Pricing in the Multi-Line Insurer with Dependent Gamma Distributed Risks allowing for Frictional Costs of Capital

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Aims of Research

• Develop a Pricing Model for a Multi-Line Insurer including frictional costs and default option value for specified model for risks.

• Implement a recently developed dependent Gamma model for lines of business.

• Develop approximations and closed form expressions for implementation.
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Introduction - Insurance Pricing

- Insurance pricing

\[ PV(\text{losses, assuming no default}) + PV(\text{surplus costs}) - PV(\text{default option}) \]


- \( PV(\text{losses, assuming no default}) \) not impacted by capital structure or frictional costs but reflects the insurance loss price of risk; \( PV(\text{surplus costs}) \) are the frictional costs - reflects capital structure - how to determine and how to allocate to line of business/policy?; \( PV(\text{default option}) \) reflects capital structure, analogy to corporate bonds, \( Q \) probabilities not \( P \) probabilities, how to allocate to line of business?
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Model for Insurer

• Insurer with portfolio of $n$ distinct insurable risks written at the beginning of the period.

• Premium $P_i$ for line of business $i$. $P_i$ allows for the expected losses, the risk loading, the risk of claims not being met due to the insolvency of the insurer and any costs of capital to be allocated to policyholders. Total premiums collected at the start of the period are $P = \sum_{i=1}^{n} P_i$.

• End of period claims for the $i^{th}$ risk denoted by $L_i$ and total claims at the end of the period $L = \sum_{i=1}^{n} L_i$. 
Model for Insurer

- Arbitrage-free model where there exists a probability measure $Q$ such that the current values of the assets and liabilities are the discounted present value of end of period random payments using a risk free discount rate.

- There exists a risk free asset such that an investment of 1 now will return $e^r$ at the end of the period for certain.
• $L_{0i}$ denotes the time 0 price or fair value for line of business $i$ ignoring default option given by 
\[ L_{0i} = E^Q \left[ e^{-r} L_i \right] \quad \text{for all } i = 1, \ldots, n \]
and total value of the initial liabilities by 
\[ L_0 = \sum_{i=1}^{n} L_{0i}. \]

• $V_0$ denotes the time 0 price or fair value for the assets given by 
\[ V_0 = E^Q \left[ e^{-r} V \right] \]
where $r$ is the risk-free continuous compounding rate of interest.

• Complete markets so that we observe $L_{0i}$ for all $i = 1, \ldots, n$, from the insurance market and $V_0$ from an asset market.
• Assuming equal priority for losses by line of business in the event of insolvency actual loss payments will be

\[
\begin{align*}
\frac{L_i V}{L} & \quad \text{if } L > V \text{ (or } \frac{V}{L} \leq 1) \\
L_i & \quad \text{if } L \leq V \text{ (or } \frac{V}{L} > 1) \\
\end{align*}
\]

or

\[
L_i \left[ 1 - \left(1 - \frac{V}{L}\right)^+ \right]
\]
Model for Insurer - Default Option Value

- Premium for line of business \( i \) allowing for default option value

\[
P_i = E^Q \left[ e^{-r} L_i \left[ 1 - \left( 1 - \frac{V}{L} \right)^+ \right] \right]
\]

\[
= L_{0i} - e^{-r} E^Q \left[ L_i \left( 1 - \frac{V}{L} \right)^+ \right]
\]

\[
= L_{0i} - D_{0i}
\]

where the line of business default value for line \( i \) is denoted \( D_{0i} \).
Denote the insurer default option value by $D_0$ so that

$$D_0 = e^{-r} \sum_{i=1}^{n} E^Q \left[ L_i \left( 1 - \frac{V_i}{L} \right)^+ \right] = \sum_{i=1}^{n} D_{0i}$$

Capital of the insurer based on a target solvency ratio of $s$ will be equal to

$$C = V_0 - P = (1 + s) L_0 - P = sL_0 + D_0$$

Beginning of period assets are

$$V_0 = (1 + s) L_0 = sL_0 + D_0 + L_0 - D_0 = C + P$$
Model for Insurer - Liability Assumptions


- \((X_0, \ldots, X_n, X_{n+1})\) are \(n + 2\) independent gamma distributed random variables with shape parameters \(\gamma_i\) and common rate parameter \(\alpha\), denoted by \(G(\gamma_i, \alpha), i = 0, \ldots, n + 1\).

