Claims Prediction with Dependence using Copula Models*

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Abstract

Classical credibility models provide for predictive claims in linear form. For example, the Bühlmann and the Bühlmann-Straub credibility models express the next period's claim as a weighted average of historical claims arising from each group's own experience and the entire portfolio's experience. The weight that is attached to the own experience reflects a credibility factor. These classical models typically assume claim independence. In this paper, we extend the notion of predicting the next period's claims by relaxing these independence assumptions. We specify claim dependence structure using the concept of copula models which in recent years, has received considerable attention for modelling dependencies. This paper extends the models offered by Frees and Wang (2005) and Yeo and Valdez (2004) that respectively used the frameworks of copula models and common effects. Here in this paper, we find that the predictive claim can be expressed as an expectation under a change of probability measure which reflects the ratio of the densities of the copulas relating to the historical claims. We examine this expectation for different families of copulas and in several instances, we find we are able to explicitly express the predictive claim in terms of historical claims. As we suspect, these predictive claims are no longer linear in form.

*Keywords: claim dependence, insurance premium, credibility models, and copula.

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1 Introduction and Motivation

In the pricing of a general insurance contract, its premium is typically determined by assessing the observable claims from a portfolio of such homogeneous contracts. These observable claims can, for example, be represented by random variables $X_{j,t}$ where $j = 1, 2, \ldots, J$ denotes the individual risk and $t = 1, 2, \ldots, T$ denotes the time period. Here, J refers to the number of contracts in the portfolio and T is the time period where historical claims are available. To simplify our illustration, the same time period will apply to all the individuals and this is typically called a balanced model. However, with a slight variation to the model, one can always extend the notion to the unbalanced case.

Insurers usually group individual risks such that the risks within each group are as homogeneous as possible in terms of certain observable risk characteristics. A common premium for the group, also known as the manual premium, is then calculated and charged. The grouping is made primarily to reach a fair and equitable premium across all individuals. Such grouping also helps isolate a large group of independent and identically distributed risks so that the law of large numbers can be invoked during the claims prediction process. This minimises variability in the claims experience within the same group. However, the grouping of the individual risks will not be precise, thereby causing the risks within each group to be not entirely homogeneous. An unknown number of unobservable traits will always contribute to the possible presence of heterogeneity among the individuals.

In premium calculation, which also requires prediction of claims, historical claims experience provides an invaluable insight into the unobservable characteristics of the individual risks. Furthermore, it is a common practice to allow for past claims experience of the insured individual in claims prediction and in premium calculation. This exercise is known as experience rating and is generally made for the purpose of reaching a fair and equitable insurance premium rate. For example, in motor insurance, a driver may have had a number of years of experience available to the insurer. Those drivers with little or no claims experience will simply be assessed an additional risk premium, and prediction of its own future claims will be based on the claims experience of the group it belongs to.

These considerations naturally point to some sort of a compromise between the two sets of experience in claims prediction: the group's claims experience and the individual's claims experience, if any. This in turn has led actuaries to utilise a pricing formula of the form

$$\text{Premium} = Z \cdot \text{Own Experience} + (1 - Z) \cdot \text{Group Experience}, \tag{1}$$

where Z, a value between 0 and 1 (inclusive), is commonly known as the "credibility factor". The credibility factor in (1) is a weight assigned to the individual's own claims experience. The vast majority of credibility models developed for almost a century now in turn leads to the computation of this weight. It is to be noted that on one hand, the group's collective experience is extensive enough for the law of large numbers to be applicable and therefore, ignores the presence of heterogeneity. On the other hand, the individual's own experience contains useful information about the risk characteristics of the individual but may be subject to random fluctuations due to lack of volume. Credibility models must therefore be able to reach an intuitively appealing formula allowing for larger credibility for larger number of years of individual experience, for example. Such a credibility factor should be close to unity when individual risk experience is abundant or where there is a high degree of heterogeneity in the overall experience. It should be close to nil if individual risk experience is lacking or unreliable, or where there is a high degree of homogeneity in overall experience.

In the construction of credibility models, it has been a common practice to assume independence of claims, although some of the models may have relaxed this assumption by assuming conditionally i.i.d. random variables that are uncorrelated instead. Using our notation introduced earlier, claims $X_{1,t}, X_{2,t}, \ldots, X_{J,t}$ are assumed to be independent across the individuals for a fixed time t, stating that claims of one insured individual do not directly impact those of other insured individuals. For a fixed individual $j = 1, 2, \ldots, J$, claims $X_{j,1}, X_{j,2}, \ldots, X_{j,T}$ are also often assumed independent across different time periods. Modelling the time dependence appears to be a more common practice when developing credibility models, but not dependence across individuals. The early paper by Gerber and Jones (1975) and the more recent ones by Frees, et al. (1999) are examples of credibility models with time dependence of claims.

