BUY AND HOLD STRATEGIES IN
OPTIMAL PORTFOLIO SELECTION PROBLEMS:
COMONOTONIC APPROXIMATIONS

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MULTIPERIOD OPTIMAL PORTFOLIO SELECTION PROBLEM

1. Investment strategies.

2. Buy and hold strategy. Terminal wealth.

3. Upper and lower bounds for the terminal wealth.

4. Optimal portfolio.
1. INVESTMENT STRATEGIES

There are \((m+1)\) securities. One of them is riskfree: \(\frac{dP^0(t)}{P^0(t)} = r dt\).

There are \(m\) risky assets: 
\[
\frac{dP^i(t)}{P^i(t)} = \mu_i dt + \sum_{j=1}^{d} \sigma_{ij} dW^j(t).
\]

By defining \(B^i(t) = \frac{1}{\sigma_i} \sum_{j=1}^{d} \bar{\sigma}_{ij} W^j(t)\),
\[
\frac{dP^i(t)}{P^i(t)} = \mu_i dt + \sigma_i dB^i(t) , \quad i = 1, \ldots, m.
\]
From the solution to this equation,

\[ P^i(t) = p_i \exp \left[ \left( \mu_i - \frac{1}{2} \sigma_i^2 \right) t + \sigma_i B^i(t) \right], \]

we obtain that the random yearly returns of asset \( i \) in year \( k \), \( Y^i_k \), are independent and have identical normal distributions with

\[
\begin{align*}
E[Y^i_k] &= \mu_i - \frac{1}{2} \sigma_i^2, \\
\text{Var}[Y^i_k] &= \sigma_i^2, \text{ and} \\
\text{Cov}[Y^i_k, Y^j_l] &= \begin{cases} 0 & \text{if } k \neq l, \\ \sigma_{ij} & \text{if } k = l. \end{cases}
\end{align*}
\]
Let $\Pi(t) = (\Pi_0(t), \Pi_1(t), \ldots, \Pi_m(t))$ denote the vector describing the proportions of wealth invested in each asset at time $t$.

In general, a vector $\Pi(t)$ will define an investment strategy.

If one unit of a security is constructed according to the investment strategy $\Pi(t)$, let $P(t)$ be the price of that unit at time $t$. Then,

$$\frac{dP(t)}{P(t)} = \sum_{i=0}^{m} \Pi_i(t) \frac{dP^i(t)}{P^i(t)} = \left[ \sum_{i=1}^{m} \Pi_i(t) (\mu_i - r) + r \right] dt + \sum_{i=1}^{m} \Pi_i(t) \sigma_i dB^i(t).$$

If $\Pi(t)$ is prefixed, $P(t)$ can be obtained by solving the stochastic differential equation above (constantly rebalanced portfolio).
2. BUY AND HOLD STRATEGY. TERMINAL WEALTH

• The new amounts of money $\alpha(t)$ are invested at time $t = 0, 1, \ldots, n - 1$ in some prefixed proportions $\bar{\pi}(t) = (\bar{\pi}_0(t), \bar{\pi}_1(t), \ldots, \bar{\pi}_m(t))$.

• Fractions $\bar{\pi}_i(t)$ are always the same. Denoting $\bar{\pi}_i(0) = \pi_i$, then $(\bar{\pi}_0(t), \ldots, \bar{\pi}_m(t)) = (\pi_0, \ldots, \pi_m)$, for every $t = 0, 1, \ldots, n - 1$.

• New quantities are invested once in a period of time (typically, once in a year), i.e.,

$$\alpha(t) = \begin{cases} 
\alpha_i & \text{if } t = i, \text{ for } i = 0, 1, \ldots, n - 1, \\
0 & \text{otherwise}.
\end{cases}$$

• The decision maker follows a buy and hold strategy, i.e., no securities are sold.
Objective: To compute the terminal wealth $W_n(\pi)$ for a given buy and hold strategy $\pi = (\pi_0, \pi_1, \ldots, \pi_m)$.