- Probability density of the \(X_i\) is
  \[
  f_{X_i}(x_i) = \frac{\alpha^{\gamma_i}}{\Gamma(\gamma_i)} e^{-\alpha x_i \gamma_i} x_i^{\gamma_i - 1}, x_i > 0
  \]
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Model for Insurer - Liability Assumptions

• End of period claims for the $n$ lines of business and the value of the assets $V$ are modelled as

$$L_i = \frac{\alpha_0}{\alpha} X_0 + X_i, \quad i = 1, ..., n$$

$$V = \frac{\alpha_0}{\alpha} X_0 + X_{n+1}$$

• Intuition: common factor $X_0$ impacts the values of each line of business claims as well as the assets (inflation)

• Each line of business and the assets have a separate independent factor impacting them denoted by $X_i, \ i = 1, ..., n$ for the line of business $i$, and $X_{n+1}$ for the assets.
Model for Insurer - Liability Assumptions

- Claims for each line of business $L_1, ..., L_n$ are gamma distributed random variables, $G(\lambda_i, \alpha)$, $i = 1, ..., n$, where $\lambda_i = \gamma_0 + \gamma_i$

- Assets $V$ are gamma distributed as $G(\gamma_0 + \gamma_{n+1}, \alpha)$, since the sum of two independent gamma random variables with the same rate parameter is also gamma with shape parameter equal to the sum of the shape parameters.
Model for Insurer - Liability Assumptions

- Total claims liability at the end of the period is

\[ L = \sum_{i=1}^{n} L_i = \frac{n\alpha_0}{\alpha}X_0 + X_., \]

where

\[ X_. = \sum_{i=1}^{n} X_i \]

- \( X_. \) is distributed \( G(\gamma., \alpha) \), with \( \gamma. = \sum_{i=1}^{n} \gamma_i \). However, the total claims liability is a sum of 2 gamma random variables, \( \frac{n\alpha_0}{\alpha}X_0 \) and \( X_. \), each with different rates and so the sum does not have a gamma distribution.
Model for Insurer - Liability Assumptions

- Furman and Landsman (2004): can represent $L$ as a mixed gamma distribution with mixed shape parameter

$$L \sim G(\gamma_0 + \gamma + \nu, \alpha),$$

where $\nu$ is a non negative integer random variable with probabilities

$$p_k = C \delta_k, \quad k \geq 0,$$

where

$$C = \frac{1}{n\gamma_0}, \quad \delta_k = k^{-1} \gamma_0 \sum_{i=1}^{k} (\frac{n-1}{n})^i \delta_{k-i}, \quad k > 0: \quad \delta_0 = 1.$$
Model for Insurer - Liability Assumptions

For the model of claims and assets that we have assumed, the $Q$ measure probability density of claims for line of business $i$ is Gamma $G(\lambda_i, \alpha)$, $i = 1, ..., n$ so that

$$f_{L_i}(y_i) = \frac{\alpha^{\lambda_i} e^{-\alpha y_i \lambda_i^{\lambda_i-1}}}{\Gamma(\lambda_i)}, y_i > 0,$$

with

$$E^Q[L_i] = \frac{\lambda_i}{\alpha}$$

and

$$Var^Q[L_i] = \frac{\lambda_i}{\alpha^2}$$
Model for Insurer - Liability Assumptions

- Note that

\[ \text{Cov}(L_i, L_j) = \frac{\gamma_0}{\alpha^2} \quad i \neq j \]

\[ \text{Cov}(L_i, V) = \frac{\gamma_0}{\alpha^2}, \quad i = 1, \ldots, n \]

and

\[ \rho(L_i, L_j) = \frac{\gamma_0}{\sqrt{\lambda_i \lambda_j}}, \quad i, j = 1, \ldots, n \]
Model for Insurer - Default Option Value

- Default option value for each line

\[
D_{0i} = e^{-r} E^Q \left[ L_i \left( 1 - \frac{V}{L} \right)^+ \right] \\
= e^{-r} E^Q \left[ \left( \frac{\alpha_0}{\alpha} X_0 + X_i \right) \left( 1 - \frac{\frac{\alpha_0}{\alpha} X_0 + X_{n+1}}{n\alpha_0 X_0 + X.} \right)^+ \right] \\
= e^{-r} \frac{\alpha_0}{\alpha} E^Q \left[ X_0 \left( 1 - \frac{\frac{\alpha_0}{\alpha} X_0 + X_{n+1}}{n\alpha_0 X_0 + X.} \right)^+ \right] \\
+ e^{-r} E^Q \left[ X_i \left( 1 - \frac{\frac{\alpha_0}{\alpha} X_0 + X_{n+1}}{n\alpha_0 X_0 + X.} \right)^+ \right], \; i = 1, \ldots, n
\]