As we have observed in the recent past, in the actuarial and insurance literature, the notion of claim dependencies is increasingly becoming an important part of the modelling process. In Wang (1998), a set of statistical tools for modelling dependencies of risks in an insurance portfolio has been suggested. Valdez and Mo (2002) and Albrecher and Kantor (2002) have both examined the impact of claim dependencies on the probability of ruin under the copula framework. The works of Heilmann (1986) and Hürlimann (1993) have investigated the effect of dependencies of risks on stop-loss premiums. Several generalisations and alternative models of dependence have since followed including, Dhaene and Goovaerts (1996, 1997) and Müller (1997), addressing their impact on stop-loss premiums. Other models have included the works of Genest, et al. (2003) and Cossette, et al. (2002) where claim dependence have been addressed in the framework of individual risk models. Furthermore, using the notion of a stochastic order, the recent papers by Purcaru and Denuit (2002, 2003) provide excellent discussion of dependencies in claim frequency for credibility models. In the credibility theory context, Frees (2003) considered multivariate credibility premiums based upon the frailty construction for different lines of business. The main driver for dependence between risks was assumed to be via the claims incidence rates only. More recently, Frees and Wang (2005) considered a generalised linear model framework for modelling marginal claims distributions and allowed for dependence using copulas. In their paper, the Student-t copula has been used to model the dependence over time for a class of risks.

In this paper, we offer additional insight into the modelling of claim dependencies, and within the framework of developing credibility premiums, or to be more precise, predicting the next period's claims given the history of claims available. In the early part of the paper, we consider the case where we only have one particular contract, or class of risks, say j, and we allow for claims dependence across time. We demonstrate that the next period's claims, given the history of claims, can be conveniently expressed as

$$E\left(X_{T+1}|X_1=x_1,\ldots,X_T=x_T\right)=\int_{-\infty}^{\infty}x_{T+1}\frac{c_{T+1}\left(\mathbf{u}_{T+1}\right)}{c_T\left(\mathbf{u}_T\right)}dF_{T+1}\left(x_{T+1}\right),$$

where $c_{T+1}(\cdot)$ and $c_T(\cdot)$ are respectively the density of the copulas associated with the claim vectors $\mathbf{X}_{T+1}=(X_1,\ldots,X_{T+1})'$ and $\mathbf{X}_T=(X_1,\ldots,X_T)'$. This ratio of densities of the copula, $\dfrac{c_{T+1}\left(\mathbf{u}_{T+1}\right)}{c_T\left(\mathbf{u}_T\right)}$, in fact induces a change of probability measure so that in effect, we can then write the claims prediction as the unconditional expectation

$$E\left(X_{T+1}|X_1=x_1,\ldots,X_T=x_T\right)=E^Q\left(X_{T+1}\right)$$

under a change of measure $dF^Q_{T+1}\left(x_{T+1}\right)=\dfrac{c_{T+1}\left(\mathbf{u}_{T+1}\right)}{c_T\left(\mathbf{u}_T\right)}dF_{T+1}\left(x_{T+1}\right)$. This change of measure allows us then to be able to construct an explicit expression for the credibility premium. With the exception of the multivariate Normal case, we no longer are able to express this credibility premium in a linear form that we commonly find in the classical credibility models such as that in expression (1). In illustrating the convenience of this formulation, we consider different families of copulas such as the familiar Normal and Student-t copulas, the Farlie-Gumbel-Morgenstern (or FGM, for short) as well as the family of Archimedean copulas.

In this work, we place importance on the modelling of claim dependence across time, that is, the claims of an individual risk over time are dependent in some sense. For example, in motor insurance, this dependence can partly be explained by the individual's proneness to accidents which may well depend on personal behaviour and driving style. In medical insurance, this dependence is influenced by the individual's health over time and some unobservable characteristics like genes and lifestyle contribute to his or her well-being. Claim dependence across individuals will be considered towards the end of this work.

