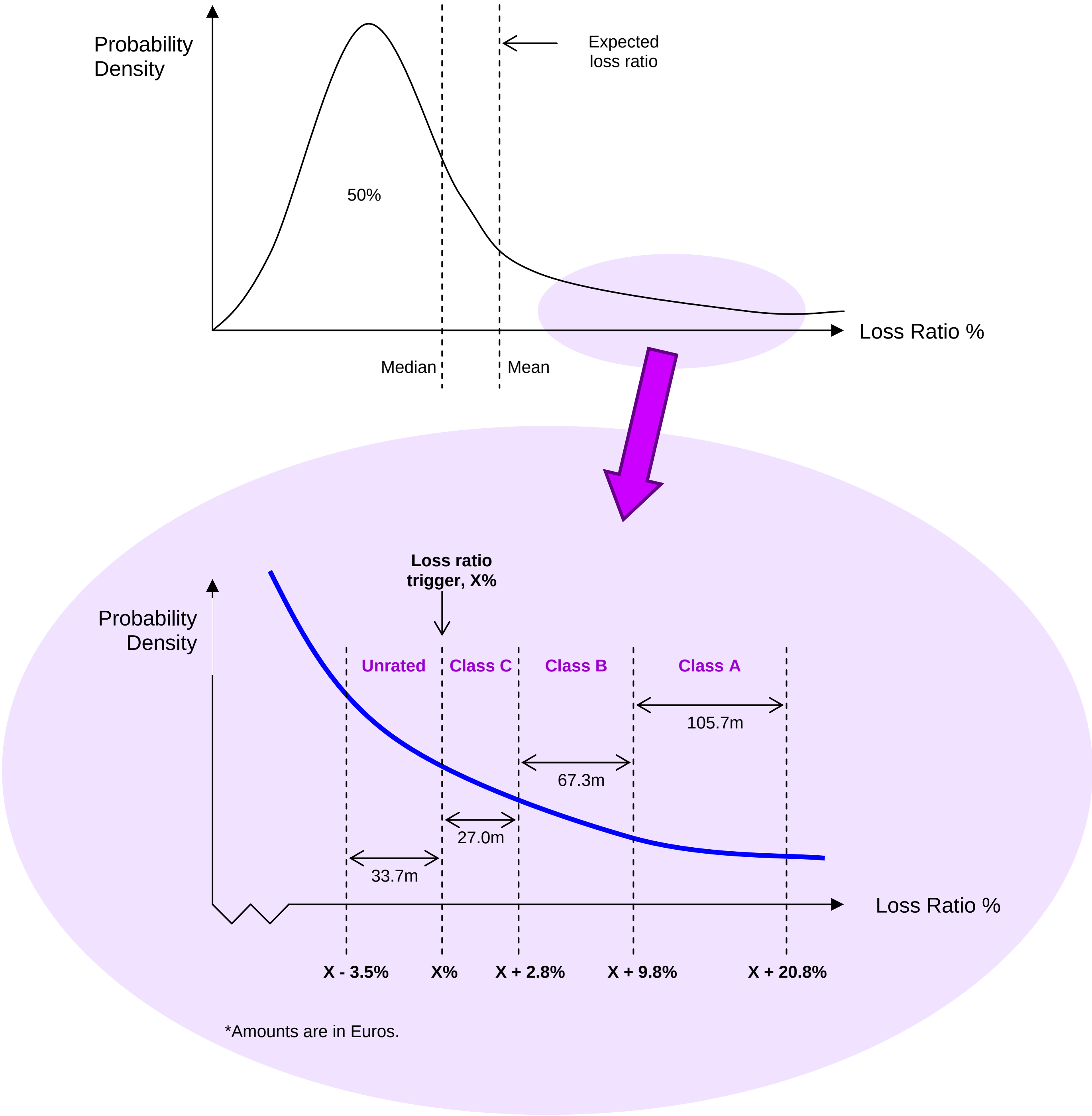
Table 1. Features of notes issued in the AXA Motor Securitization[1]

	Equity Tranche	C Notes	B Notes	A Notes
Amount	Euro 33.7 million	Euro 27.0 million	Euro 67.3 million	Euro 105.7 million
Rating (S&P/Fitch)	NR	BBB/BBB-	A/A	AAA/AAA
Risk Transfer Thresholds*	Loss ratio trigger –3.5%	Loss ratio trigger	Loss ratio trigger +2.8%	Loss ratio trigger +9.8%
Tranche Size*	3.5 points of loss ratio [-3.5%; 0%]	2.8 points of loss ratio [0%; +2.8%]	7.0 points of loss ratio [+2.8%; +9.8%]	11.0 points of loss ratio [+9.8%;+20.8%]

*2005 thresholds = Loss ratio trigger

[1] Features are from AXA Financial Protection (2005).

Figure 2. Note Tranching by Loss Ratio (AXA)



3 Benefits of Motor Insurance Securitization

A. The motor insurance company

- Securitization allows the motor insurance company to hedge their underlying loss risk, transferring it to investors.
- It is an additional source of funds for the party securitizing the risk, freeing up capital and improving the solvency of the company.
- This development can potentially diminish the expected loss rate assumed by the motor insurer, as investors holding the bonds have incentive to reduce loss for higher coupons.

B. The investors

- These new securities, with their higher coupon return compared to other general bonds, can be seen as a promising investment opportunity.
- The investors who are able to effectively decrease the company’s loss rate, have greater potential in maximising their return.
- Even the insurance policy holders who can get protection against motor accidents, still have at least two incentives to invest in this security: the insurance premium discounts next time and higher coupon payments by reducing the loss rates.