where \( Q \sim \prod_{i=0}^{n+1} G(\gamma_i, \alpha) \).
• Expectation in the first term (see paper for proofs)

\[
E^Q \left[ X_0 \left( 1 - \frac{\alpha_0}{\alpha} \frac{X_0 + X_{n+1}}{X_0 + X} \right)^+ \right] = \frac{\gamma_0}{\alpha} E^Q_0 \left[ \left( 1 - \frac{\alpha_0}{n\alpha_0} \frac{X_0 + X_{n+1}}{X_0 + X} \right)^+ \right]
\]

where

\[
dQ^0 = \left( \frac{\alpha^{\gamma_0+1}}{\Gamma(\gamma_0+1)} e^{-\alpha x_0 x_0^{\gamma_0}} \right) \prod_{i=1}^{n+1} \left( \frac{\alpha^{\gamma_i}}{\Gamma(\gamma_i)} e^{-\alpha x_i x_i^{\gamma_i-1}} \alpha^{\gamma_i} \right) dx_0 \ldots dx_{n+1}
\]
Model for Insurer - Default Option Value

- Expectation in the second term

\[ E^Q \left[ X_i \left( 1 - \frac{\alpha_0}{\alpha} \frac{X_0 + X_{n+1}}{X_0 + X} \right)^+ \right] = \frac{\gamma_i}{\alpha} E^Q_i \left[ \left( 1 - \frac{\alpha_0}{\alpha} \frac{X_0 + X_{n+1}}{X_0 + X} \right)^+ \right] \]

where

\[ dQ^i = \left( \frac{\alpha \gamma_i + 1}{\Gamma(\gamma_i + 1)} e^{-\alpha x_i x_i^{\gamma_i} \alpha \gamma_i + 1} \right) \prod_{j=0, j \neq i}^{n+1} \left( \frac{\alpha \gamma_j}{\Gamma(\gamma_j)} e^{-\alpha x_j x_j^{\gamma_j - 1}} \right) dx_0 \ldots dx_{n+1} \]

\[ i = 1, \ldots, n. \]

Thus \( Q_i \sim G(\gamma_i + 1, \alpha) \prod_{j=0, j \neq i}^{n+1} G(\gamma_i, \alpha), i = 1, \ldots, n. \)
Model for Insurer - Default Option Value

• To evaluate the expression

\[ E^{Q_i} \left[ \left( 1 - \frac{\alpha_0}{\alpha} X_0 + X_{n+1} \right)^+ \right], \; i = 1, \ldots, n \]

• Note that under \( Q_0 \)

\[ V = \frac{\alpha_0}{\alpha} X_0 + X_{n+1} \sim G(\gamma_0 + 1 + \gamma_{n+1}, \alpha) \]

and under \( Q_i, \; i = 1, \ldots, n \)

\[ V = \frac{\alpha_0}{\alpha} X_0 + X_{n+1} \sim G(\gamma_0 + \gamma_{n+1}, \alpha) \]
Model for Insurer - Default Option Value

- Under $Q_0$ and $Q_i$, $i = 1, \ldots, n$, the distribution of $\frac{n\alpha_0}{\alpha}X_0 + X$ will not be Gamma.

- Its distribution can be represented as a mixture of Gamma random variables - Furman and Landsman (2004).
Model for Insurer - Default Option Value

- Under $Q_0$

\[ L = \frac{n\alpha_0}{\alpha} X_0 + X. \sim G(\gamma_0 + 1 + \gamma + \tilde{\nu}, \alpha) \]

where $\tilde{\nu}$ is a non-negative integer random variable defined in

\[ \tilde{p}_k = \tilde{C}\tilde{\delta}_k, \quad k \geq 0, \]

\[ \tilde{C} = \frac{1}{n^{\gamma_0+1}}: \quad \tilde{\delta}_k = k^{-1}(\gamma_0 + 1) \sum_{i=1}^{k} \left( \frac{n-1}{n} \right)^i \tilde{\delta}_{k-i}, \quad k > 0: \quad \tilde{\delta}_0 = 1. \]
Model for Insurer - Default Option Value

• Under $Q_i$, $i = 1, \ldots, n,$

$$L = \frac{n\alpha_0}{\alpha} X_0 + X. \sim G(\gamma_0 + 1 + \gamma + \nu, \alpha),$$

where $\nu$ is a non-negative integer random variable defined previously.
Model for Insurer - Some Functions