The rest of the paper has been organised as follows. In the next section, we derive generic formulas for the credibility premium when claims experience over time is modelled by copulas. Here, we find that the copula density ratio plays an important role in computing the credibility premium. The subsequent sections examine cases of various copulas. In Section 3, we consider the case of the Gaussian and the Student-t copulas. In Section 4, we consider the family of Archimedean copulas which are generated by a single-valued Archimedean generator. Section 5 develops the credibility premiums for the case of the Farlie-Gumbel-Morgenstern copulas. To visualize the effects of tilting induced by the copula density ratio, we provide some illustrative examples in Section 6. Section 7 provides a brief sketch of how to extend the work to include simultaneous dependence of claims across individuals. We provide concluding statements in Section 8.

2 Relaxing the independence assumption across time periods

This section considers the case of only a single contract j, or a single class of risks, and assumes that claim dependencies are specified using copula functions. Effectively, we continue to assume independence across individual contracts, although we address this issue in a later section. To avoid the cumbersome notation, we drop the part of the subscript referring to the contract so that the observed claims vector is denoted by $\mathbf{X}_T = (X_1, \ldots, X_T)'$.

There has been a number of papers on copulas, hence we do not provide introduction here. Rather, we only give a short discussion about copulas as a matter of introducing the notation. We also advise the reader to examine textbooks and references therein found in Mari and Kotz (2001) and Nelsen (1999). Copulas are functions that join or couple multivariate distribution functions to their one-dimensional marginal distributions functions. Specifically, we have

$$H_T(x_1, \ldots, x_T) = P(X_1 \leq x_1, \ldots, X_T \leq x_T) = C_T(F_1(x_1), \ldots, F_T(x_T)),$$

where $F_t(\cdot)$ refers to the marginal distribution associated with X_t, H_T is their joint distribution function, and C_T is the corresponding copula function where subscripts of H and C refer to the dimension of the claim vector. Sklar (1959) proved the existence of copulas for every joint distribution function and demonstrated that they are indeed unique if the marginal distribution functions are continuous. It is also sometimes convenient to write this as

$$H_T(x_1, \ldots, x_T) = C_T(u_1, \ldots, u_T), \tag{2}$$

where $\mathbf{u}_T = (u_1, \ldots, u_T)'$ and $u_t = F_t(x_t)$ refers to the distribution function for $t = 1, \ldots, T$.

Vectors shall be written in bold letters. For example, we shall denote the observed values of $\mathbf{X}_T$ by $\mathbf{x}_T = (x_1, \ldots, x_T)'$ and similarly for $\mathbf{X}_{T+1}$ by $\mathbf{x}_{T+1} = (x_1, \ldots, x_T, x_{T+1})'$. For ease of notation, we will also denote the vectors $\mathbf{u}_T = (u_1, \ldots, u_T)'$ and $\mathbf{u}_{T+1} = (u_1, \ldots, u_T, u_{T+1})'$. We shall assume that the densities of the copulas exist and are respectively denoted by

$$c_T(\mathbf{u}_T) = \frac{\partial^T C_T(\mathbf{u}_T)}{\partial u_1 \ldots \partial u_T} \tag{3}$$

and

$$c_{T+1}(\mathbf{u}_{T+1}) = \frac{\partial^{T+1} C_{T+1}(\mathbf{u}_{T+1})}{\partial u_1 \ldots \partial u_T \partial u_{T+1}}. \tag{4}$$

Notice that the marginal distribution functions have been denoted by

$$F_t(x_t) = P(X_t \leq x_t) \text{ for } t = 1, \ldots, T, T+1$$

and if the corresponding density functions exist, we denote them by

$$f_t(x_t) = \frac{dF_t(x_t)}{dx_t} \text{ for } t = 1, \ldots, T, T+1.$$

Similarly, the multivariate density functions, if they exist, will be respectively denoted by

$$h_T(\mathbf{x}_T) = \frac{\partial^T H_T(\mathbf{x}_T)}{\partial x_1 \ldots \partial x_T}$$

and

$$h_{T+1}(\mathbf{x}_{T+1}) = \frac{\partial^{T+1} H_{T+1}(\mathbf{x}_{T+1})}{\partial x_1 \ldots \partial x_T \partial x_{T+1}}.$$

Our primary motivation here is to construct predictors for the next period's claim of an individual contract based on all its observed claims $X_1, \ldots, X_T$, for which the next period's claim is to be denoted by the random variable X_{T+1}. As previously stated, we assume that there is dependence only over time periods for the same individual and this dependency structure is specified with a copula function as in (2). In this work, we incorporate the copula dependence structure into the predictor for claims, which may result in other forms of explicit credibility formulas. It should be noted that the credibility premium may no longer be linear with respect to the historical claims, but rather be non-linear in form. Hence, it may not be possible to have the linear credibility form that we have been used to.