C. Government (as an investor alos).

- With this approach, the Government may gain greater initiative to focus on reducing motor hazards, by either imposing new laws/regulations or allocating a greater budget to related expenditure on, for example, free driving classes or public information on safe driving.

D. Other companies or organisations

- Those that are related to the motor business and motor insurance industry can be another group of investors who could potentially benefit in great amounts from this product.

E. Drive Safe and Get Higher Coupons!!!.

Also the general investors in the capital market who seek higher yield securities can be potential investors. In any case the basic principle is “drive safe and get higher coupons”.

4 Pricing Model for Securitization of Motor Insurance Risks

- Assumptions :
- Here we assume that the aggregate claims follow Compound Poisson distribution.

-We then derive the pricing formulas for the stop loss premiums that will be used for the pricing of the securitisation.

-We consider two pricing methods for the securitisation: the cumulative loss method and the periodic loss method.

4.1 Cumulative Loss Method

- For modelling the aggregate losses, we consider the losses accumulated from the issue date until maturity.

- Let N_t be a Poisson process with constant parameter λ. We assume an aggregate loss distribution follows a discounted compound Poisson process with risk free force of interest *r* defined on a probability space (Ω, F, P):

$$S_t = \int_{(0,t]}\int_{R^+} e^{-ru} xN(du,dx) = \sum_{i=1}^{N} e^{-rt} X_i \ , \qquad (1)$$

where T_i's are jump times of the Poisson process N_t and X_i's are i.i.d claim size distribution (non-negative) with pdf of $f_X(x)$. Note that *N(du,dx)* is the counting measure for the point process.

The following theorem can be shown by using standard machine in probability theory and the proof for the case of infinite time horizon can be found in Paulsen (1993).

Theorem 1 *For each $t \in (0, T]$ and every measurable set B such that $N((0, T] , B) < \infty$, and for any function f(u) and g(x) such that* $\int_{(0,T]}\int_B f(u)g(x)N(du,dx) < \infty$, *the following holds,*

$$E\left[\exp\left\{\int_{(0,T]}\int_B f(u)g(x)N(du,dx)\right\}\right] = \exp\left[\lambda\int_{(0,T]}\int_B \left(e^{f(u)g(x)} - 1\right)dF_X(x)du\right]. \qquad (2)$$

We need the Fourier transform of the distribution of S_t for the calculation of the stop loss premium and it is calculated using the above Theorem 1.

Corollary 2 *Let us denote the probability distribution function (pdf) of S_t by $f_S(x,t)$. The Fourier transform of the distribution of S_t for a given t is expressed as follows.*

$$\hat{f}_S(u,t) = \int_{R^+} e^{iux} f_S(x,t)dx = E[e^{iuS_t}]$$

$$= \exp\left\{\lambda\int_0^t \left(\hat{f}_X(ue^{-rs}) - 1\right)ds\right\}, \qquad (3)$$

where $\hat{f}_X(u)$ is the Fourier transform of the distribution of a claim size random variable X.

- Since we are working in an incomplete market, there are infinite numbers of risk neutral measures such that each choice of measure yields no arbitrage price of insurance risks.

- Esscher transform is one of the possible choices of such probability measure changes.

- Also it is known that for geometric Lévy process model Esscher transform is actually minimal entropy martingale measure and it is also the measure which maximizing the expected power utility function.[2]

- Risk adjustment parameter h in Esscher transform plays a key role to explain the attitude of market to the insurance risk and it can be obtained under the martingale condition.

- Specifically, we define a probability measure Q whose Radon-Nikodym derivative is

$$\frac{dQ}{dP}|_{F_t} = \frac{e^{h(t)S_t}}{E^P[e^{h(t)S_t}]}, \ 0 < t \le T, \tag{4}$$

equivalently,

$$f_S^Q(x,t) = \frac{e^{h(t)x}}{E^P[e^{h(t)S_t}]} f_S^P(x,t),$$

provided that $E^P[e^{h(t)S_t}]$ exists for all $0 < t \le T$. Note that *h(t)* is non-negative deterministic function which satisfies martingale condition described below.

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[2] See Gerber and Shiu (1994) or Miyahara and Fujiwara (2003) for details.

- The main goal of this study is associated with the calculation of a risk adjusted price of stop loss contract with retention level of *d*, in other words,

$$E^Q[(S_t - d)_+] = \int_d^\infty (x-d) f_S^Q(x,t)dx \tag{5}$$

- Direct application of Theorem 3.4 in Dufresne, Garrido and Morales (2006) gives the following:

$$E^Q[(S_t - d)_+] = \frac{E^Q[S_t]}{2} + \frac{1}{\pi} PV \int_0^\infty \mathrm{Re}\left[\frac{e^{-iud}(\hat{f}_S^Q(u,t) - 1)}{(iu)^2}\right] du \tag{6}$$

$$= \frac{1}{2\pi} PV \int_{-\infty}^\infty \frac{e^{-iud}[\hat{f}_S^Q(u,t) - 1 - iuE^Q[S_t]]}{(iu)^2} du \tag{7}$$

where PV $\int$ refers to Cauchy principle value integral.