- Define the following functions (evaluate numerically)

\[ \psi(\gamma) = \frac{\Gamma'(\gamma)}{\Gamma(\gamma)} = \frac{d}{d\gamma} \ln \Gamma(\gamma) \] is the Psi (Digamma) function

\[ \psi(\gamma; a) = \frac{1}{\Gamma(\gamma)} \int_0^\infty \ln(a + x)x^{\gamma-1} \exp(-x)dx \] where \( \psi(\gamma; 0) \) is simply the digamma function \( \psi(\gamma) \)

\[ \phi(\gamma; a) = \frac{1}{\Gamma(\gamma)} \int_0^\infty \frac{1}{a + x}x^{\gamma-1} \exp(-x)dx, \gamma > 1 \text{ so that } \phi(\gamma; 0) = \frac{1}{\gamma - 1}. \]
Model for Insurer - Log-Normal Approximation

- Log normal approximation

\[ D_{0i} = e^{-r \frac{\alpha_0 \gamma_0}{\alpha^2}} \left[ \frac{1}{2} - e^{\mu L_0 + \frac{1}{2} \sigma^2 \Lambda_0} \phi \left( \frac{-\left(\mu \Lambda_0 + \sigma^2 \Lambda_0\right)}{\sigma \Lambda_0} \right) \right] \]

\[ + e^{-r \frac{\gamma_i}{\alpha}} \left[ \frac{1}{2} - e^{\mu \Lambda_i + \frac{1}{2} \sigma^2 \Lambda_i} \phi \left( \frac{-\left(\mu \Lambda_i + \sigma^2 \Lambda_i\right)}{\sigma \Lambda_i} \right) \right], \quad i = 1, \ldots, n. \]
Model for Insurer - Moments of $\Lambda$

- Under $Q_0$

$$\mu^0_\Lambda = \psi (\gamma_0 + \gamma_{n+1} + 1) - \left( \sum_{k=0}^{\infty} \psi (\gamma_0 + 1 + \gamma + k) \tilde{p}_k \right)$$

$$\sigma^2_{\Lambda^0} = \frac{\Gamma'' (\gamma_0 + \gamma_{n+1} + 1)}{\Gamma (\gamma_0 + \gamma_{n+1} + 1)} - \psi (\gamma_0 + \gamma_{n+1} + 1)^2$$

$$+ \sum_{k=0}^{\infty} \left[ \frac{\Gamma'' (\gamma_0 + 1 + \gamma + k)}{\Gamma (\gamma_0 + 1 + \gamma + k)} - \psi (\gamma_0 + 1 + \gamma + k)^2 \right] \tilde{p}_k$$

$$- 2 Cov_{Q_0} [\ln V, \ln L]$$
Model for Insurer - Moments of $\Lambda$ (approximation)

- We also have

$$\text{Cov}_{Q_0}^{0} [\ln V, \ln L]$$

$$= E[\psi(\gamma.; n\alpha_0 X_0)\psi(\gamma_{n+1}; \alpha_0 X_0)] - E\psi(\gamma.; n\alpha_0 X_0)E\psi(\gamma_{n+1}; \alpha_0 X_0)$$

- For large $\gamma.$ and $\gamma_{n+1}$

$$\text{Cov}_{Q_0}^{0} [\ln V, \ln L] \approx \frac{n(\gamma_0 + 1)}{(\gamma. - 1)(\gamma_{n+1} - 1)}.$$
Model for Insurer - Moments of $\Lambda$

- Under $Q_i$, $i = 1, \ldots, n$

$$\mu^i_\Lambda = \psi (\gamma_0 + \gamma_{n+1}) - \left( \sum_{k=0}^{\infty} \psi (\gamma_0 + 1 + \gamma + k) p_k \right)$$

$$\sigma^2_\Lambda^i = \frac{\Gamma''(\gamma_0 + \gamma_{n+1})}{\Gamma(\gamma_0 + \gamma_{n+1})} - \psi (\gamma_0 + \gamma_{n+1})^2$$

$$+ \sum_{k=0}^{\infty} \left[ \frac{\Gamma''(\gamma_0 + 1 + \gamma + k)}{\Gamma(\gamma_0 + 1 + \gamma + k)} - \psi (\gamma_0 + 1 + \gamma + k)^2 \right] p_k$$

$$- 2 \text{Cov}_{Q_i} \left[ \ln V, \ln L \right], i = 1, \ldots, n.$$
Model for Insurer - Moments of $\Lambda$ (approximation)

- We also have

$$Cov_{Q_i} [\ln V, \ln L] = E[\psi(\gamma_i + 1; n\alpha_0 X_0)\psi(\gamma_{n+1}; \alpha_0 X_0)] - E\psi(\gamma_i + 1; n\alpha_0 X_0)E\psi(\gamma_{n+1}; \alpha_0 X_0)$$

- Approximation

$$Cov_{Q_i} [\ln V, \ln L] \approx \frac{n\gamma_0}{\gamma_i(\gamma_{n+1} - 1)}, \quad i = 1, ..., n.$$
Model for Insurer - Frictional Costs

- Frictional costs (transactions costs and other costs such as additional taxation and agency costs are assumed to occur as a percentage of the end of period surplus and are only incurred provided the insurer is solvent.