It is well-known in statistics that the best predictor of X_{T+1}, based on all its observed claims $X_1, \ldots, X_T$, in the sense of the "mean squared prediction error" is the conditional expectation

$$E(X_{T+1}|\mathbf{X}_T = \mathbf{x}_T). \tag{5}$$

In other words, the predictor in (5) is the required functional $g(\mathbf{X}_T)$ that minimises the following mean squared prediction error:

$$E[X_{T+1} - g(\mathbf{X}_T)]^2.$$

See, for example, Shao (2003) for a nice proof on page 40. This same result is well-known in the actuarial and insurance literature, for example, see Dannenburg, Kaas, and Goovaerts (1996).

Proceeding now, the conditional density, $f_{X_{T+1}|\mathbf{X}_T}(x_{T+1}|\mathbf{x}_T)$, is desired as the conditional expectation in (5) gives our best estimate of the next period's claims and also gives our desired credibility premium. This conditional density has been referred to in Frees and Wang (2005) as the predictive density and we shall assume it exists. The predicted claims can be conveniently expressed as

$$E(X_{T+1}|\mathbf{X}_T = \mathbf{x}_T) = \int_{-\infty}^{\infty} x_{T+1} \cdot f_{X_{T+1}|\mathbf{X}_T}(x_{T+1}|\mathbf{x}_T)\, dx_{T+1}, \tag{6}$$

where the integral is the Riemann-Stieltjes integral. Notice that the limits of the integral is the entire real line so that we are not confined to the assumption that claims are non-negative. We now give the following result for the conditional expectation as a proposition.

Proposition 1 *Consider the copula model satisfying the assumptions described in this section. The conditional expectation of $X_{T+1}|\mathbf{X}_T$ can be expressed in the following manner:*

$$E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right)=\int_{-\infty}^{\infty}x_{T+1}\cdot\frac{c_{T+1}\left(\mathbf{u}_{T+1}\right)}{c_T\left(\mathbf{u}_T\right)}dF_{T+1}\left(x_{T+1}\right), \tag{7}$$

where $c_{T+1}\left(\mathbf{u}_{T+1}\right)$ and $c_T\left(\mathbf{u}_T\right)$ are respectively defined in (3) and (4), and that F_{T+1} is a known marginal distribution function of X_{T+1}.

Proof. It is clear from the definition of conditional density, that if it exists, we must have

$$f_{X_{T+1}|\mathbf{X}_T}\left(x_{T+1}|\mathbf{x}_T\right)=\frac{h_{T+1}\left(\mathbf{x}_{T+1}\right)}{h_T\left(\mathbf{x}_T\right)}.$$

The numerator can be written as

$$\begin{aligned}
h_{T+1}\left(\mathbf{x}_{T+1}\right) &= \frac{\partial^{T+1}C_{T+1}\left(\mathbf{u}_{T+1}\right)}{\partial u_1\ldots\partial u_T\partial u_{T+1}}\times\prod_{t=1}^{T+1}f_t\left(x_t\right)\\
&= c_{T+1}\left(\mathbf{u}_{T+1}\right)\times\prod_{t=1}^{T+1}f_t\left(x_t\right),
\end{aligned}$$

where $\mathbf{u}_{T+1}$ is understood to be evaluated at the respective marginals $F_t\left(x_t\right)$ for $t=1,\ldots,T,T+1$. Similarly, we have

$$h_T\left(\mathbf{x}_T\right)=c_T\left(\mathbf{u}_T\right)\times\prod_{t=1}^{T}f_t\left(x_t\right).$$