Let $Z^i_j$ be the sum of returns of 1 unit of capital invested at time $t = j$ of asset $i$ from time $t = j$ to the final time $t = n$,

$$Z^i_j = \sum_{k=j+1}^{n} Y^i_k.$$

The terminal wealth invested in asset $i$ is

$$W^i_n(\pi) = \sum_{j=0}^{n-1} \pi_i \alpha_j e^{Z^i_j},$$

whereas the terminal wealth will be given by

$$W(\pi) = \sum_{i=0}^{m} W^i(\pi) = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \alpha_j e^{Z^i_j}.$$
3. UPPER AND LOWER BOUNDS FOR THE TERMINAL WEALTH

Let $X = (X_1, X_2, \ldots, X_n)$ and let $S = X_1 + X_2 + \cdots + X_n$. It can be shown that

$$S^l \leq_{cx} \sum_{i=1}^{n} X_i \leq_{cx} S^c,$$

where $S^c = \sum_{i=1}^{n} F_{X_i}^{-1}(U)$ and $S^l = \sum_{i=1}^{n} E[X_i | \Lambda]$. 
If \( S = \sum_{i=1}^{n} \bar{\alpha}_i e^{\bar{Z}_i} \) with \( \bar{\alpha}_i \geq 0 \),

\[
S^c = \sum_{i=1}^{n} F^{-1}_{\bar{\alpha}_i e^{\bar{Z}_i}}(U) = \sum_{i=1}^{n} \bar{\alpha}_i e^{E[\bar{Z}_i] + \sigma_{\bar{Z}_i} \Phi^{-1}(U)}.
\]

For a given \( \Lambda = \sum_{j=1}^{n} \gamma_j \bar{Z}_j \),

\[
S^l = \sum_{i=1}^{n} \bar{\alpha}_i E[e^{\bar{Z}_i} | \Lambda] = \sum_{i=1}^{n} \bar{\alpha}_i e^{E[\bar{Z}_i] + \frac{1}{2} (1 - r_i^2) \sigma_{\bar{Z}_i}^2 + r_i \sigma_{\bar{Z}_i} \Phi^{-1}(U)}.
\]

We need values of \( \gamma_j \) that minimize of the “distance” between \( S \) and \( S^l \).
'Maximal Variance' lower bound approach. As we have that $\text{Var}[S] = \text{Var}[S^l] + \mathbb{E}[	ext{Var}[S | \Lambda]]$, it seems reasonable to choose the coefficients $\gamma_j$ such that the variance of $S^l$ is maximized:

$$\gamma_k = \bar{\alpha}_k e^{E[\bar{Z}_k]} + \frac{1}{2} \sigma^2_{\bar{Z}_k}.$$ 

'Taylor-based' lower bound approach. $\Lambda$ is a linear transformation of a first order approximation to $S$:

$$\gamma_k = \bar{\alpha}_k e^{E[\bar{Z}_k]}.$$
Comonotonic Upper Bound B&H strategy:

\[ W^c(\pi) = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \alpha_j e^{(n-j)(\mu_i - \frac{1}{2} \sigma_i^2)} + \sqrt{n-j} \sigma_i \Phi^{-1}(U). \]

Note that \( W^c(\pi) \) is a linear combination of fractions \( \pi_i \), \( i = 0, \ldots, m \).

Comonotonic Lower Bound B&H strategy:

\[ W^l(\pi) = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \alpha_j e^{(n-j)(\mu_i - \frac{1}{2} r_{ij}^2 \sigma_i^2)} + r_{ij} \sqrt{n-j} \sigma_i \Phi^{-1}(U) \]

where the correlation coefficients \( r_{ij} \) are given by

\[ r^M_{ij} = \frac{\sum_{k=0}^{m} \sum_{l=0}^{n-1} \pi_k \alpha_l (n - \max(j, l)) \sigma_{ik} e^{(n-l)\mu_k} \sigma_i \left[ (n-j) \sum_{s,k=0}^{m} \sum_{t,l=0}^{n-1} \pi_s \pi_k \alpha_t \alpha_l (n - \max(t, l)) \sigma_{sk} e^{(n-t)\mu_s + (n-l)\mu_k} \right]^{1/2}}{\sigma_i \left[ (n-j) \sum_{s,k=0}^{m} \sum_{t,l=0}^{n-1} \pi_s \pi_k \alpha_t \alpha_l (n - \max(t, l)) \sigma_{sk} e^{(n-t)\mu_s + (n-l)\mu_k} \right]^{1/2}}. \]
and