- Then, for each t,

$$E^Q[S_t] = \int_0^\infty \frac{xe^{h(t)x}}{E^P[e^{h(t)S_t}]} f_S^P(x,t)dx = \frac{E^P[S_t e^{h(t)S_t}]}{E^P[e^{h(t)S_t}]} = \frac{\partial}{\partial h(t)}\{\log E^P[e^{h(t)S_t}]\}. \tag{8}$$

Since $E^P[e^{h(t)S_t}] = \hat{f}_S^P(-ih(t),t)$, the latter would be reduced to

$$E^Q[S_t] = \lambda \int_0^t \frac{\partial}{\partial h(t)} \hat{f}_X^P(-ih(t)e^{-rs})ds$$

$$= \frac{\lambda}{rh(t)}\left\{E^Q\left[e^{h(t)X}\right] - E^Q\left[e^{h(t)e^{-rt}X}\right]\right\} \tag{9}$$

- The second equality can be obtained by using Corollary 2 and changing order of integration and expectation. Also,

$$\hat{f}_S^Q(u,t) = \int_0^\infty e^{iux} \frac{e^{h(t)x}}{E^P[e^{h(t)S_t}]} f_S^P(x,t)dx = \frac{\hat{f}_S^P(u - ih(t),t)}{\hat{f}_S^P(-ih(t),t)}$$

$$= \exp\left\{\lambda \int_0^t \left\{\hat{f}_X^P((u - ih(t))e^{-rs}) - \hat{f}_X^P(-ih(t)e^{-rs})\right\} ds\right\} \tag{10}$$

By substituting (9) and (10) to (6) or (7), we have a risk adjusted price formula of stop loss contract.

- Under the assumption of no arbitrage between insurance market and capital market, the discounted surplus process should be a martingale under a risk neutral measure *Q*. Specifically we define the accumulated surplus process as follow,

$$U_t = u_0 e^{rt} + G e^{rt} \bar{a}_{\bar{t}|} - e^{rt} S_t , \qquad (11)$$

where u_0 is the initial surplus and G is the premium rate which is typically larger than $\lambda E^P[X]$.

- We can find an equivalent martingale measure *Q* that satisfies the following,

$$E^Q\left[e^{-rt} U_t \mid F_s\right] = e^{-rs} U_s , \ \ Q\text{-}a.s.$$

for any $0 \le s \le t \le T$. By (9), for each t, we can show that *h(t)* is the solution of the following equation which also satisfies the existence of Esscher transform (4),

$$E^Q\left[S_t\right] = \frac{\lambda}{rh(t)} \left\{ E^P\left[e^{h(t)X}\right] - E^P\left[e^{h(t)e^{-rt}X}\right] \right\} = G\bar{a}_{\bar{t}|} . \qquad (12)$$

Example 1. *Suppose X follows exponential distribution with parameter* β *such that* $E^P[X]=\beta$.

Then the followings holds:

$$\hat{f}_X^P(u)=(1-i\beta u)^{-1}$$

$$E^Q[S_t]=\frac{\lambda}{rh(t)}\{(1-\beta h(t))^{-1}-(1-\beta h(t)e^{-rt})^{-1}\}$$

$$=\frac{\lambda\beta\bar{a}_{\bar{t}|}}{(1-\beta h(t))(1-\beta h(t)e^{-rt})},\quad h(t)<\frac{1}{\beta}. \tag{13}$$

By solving (12) we can get the following explicit expression for h(t),

$$h(t)=\frac{1}{2\beta}\left(1+e^{rt}-\sqrt{(1+e^{rt})^2-4e^{rt}(1-\lambda\beta/G)}\right). \tag{14}$$

The Fourier transform of pdf of S_t *is*

$$\hat{f}_S^Q(u,t)=A(u)e^{iB(u)} \tag{15}$$

where

$$A(u)=\left(\frac{e^{rt}-\beta h(t)}{1-\beta h(t)}\right)^{-\frac{\lambda}{r}}\left(\frac{\beta^2h(t)^2-2\beta h(t)e^{rt}+e^{2rt}+\beta^2u^2}{\beta^2h(t)^2-2\beta h(t)+1+\beta^2u^2}\right)^{\frac{\lambda}{2r}}$$

and

$$B(u)=\frac{\lambda}{r}\left\{\arctan\left(\frac{\beta h(t)-1}{\beta u}\right)-\arctan\left(\frac{\beta h(t)-e^{rt}}{\beta u}\right)\right\}.$$

By substituting (13) and (15) into (6), we get the following formula,

$$E^Q[(S_t-d)_+]=\frac{\lambda}{2rh(t)}\{(1-\beta h(t))^{-1}-(1-\beta h(t)e^{-rt})^{-1}\}$$

$$+\frac{1}{\pi}\int_0^\infty\frac{1}{u^2}\{\cos(ud)-A(u)\cos(B(u)-ud)\}du \tag{16}$$

- The actual loss ratio is defined as the actual aggregate loss divided by the total premium over a period of time [0, t].

- The annual insurance premium G is measured using the mean value premium principle which is expressed as

$$G = (1+\theta)E[S_1] \tag{17}$$

where S_1: is the aggregate claim on [0,1], and θ is a loading factor.

- Let us denote the actual cumulative loss rate by q_t,

$$q_t = \frac{S_t}{G\bar{a}_{\bar{t}|}} = \frac{L_t}{G\bar{s}_{\bar{t}|}}, \tag{18}$$

where $L_t = S_t e^{rt}$ is the cumulative losses until time t, and $\bar{a}_{\bar{t}|}$ and $\bar{s}_{\bar{t}|}$ are the present value and the accumulated value of continuously paying annuities with unit of annual payment respectively.

- The estimated target loss ratio is denoted by $\hat{q}$.