From these, we now have

$$\begin{aligned}
E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right) &= \int_{-\infty}^{\infty}x_{T+1}\cdot\frac{h_{T+1}\left(\mathbf{x}_{T+1}\right)}{h_T\left(\mathbf{x}_T\right)}dx_{T+1}\\
&= \int_{-\infty}^{\infty}x_{T+1}\cdot\frac{c_{T+1}\left(\mathbf{u}_{T+1}\right)\times\prod_{t=1}^{T+1}f_t\left(x_t\right)}{c_T\left(\mathbf{u}_T\right)\times\prod_{t=1}^{T}f_t\left(x_t\right)}dx_{T+1}\\
&= \int_{-\infty}^{\infty}x_{T+1}\cdot\frac{c_{T+1}\left(\mathbf{u}_{T+1}\right)}{c_T\left(\mathbf{u}_T\right)}f_{T+1}\left(x_{T+1}\right)dx_{T+1},
\end{aligned}$$

and the result given in (7) follows. ■

We observe that the ratio $c_{T+1}\left(\mathbf{u}_{T+1}\right)/c_T\left(\mathbf{u}_T\right)$ induces a change of measure of the original probability measure corresponding to the observable claim X_{T+1} in the next period. The above conditional expectation can be rewritten as an unconditional expectation with respect to the new marginal density

$$dF_{T+1}^{Q}\left(x_{T+1}\right)=\frac{c_{T+1}\left(\mathbf{u}_{T+1}\right)}{c_T\left(\mathbf{u}_T\right)}dF_{T+1}\left(x_{T+1}\right),$$

that is,

$$E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right)=\int_{-\infty}^{\infty}x_{T+1}\cdot f_{T+1}^{c}\left(x_{T+1}\right)dx_{T+1}=E^{Q}\left(X_{T+1}\right), \tag{8}$$

where E^Q denotes taking the expectation under the said new marginal density. As a matter of fact, the copula density ratio is the conditional density

$$c\left(u_{T+1}|\mathbf{u}_T\right)=\frac{c_{T+1}\left(\mathbf{u}_{T+1}\right)}{c_T\left(\mathbf{u}_T\right)},$$

so that one may effectively call this *tilting induced by the conditional density copula.* In the case where we have independence, notice that we have $c_{T+1}\left(\mathbf{u}_{T+1}\right)/c_T\left(\mathbf{u}_T\right)=1$ so that we have $E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right)=E\left(X_{T+1}\right)$. Thus, the next period's claim is not influenced by previous claims.

The use of copulas to model the time dependence of claims for a single individual risk is similar in structure to that discussed in Frees and Wang (2005). Nonetheless, there are vital differences between our respective works. The most important difference lies in the representation of the conditional expectation. In our work, we show that the conditional expectation can be interpreted as a change of measure and hence succinctly written as an unconditional expectation under a revised marginal density. This revision of the marginal density can be interpreted as a "re-weighting" of the density function after observing the previous claims. Therefore, the observed claims have a direct influence on the next period's claim. The magnitude of this impact is affected by the structure of the specified copula and is measured by how the ratio of the densities of the copula affects the re-shaping of the distribution.

Another clear difference in our respective works has to do with the explicit construction of the resulting unconditional expectation. We show, in the next few sections, that for special families of copulas, we are able to construct explicit expressions for the predicted claims. We even find that in the case of the Normal copula, the resulting unconditional expectation has the form of the Wang transformation.

Finally, unlike that of the work of Frees and Wang (2005), we do not limit the application of the formulas to the family of Student-t copulas. We extend the applications to various families of copulas such as familiar elliptical copulas like the Normal and Student-t copulas, the Archimedean family of copulas, as well as the FGM family of copulas. We consider these various copulas separately in the next few sections to demonstrate appreciation of the applicability of Proposition 1.

3 The Gaussian and Student-t copulas

This section considers the Normal and Student-t copulas. For both of these types of copulas, we shall denote the correlation matrix by $\mathbf{\Sigma}_T$ with representation $\mathbf{\Sigma}_T=\left(\rho_{ij}\right)$ where $\rho_{ij}=1$ for all $i=j$, $i,j=1,\ldots,T$ so that $\mathrm{diag}(\mathbf{\Sigma}_T)=1$.

3.1 The case of the Gaussian copula

Consider the Gaussian copula with correlation matrix $\mathbf{\Sigma}_T$ which has the form

$$C_T(\mathbf{u}_T) = \Phi_{\Sigma_T}\left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_T)\right) \tag{9}$$

where Φ_{Σ_T} is a standardised T-dimensional Normal distribution function and Φ^{-1} refers to the quantile function of the standard one-dimensional Normal distribution. It can be shown that the corresponding density of this Gaussian copula has the form

$$c_T(\mathbf{u}_T) = \frac{\exp\left(-\frac{1}{2}\boldsymbol{\varsigma}'_T \mathbf{\Sigma}_T^{-1} \boldsymbol{\varsigma}_T\right)}{\sqrt{(2\pi)^T |\mathbf{\Sigma}_T|} \prod_{k=1}^T \phi\left(\Phi^{-1}(u_k)\right)} \tag{10}$$

where $\boldsymbol{\varsigma}'_T = \left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_T)\right)$ and $\phi(z) = (2\pi)^{-1/2} \exp\left(-\frac{1}{2}z^2\right)$, the density of a standard Normal.