- Denote the value of these as \( C_c \) then

\[
C_c = E^Q \left[ c_c e^{-r} (V - L)^+ \right] = \sum_{i=1}^{n} E^Q \left[ c_c e^{-r} L_i (\Lambda - 1)^+ \right]
\]

where \( c_c \) is the costs of capital as a percentage of the surplus, provided it is positive.
If we denote the costs of capital allocated to line of business \( i \in \{1, \ldots, n\} \) by \( C^i_c \) we then have

\[
C^i_c = c_c e^{-r} E^Q [L_i (\Lambda - 1)^+] \\
= c_c D_0 i - c_c e^{-r} E^Q [L_i (1 - \Lambda)] \\
= c_c D_0 i - c_c L_i \Lambda
\]
Model for Insurer - Frictional Costs

We have

\[ L_{\Lambda i} = e^{-r} E^Q [L_i (1 - \Lambda)] \]
\[ = e^{-r} \left[ \frac{\gamma_0}{\alpha} E^{Q_0} (1 - \Lambda) + \frac{\gamma_i}{\alpha} E^{Q_i} (1 - \Lambda) \right] \]
\[ = e^{-r} \left[ \frac{\gamma_0}{\alpha} (1 - E^{Q_0} [\Lambda]) + \frac{\gamma_i}{\alpha} (1 - E^{Q_i} [\Lambda]) \right], \; i = 1, \ldots, n. \]
Model for Insurer - Frictional Costs

- Approximation for the expectation of $\Lambda$

$$E^{Q_0}[\Lambda] \approx (\gamma_0 + 1 + \gamma_{n+1})(\frac{1}{\gamma. - 1} - \frac{n(\gamma_0 + 1)}{(\gamma. - 1)(\gamma. - 2)}),$$

and

$$E^{Q_i}[\Lambda] \approx (\gamma_0 + \gamma_{n+1})(\frac{1}{\gamma.} - \frac{n(\gamma_0 + 1)}{\gamma.(\gamma. - 1)}), \; i = 1, \ldots, n$$

and under the log normal assumption

$$E^{Q_i}[\Lambda] = \exp(\mu_i^{\Lambda} + \frac{1}{2}\sigma^2_{\Lambda_i}), \; i = 0, 1, \ldots, n,$$
Model for Insurer - Frictional Costs

Following Estrella (2004) we assume that financial distress costs are a percentage of the absolute value of the shortfall of assets over liabilities in the event of insolvency. Denoting the value of these financial distress costs by $C_f$ we have

$$C_f = E^Q \left[ c_f e^{-r} (L - V)^+ \right] = E^Q \left[ c_f e^{-r} L (1 - \Lambda)^+ \right]$$

where $c_f$ is the financial distress costs as a percentage of the asset shortfall in the event of insolvency.
Model for Insurer - Frictional Costs

- Allocation to line of business $i$

\[ C_f^i = c_f e^{-r} E^Q \left[ L_i (1 - \Lambda)^+ \right] = c_f D_{0i} \]

where $C_f^i$ is the allocation of the financial distress costs to line of business $i$. 

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Model for Insurer - By Line Price

- An expression for the price by line of business, including the frictional costs of capital and default option value, for a multi-line insurer is given by

\[ L_{0i} + C_c^i + C_f^i - D_{0i} = e^{-r \frac{\lambda_i}{\alpha}} - c_c L_{\Lambda i} - \left(1 - c_f - c_c\right) D_{0i} \]

- Closed form expressions for \( L_{\Lambda i} \) and \( D_{0i} \) are derived in this paper under a recently developed dependent gamma distribution model.
Conclusions and Summary

- Results for a model for multiline insurer with dependent gamma distribution allowing for default put and frictional costs.

- Closed form and approximate expressions for default option value by line as well as allocation of frictional costs.

- Issues - Optimal capital structure (see Chandra and Sherris, 2005).

- Issues - Marginal versus full balance sheet allocations.