The loss event (triggering) occurs in the case when the actual loss ratio exceeds the predetermined target loss ratio, $q_t > \hat{q}$. That is $q_t > \hat{q}$ implies that $L_t > \hat{q}\ G\ \bar{s}_{\bar{t}|}$ and loss event is triggered, and $q_t \leq \hat{q}$ implies no loss trigger

The cumulative excess loss amount at time t, $L^e{}_t$, can be described by multiplying the total premiums and the difference between the actual loss ratio and the target loss ratio, which can be expressed by the following formula,

$$L^e{}_t = \begin{cases} L_t - \hat{q}G\bar{s}_{\bar{t}|} = (q_t - \hat{q})G\bar{s}_{\bar{t}|}, & if \quad q_t > \hat{q} \\ 0, & otherwise \end{cases}. \tag{19}$$

- Now we design the securitization of motor insurance loss rates and consider the pricing method. We show the details of the tranches of the security in Table 2.

Table 2. Features of Tranches in Motor Insurance Securitisation

Tranches (j)	Equity Tranche (1)	C Notes (2)	B Notes (3)	A Notes (4)
Rating	NR	BBB/BBB-	A/A	AAA/AAA
Weight	$W^{(1)}$	$W^{(2)}$	$W^{(3)}$	$W^{(4)}$
Notional Amount	$N^{(1)} = N^{(T)}W^{(1)}$	$N^{(2)} =N^{(T)}W^{(2)}$	$N^{(3)} = N^{(T)}W^{(3)}$	$N^{(4)} =N^{(T)}W^{(4)}$
Spread on Incomes	$s^{(1)}$	$s^{(2)}$	$s^{(3)}$	$s^{(4)}$
Tranche Size	$[q^{(0)}, q^{(1)}]$	$[q^{(1)}, q^{(2)}]$	$[q^{(2)}, q^{(3)}]$	$[q^{(3)}, q^{(4)}]$

- The total notional amount is denoted by $N^{(T)}$ which should be determined by SPC. The factors that need to be considered for the tranches' weights $W^{(j)}$ (or equivalently the notional amounts of $N^{(j)}$) are the customers' preference, market reaction and the competition in the financial sector.

- However, this study explores the tranche weight only from the mathematical view point, assuming all the external factors are negligible.

- The spreads on the coupon rates can be determined according to the tranche sizes. Or we can find the tranche sizes give the spreads. Our pricing purpose is to determine the spreads or the tranche sizes.

The jth tranche loss amount $L_t^{(j)}$ is

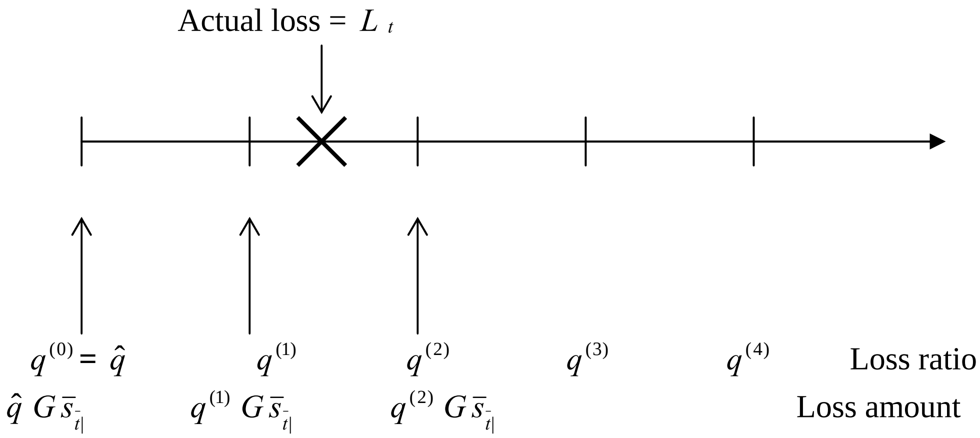
$$L_t^{(j)} = Min[(q^{(j)} - q^{(j-1)}) \cdot G\,\bar{s}_{\bar{t}|},\ Max\{0, L^e{}_t - (q^{(j-1)} - \hat{q}) \cdot G\,\bar{s}_{\bar{t}|}\}]$$

$$= (L_t - l_t^{(j-1)})_+ - (L_t - l_t^{(j)})_+ \qquad (20)$$

where $l_t^{(j)} = q^{(j)}\ G\,\bar{s}_{\bar{t}|}$, and $q^{(0)} = \hat{q}$ = the target loss ratio[3].

- For tranche 2 (C note), as an example, when actual cumulative loss occurs between $l_t^{(1)} = q^{(1)}\ G\,\bar{s}_{\bar{t}|}$ and $l_t^{(2)} = q^{(2)}\ G\,\bar{s}_{\bar{t}|}$, the holder of the equity tranche (tranche 1) will lose their whole equity and the C note holders will lose a portion of their coupons or notional amounts. See Figure 3.

Figure 3. Actual Losses in Tranche Modelling



- Let us denote the discounted value of the cumulative loss of a tranche j by $S_t^{(j)}$, with the discounted attachment point $d_t^{(j-1)} = e^{-rt} l_t^{(j-1)}$ and discounted detachment point $d_t^{(j)} = e^{-rt} l_t^{(j)}$ with $d_t^{(0)} = e^{-rt} l_t^{(0)} = \hat{q}\ G\,\bar{a}_{\bar{t}|}$,

$$S_t^{(j)} = L_t^{(j)} e^{-rt}.$$

[3] The target loss ratio can be different according to security design, i.e. $\hat{q} = q^{(j)}$ for some j>0.