\[ r_{ij}^T = \frac{\sum_{k=0}^{m} \sum_{l=0}^{n-1} \pi_k \alpha_l (n - \max(j, l)) \sigma_{ik} e^{(n-l)[\mu_k - \frac{1}{2} \sigma_k^2]}}{\sigma_i (n - j)^{1/2}} \cdot \left[ \sum_{s,k=0}^{m} \sum_{t,l=0}^{n-1} \pi_s \pi_k \alpha_t \alpha_l (n - \max(t, l)) \sigma_{sk} e^{(n-t)[\mu_s - \frac{1}{2} \sigma_s^2] + (n-l)[\mu_k - \frac{1}{2} \sigma_k^2]} \right]^{-1/2} \]
**Numerical illustration:** 2 risky, 1 risk-free. $\mu_1 = 0.06$, $\mu_2 = 0.1$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, Pearson’s correlation: 0.5, $r = 0.03$.

Every period $\alpha_i = 1$, invested in proportions: 19% risk-free asset, 45% first risky asset, 36% in the second risky asset. This amount is invested for $i = 0, \ldots, 19$, whereas in $i = 20$ the invested amount is $\alpha_{20} = 0$. The simulated results were obtained with 500,000 random paths.
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4. OPTIMAL PORTFOLIO

Possible criteria: maximizing an expected utility, Yaari’s dual theory of choice under risk:

$$\max_{\pi} \rho_f [W_n(\pi)] = \max_{\pi} \int_0^\infty f(\Pr(W_n(\pi) > x))dx,$$

risk measures (some of them correspond to distorted expectations $\rho_f [W_n(\pi)]$ for appropriate choices of the distortion function $f$).
Value at Risk at level $p$:

$Q_p[X] = F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}$. If $F_X$ is an strictly increasing function, then it coincides with the related risk measure $Q_p^+[X] = \sup\{x \in \mathbb{R} \mid F_X(x) \leq p\}$, $p \in (0,1)$.

Additive for sums of comonotonic risks.

Conditional Left Tail Expectation at level $p$ ($CLTE_p[X]$):

$$CLTE_p[X] = E \left[ X \mid X < Q_p^+[X] \right] , \quad p \in (0,1) .$$
For the upper and lower bounds in B&H strategy:

\[
Q_p[W^c(\pi)] = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \alpha_j e^{(n-j)(\mu_i - \frac{1}{2} \sigma_i^2) + \sqrt{n-j} \sigma_i \Phi^{-1}(p)},
\]

\[
Q_p[W^l(\pi)] = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \alpha_j e^{(n-j)(\mu_i - \frac{1}{2} r_{ij}^2 \sigma_i^2) + r_{ij} \sqrt{n-j} \sigma_i \Phi^{-1}(p)},
\]

\[
CLTE_p[W^c(\pi)] = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \alpha_j e^{\mu_i(n-j)} \frac{1 - \Phi(\sqrt{n-j} \sigma_i - \Phi^{-1}(p))}{p},
\]

\[
CLTE_p[W^l(\pi)] = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \alpha_j e^{\mu_i(n-j)} \frac{1 - \Phi(\sqrt{n-j} r_{ij} \sigma_i - \Phi^{-1}(p))}{p}.
\]
Maximizing the Value at Risk: for a given probability $p$ and a given investment strategy, let $K_p(\pi)$ be the $p$-target capital defined as the $(1 - p)$-th order ‘+’-quantile of terminal wealth,

$K_p(\pi) = Q_{1-p}^+ [W(\pi)]$.

For the optimal case:

$$K^*_p = \max_{\pi} Q_{1-p}^+ [W(\pi)].$$

Alternatives:

$$K^c_p = \max_{\pi} Q_{1-p}^+ [W^c(\pi)]$$

or

$$K^l_p = \max_{\pi} Q_{1-p}^+ [W^l(\pi)].$$
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Buy & Hold Strategies  Comonotonic Approximations
Maximizing the CLTE: \( \max_{\pi} CLTE_{1-p}[W(\pi)] \). This optimization problem describes decisions of risk averse investors.

The \( CLTE_{1-p} \) has the following nice property (lacking with the VaR):

\[
CLTE_{1-p}[W^c(\pi)] \leq CLTE_{1-p}[W(\pi)] \leq CLTE_{1-p}[W^l(\pi)].
\]

Alternative: \( \max_{\pi} CLTE_{1-p}[W^l(\pi)] \).
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Buy & Hold Strategies

Ccomonotonic Approximations