Using the price formulas (6) or (7), we can calculate the risk adjusted price of $S_t^{(j)}$,

$$E^Q[S_t^{(j)}] = E^Q[(S_t - d_t^{(j-1)})_+] - E^Q[(S_t - d_t^{(j)})_+]. \tag{21}$$

Note that

$$d_t^{(j)} = q^{(j)}\, G\, \bar{a}_{\bar{t}|}, \text{ with } d_t^{(0)} = \hat{q}\, G\, \bar{a}_{\bar{t}|}. \tag{22}$$

- Let us consider the coupon payments for the investors in the jth tranche. If $S_t < d_t^{(j-1)}$ then the investors in the jth tranche experience no losses and their coupons are not affected. If $d_t^{(j-1)} < S_t < d_t^{(j)}$ then they should have some coupon (or principal) reductions. If $S_t > d_t^{(j)}$, then they receive no coupons.

- We want to find a spread[4], $s^{(j)}$, for the jth tranche such that the expectation, with respect to a risk adjusted probability measure *Q*, of the loss amounts for the jth tranche is equivalent to the expectation of the incomes of the tranche j investors,

$$E^Q\left[\int_0^T e^{-rt}\, dL_t^{(j)}\right] = E^Q\left[\int_0^T e^{-rt}\, P_t^{(j)}\, dt\right], \tag{23}$$

where $P_t^{(j)}$ is the income payment for the tranche j investors at time t.

- In this paper, we assume that the income payments are determined by the level of the loss amounts and that they are paid on discrete time periods like at the end of each year[5],

$$P_t^{(j)} = (l_t^{(j)} - l_t^{(j-1)} - L_t^{(j)})\, s^{(j)}. \tag{24}$$

The discrete version of the pricing formula is,

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[4] The spread can be varied according to time t, $s^{(j)} = s_t^{(j)}$, but we assume that it is a constant for each tranche j.

[5] We can adjust periodic coupon payments or coupon rates such as semi-annually or quarterly paid coupons.

$$E^Q\left[\sum_{t=1}^{T}(L_t^{(j)} - L_{t-1}^{(j)})e^{-rt}\right] = E^Q\left[\sum_{t=1}^{T}(l_t^{(j)} - l_t^{(j-1)} - L_t^{(j)})s^{(j)}e^{-rt}\right] = N^{(j)},$$

where $N^{(j)}$ is the notional amount of tranche j.

Equivalently we have,

$$E^Q\left[\sum_{t=1}^{T}(S_t^{(j)} - S_{t-1}^{(j)}e^{-r})\right] = E^Q\left[\sum_{t=1}^{T}(d_t^{(j)} - d_t^{(j-1)} - S_t^{(j)})s^{(j)}\right] \tag{25}$$

The spread, $s^{(j)}$, for the jth tranche is calculated by,

$$s^{(j)} = \frac{E^Q\left[\sum_{t=1}^{T}(S_t^{(j)} - S_{t-1}^{(j)}e^{-r})\right]}{E^Q\left[\sum_{t=1}^{T}(d_t^{(j)} - d_t^{(j-1)} - S_t^{(j)})\right]} \tag{26}$$

For a pre-fixed spread $s^{(j)}$ for jth tranche, the market price of the tranche is the difference between expected loss and expected premium payments.

$$V_0^{(j)} = E^Q\left[\sum_{t=1}^{T}(S_t^{(j)} - S_{t-1}^{(j)}e^{-r})\right] - s^{(j)}E^Q\left[\sum_{t=1}^{T}(d_t^{(j)} - d_t^{(j-1)} - S_t^{(j)})\right]. \tag{27}$$

We notice that the initial investment amount $N^{(j)}$ of tranche j can be affected according to the level of losses in the design of the above security.

- Some investors may want the initial investment at maturity. To satisfy these investors we design the security as follows.

The investors may want higher yield so we assume that the coupon rates have spreads over an interest rates of a very high quality (risk free) security.

$$N^{(j)} = E^Q\left[\sum_{t=1}^{T}(L_t^{(j)} - L_{t-1}^{(j)})e^{-rt}\right] = E^Q\left[\sum_{t=1}^{T}\tilde{C}_t^{(j)}e^{-rt} + N^{(j)}e^{-rT}\right] \tag{28}$$

where $\tilde{C}_t^{(j)}$ is the coupon for tranche j investors, which is a random amount, at time t.

- We assume that $\tilde{C}_t^{(j)}$ is inversely proportional to the tranche j loss amount and defined by,

$$\tilde{C}_t^{(j)} = (1-\Lambda_t^{(j)})N^{(j)}(R_t + s^{(j)}) \tag{29}$$

where R_t is the coupon rate of a high quality security (such as LIBOR), $s^{(j)}$ is the spread for jth tranche assumed to be a constant for each j, and

$$\Lambda_t^{(j)} = \frac{L_t^{(j)}}{l_t^{(j)} - l_t^{(j-1)}} = \frac{S_t^{(j)}}{d_t^{(j)} - d_t^{(j-1)}} \tag{30}$$

is the proportion of losses in tranche j with $0 \le \Lambda_t^{(j)} \le 1$.

Substituting (29) into (28) and then taking expectation with respect to risk adjusted probability measure Q, we have the formula for the spread of jth tranche,

$$s^{(j)} = \frac{1 - e^{-rT} - E^Q\left[\sum_{t=1}^{T}(1-\Lambda_t^{(j)})\, R_t\, e^{-rt}\right]}{E^Q[\sum_{t=1}^{T}(1-\Lambda_t^{(j)})e^{-rt}]}. \tag{31}$$

To find the spread we may need a model for the rate of a high quality security (such as LIBOR). When the coupon rate of a high quality security (such as LIBOR) is constant, $R_t = R$, the spread is expressed by

$$s^{(j)} = \frac{1 - e^{-rT}}{E^Q[\sum_{t=1}^{T}(1-\Lambda_t^{(j)})e^{-rt}]} - R .$$

That is, the spread decreases when the coupon rate of a high quality security increases and it will be negative if R is very large compared to the risk free rate r.

If we calculate the coupons by defining

$$\tilde{C}_t^{(j)} = (1-\Lambda_t^{(j)})N^{(j)}s^{(j)}, \tag{32}$$

then the coupon rate $s^{(j)}$ is expressed as follows,

$$s^{(j)} = \frac{1-e^{-rT}}{E^Q[\sum_{t=1}^{T}(1-\Lambda_t^{(j)})e^{-rt}]}. \tag{33}$$

Since the coupon rate of a high quality security (such as LIBOR) can be random, by removing it from the formula, we don't need to worry about the term structure of the rates.

4.2 Periodic Loss Method

- The cumulative loss method discussed above may not appeal to some investors because of a few demerits.

- For example, an initial start with large losses can establish whether investors will not receive any of their coupons after a certain time, before the maturity date.

- Also we can design the security such that the note holders are guaranteed their principal at maturity. This means that the maximum loss each period is the coupon amount only and the initially invested notional amount is not affected.

- The actual loss ratio, $q_{t-1,t}$, is defined as the actual aggregate loss divided by the total premium over a period of time [t-1, t] and the target loss ratio, $\hat{q}$, is predetermined by SPC. The annual insurance premium, $G_{t-1,t}$, over a period of time [t-1, t] is calculated by

$$G_{t-1,t} = (1+\theta)E[L_{t-1,t}] \tag{34}$$

where $L_{t-1,t}$ is the aggregate claim on [*t*-1, *t*], i.e. periodically based aggregate claim amount, and θ is a loading factor.

- The actual loss ratio on [t-1,t] is

$$q_{t-1,t} = = \frac{L_{t-1,t}}{G_{t-1,t}}. \tag{35}$$

-The loss event (triggering) occurs in the case when the actual loss ratio exceeds the predetermined target loss ratio: $q_{t-1,t} > \hat{q}$ implies that $L_{t-1,t} > \hat{q}\ G_{t-1,t}$ and the loss event is triggered. $q_{t-1,t} \leq \hat{q}$ implies no loss trigger.

The excess loss amount, $L^e{}_{t-1,t}$, on [t-1,t] can be described by multiplying the annual premium and the difference between the actual loss ratio and the target loss ratio,

$$L^e{}_{t-1,t} = \begin{Bmatrix} (q_{t-1,t} - \hat{q})G_{t-1,t} & if \quad q_{t-1,t} > \hat{q} \\ 0, & otherwise \end{Bmatrix}. \tag{36}$$

We want to find the spreads or the tranche sizes based on the periodic losses.

The jth tranche loss amount $L^{(j)}_{t-1,t}$ is

$$\begin{aligned} L^{(j)}_{t-1,t} &= Min[(q^{(j)} - q^{(j-1)}) \cdot G_{t-1,t}, Max\{0, L^e{}_{t-1,t} - (q^{(j-1)} - \hat{q}) \cdot G_{t-1,t}\}] \\ &= (L_{t-1,t} - l^{(j-1)}_{t-1,t})_+ - (L_{t-1,t} - l^{(j)}_{t-1,t})_+ \end{aligned} \tag{37}$$

where $l^{(j)}_{t-1,t} = q^{(j)} \cdot G_{t-1,t}$, with $l^{(0)}_{t-1,t} = \hat{q} \cdot G_{t-1,t}$.

Assume that the notional amount of tranche j, $N^{(j)}$, is invested into a high quality (risk free) security. The notional amount of tranche j should be equal to the summation of the discounted coupons plus the discounted face value (notional amount) of the jth tranche.

$$N^{(j)} = E^Q\left[\sum_{t=1}^{T} C_t e^{-rt} + N^{(j)} e^{-rT}\right] \tag{38}$$

$$= E^Q\left[\sum_{t=1}^{T} \tilde{C}^{(j)}_t e^{-rt} + N^{(j)} e^{-rT}\right], \tag{39}$$

where C_t is the coupon received from high quality security at time t, and $\tilde{C}_t^{(j)}$ is the coupon for tranche j investors, which is a random amount, at time t.

We assume that the coupon payments for tranche j paid in the interval [t-1, t] depend on the tranche j loss amount and defined by,

$$\tilde{C}_t^{(j)} = \frac{l_{t-1,t}^{(j)} - l_{t-1,t}^{(j-1)} - L_{t-1,t}^{(j)}}{l_{t-1,t}^{(j)} - l_{t-1,t}^{(j-1)}} N^{(j)}(R_t + s^{(j)})$$

$$= (1 - \Lambda_{t-1,t}^{(j)}) N^{(j)}(R_t + s^{(j)}) \qquad (40)$$

where R_t is the coupon rate of the high quality security, $s^{(j)}$ is the spread for jth tranche assumed to be a constant for each j, and

$\Lambda_{t-1,t}^{(j)} = \frac{L_{t-1,t}^{(j)}}{l_{t-1,t}^{(j)} - l_{t-1,t}^{(j-1)}}$ is the proportion of losses in tranche j with $0 \le \Lambda_{t-1,t}^{(j)} \le 1$.

From (40), we can notice that the maximum coupon amount is $N^{(j)}(R_t + s^{(j)})$ and the minimum coupon amount is 0.

Substituting (40) into (39) produces

$$N^{(j)} = \sum_{t=1}^{T} (1 - \Lambda_{t-1,t}^{(j)}) N^{(j)}(R_t + s^{(j)}) e^{-rt} + N^{(j)} e^{-rT} \ . \qquad (41)$$

Now, it is possible to determine spread of jth tranche ,

$$s^{(j)} = \frac{1 - e^{-rT} * E^Q \left\lfloor \sum_{t=1}^{T} (1 - \Lambda_{t-1,t}^{(j)}) R_t e^{-rt} \right\rfloor}{E^Q [\sum_{t=1}^{T} (1 - \Lambda_{t-1,t}^{(j)}) e^{-rt}]} . \qquad (42)$$

Note that the spread $s^{(j)}$ is the actual spread over the coupon rate R_t (such as LIBOR) of a high quality (risk free) security, which can be negative. And we may need the term structure of R_t. So we may define the coupons by

$$\tilde{C}_t^{(j)} = (1-\Lambda_{t-1,t}^{(j)})N^{(j)}s^{(j)}, \tag{43}$$

then the coupon rate $s^{(j)}$ is expressed by,

$$s^{(j)} = \frac{1-e^{-rT}}{E^Q[\sum_{t=1}^{T}(1-\Lambda_{t-1,t}^{(j)})e^{-rt}]}. \tag{44}$$

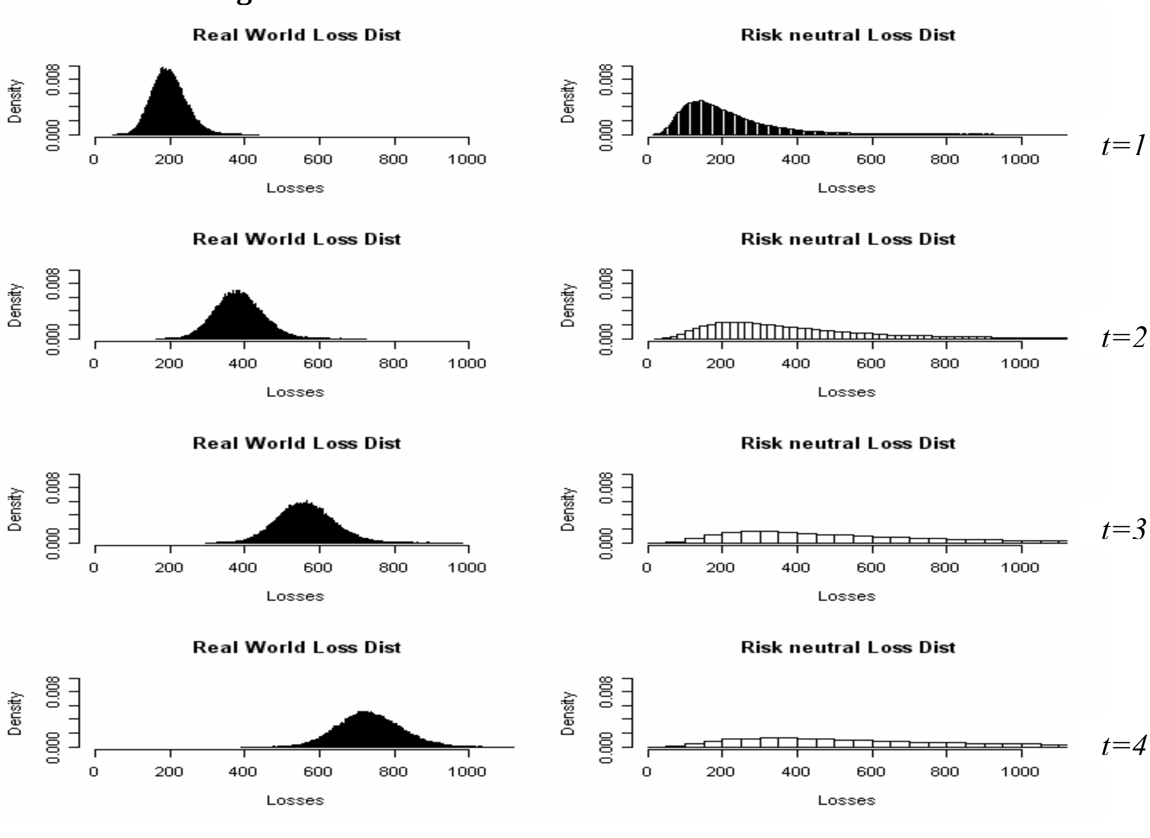
5 Numerical Examples [6]

Here we show numerical examples under the specified assumptions on the distribution of the discounted loss and its parameters. The results may vary on the changes of the assumptions.

Model assumption : Compound Poisson / Exponential

We assume the Poisson parameter $\lambda=40$, $E^P[X]=\beta=5$, security loading $\theta=0.1$, the continuously compounding risk free interest rate $r=0.045$, and maturity $T=4$. Then the constant premium rate $G=(1+\theta)E^P[S_1]=215.123$. The following figure shows evolution of discounted loss distribution S_t over time and compares two densities $f_S^P(x,t)$ and $f_S^Q(x,t)$.[7]

Figure 4. Evolution of Loss Distribution over Time



[6] Numerical integration is implemented by Mathematica 6.0. and we use R for simulation.
[7] Obtained by 100,000 simulation in R.

- We can see that the distributions of discounted loss under Esscher transform have heavy tails on both side which means market put more weight on extreme values.

- The following table contains summary statistics of loss ration distribution, $q_t = S_t / G\bar{a}_{\bar{t}|}$, by which we will define loss ratio trigger and tranche sizes.

Table 3. Summary Statistics of Loss Ration Distribution

		Min	**1st qu.**	**Median**	**Mean**	**3rd qu.**	**Max.**
T=1	Under P	0.2299	0.7838	**0.9178**	0.9296	1.0620	2.0780
	Under Q	0.0742	0.5939	**0.8547**	0.9998	1.2340	11.5400
T=2	Under P	0.3999	0.8285	**0.9246**	0.9301	1.0260	1.7610
	Under Q	0.0704	0.5452	**0.8178**	1.0000	1.2400	20.4200
T=3	Under P	0.4861	0.8471	**0.9259**	0.9300	1.0090	1.6280
	Under Q	0.0535	0.4925	**0.7745**	1.0010	1.2350	34.7200
T=4	Under P	0.4943	0.8577	**0.9262**	0.9292	0.9979	1.4310
	Under Q	0.0277	0.4472	**0.7355**	0.9994	1.2310	25.3200

Note that the mean of q_t under risk neutral measure is set to be 1.[8]

[8] The errors in the table are due to the simulations.

- It seems to be reasonable to consider selling loss risks in between median to the 3rd quartile and the following table gives details on the order statistics for given percentiles.

Table 4. Order Statistics for given Percentiles

		45%-tile	50%-tile	55%-tile	60%-tile	75%-tile	
T=1	Under P	**0.8918**	**0.9178**	**0.9442**	**0.9709**	**1.0620**	(***)
	Under Q	**0.7979**	**0.8547**	**0.9157**	**0.9811**	**1.2340**	(****)
T=2	Under P	0.9063	0.9246	0.9429	0.9620	1.0260	
	Under Q	0.7576	0.8178	0.8823	0.9546	1.2400	
T=3	Under P	0.9107	0.9259	0.9413	0.9567	1.0090	
	Under Q	0.7098	0.7745	0.8452	0.9224	1.2350	
T=4	Under P	**0.9133**	**0.9262**	**0.9392**	**0.9528**	**0.9979**	(*)
	Under Q	**0.6700**	**0.7355**	**0.8077**	**0.8911**	**1.2310**	(**)

We consider four tranches where tranches are defined by the percentiles of loss ratio distribution at time *T=4* for both under measure *P* and *Q*.

Table 5. Spreads under the Cumulative Loss Method (based on formula (26))

Tranches (j)		(1)	(2)	(3)	(4)
(*)	$[p^{(j-1)}, p^{(j)}]$	[0.9133,0.9262]	[0.9262, 0.9392]	[0.9392, 0.9528]	[0.9528, 0.9979]
	$s^{(j)}$	1.1412	0.9748	0.8037	0.5495
(**)	$[q^{(j-1)}, q^{(j)}]$	[0.6700,0.7355]	[0.7355, 0.8077]	[0.8077, 0.8911]	[0.8911, 1.2310]
	$s^{(j)}$	27.2692	9.0511	2.8925	0.2225

We consider four tranches where tranches are defined by the percentiles of loss ratio distribution at time *T=4* for both under measure *P* and *Q*.

Table 6. Spreads under the Cumulative Loss Method (based on formula (31))

Tranches (j)		(1)	(2)	(3)	(4)
(*)	$[p^{(j-1)}, p^{(j)}]$	[0.9133,0.9262]	[0.9262, 0.9392]	[0.9392, 0.9528]	[0.9528, 0.9979]
	$s^{(j)}$	0.001241	0.001231	0.001226	0.001190
(**)	$[q^{(j-1)}, q^{(j)}]$	[0.6700,0.7355]	[0.7355, 0.8077]	[0.8077, 0.8911]	[0.8911, 1.2310]
	$s^{(j)}$	0.003088	0.003008	0.002893	0.002722

We assume the 1yr-LIBOR rate=0.045. Tranches are defined by the percentiles of loss ratio distribution at time *T=1* for both under measure *P* and *Q*.

Table 7. Spreads under the Periodic Loss Method (based on formula (42))

Tranches (j)		(1)	(2)	(3)	(4)
(***)	$[p^{(j-1)}, p^{(j)}]$	[0.8918,0.9178]	[0.9178, 0.9442]	[0.9442, 0.9709]	[0.9709, 1.0620]
	$s^{(j)}$	0.001453	0.001414	0.001384	0.001252
(****)	$[q^{(j-1)}, q^{(j)}]$	[0.7979,0.8547]	[0.8547, 0.9157]	[0.9157, 0.9811]	[0.9811, 1.2340]
	$s^{(j)}$	0.002266	0.002114	0.001933	0.001874

6 Conclusion

- Motor insurance securitization is ground breaking. It is an alternative channel for motor insurers, who once were limited to only traditional reinsurance methods to manage risk. Although motor insurance companies, such as AXA, have recognised the potential of securitization as a risk management tool, that role can be further enhanced through innovative private placements.

- From the two possible pricing methods discussed, it is more desirable to model the loss ratio periodically as opposed to using the cumulative method. Both of these methods are analytically correct, though when the bond is considered from the investors' point of view, the periodic method is best. This is because when a substantial loss is incurred in any given period the bond holder will either forfeit or sell their contract in the following period. As a consequence of this, there is a possibility of the securitization structure collapsing. T

Motor insurance securitization is a very new development and is currently in its experimental phase. At the present time, there is an incomplete market for these new securities due to the rarity of this transaction. Hence, there are concerns about the liquidity of these securities which can be improved by multiple transactions, therefore broadening the investor base. Although, if this trend of using securitization as a method of motor insurance risk transfer continues to move forward in the future, these problems should improve.

Drive safe and get more coupons !!!

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Questions??

Thank you very much!!!