Proposition 2 *Let $\mathbf{X}_T$ be a claims vector with a Normal copula structure as specified in (9). The ratio of the densities of the Normal copula can be expressed as*

$$\frac{c_{T+1}(\mathbf{u}_{T+1})}{c_T(\mathbf{u}_T)} = \frac{1}{\sigma_{Z.T+1}} \cdot \frac{\phi\left(\Phi^{-1}(\breve{u}_{T+1})\right)}{\phi\left(\Phi^{-1}(u_{T+1})\right)}$$

where $\Phi^{-1}(\breve{u}_{T+1}) = \left(\Phi^{-1}(u_{T+1}) - \mu_{Z.T+1}\right)/\sigma_{Z.T+1}$, $\mu_{Z.T+1} = \boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T$ *and* $\sigma_{Z.T+1} = 1 - \boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}$.

Proof. First, from (10), we find the ratio of the densities is given by

$$\frac{c_{T+1}(\mathbf{u}_{T+1})}{c_T(\mathbf{u}_T)} = \frac{\exp\left[-\frac{1}{2}\left(\boldsymbol{\varsigma}'_{T+1}\mathbf{\Sigma}_{T+1}^{-1}\boldsymbol{\varsigma}_{T+1} - \boldsymbol{\varsigma}'_T\mathbf{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T\right)\right]}{\sqrt{2\pi |\mathbf{\Sigma}_{T+1}| / |\mathbf{\Sigma}_T|}\, \phi\left(\Phi^{-1}(u_{T+1})\right)}$$

From (27) of Appendix A, we find that

$$\boldsymbol{\varsigma}'_{T+1}\mathbf{\Sigma}_{T+1}^{-1}\boldsymbol{\varsigma}_{T+1} - \boldsymbol{\varsigma}'_T\mathbf{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T = \left(\frac{\Phi^{-1}(u_{T+1}) - \mu_{Z.T+1}}{\sigma_{Z.T+1}}\right)^2,$$

where we write $\mu_{Z.T+1} = \boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T$ and $\sigma_{Z.T+1} = 1 - \boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}$. By noting the properties of determinant, we have

$$|\mathbf{\Sigma}_{T+1}| / |\mathbf{\Sigma}_T| = \left(1 - \boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}\right) = \sigma^2_{Z.T+1}.$$

Thus we can write

$$\begin{aligned} & \frac{\exp\left[-\frac{1}{2}\left(\boldsymbol{\varsigma}'_{T+1}\mathbf{\Sigma}_{T+1}^{-1}\boldsymbol{\varsigma}_{T+1} - \boldsymbol{\varsigma}'_T\mathbf{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T\right)\right]}{\sqrt{2\pi |\mathbf{\Sigma}_{T+1}| / |\mathbf{\Sigma}_T|}} \\ = & \frac{\exp\left[-\frac{1}{2}\left(\frac{\Phi^{-1}(u_{T+1}) - \mu_{Z.T+1}}{\sigma_{Z.T+1}}\right)^2\right]}{\sqrt{2\pi}\sigma_{Z.T+1}} \\ = & \phi\left[\left(\Phi^{-1}(u_{T+1}) - \mu_{Z.T+1}\right)/\sigma_{Z.T+1}\right]/\sigma_{Z.T+1}. \end{aligned}$$

The results in the proposition then follow. ■

Using Proposition 2, the following result becomes immediate.

Corollary 1 *Let $\mathbf{X}_T$ be a claims vector with a Normal copula structure as specified in (9). Then the next period's predicted claims has the representation*

$$E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right)=E_Z\left\{F_{T+1}^{-1}\left[\Phi\left(\mu_{Z.T+1}+\sigma_{Z.T+1}Z\right)\right]\right\},$$

where the expectation on the right-hand side is computed for a standard Normal random variable Z.

Proof. From propositions 1 and 2, we find the conditional expectation is

$$\begin{aligned}&E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right)\\ =&\int_{-\infty}^{\infty}x_{T+1}\frac{\phi\left[\left(\Phi^{-1}\left(u_{T+1}\right)-\mu_{Z.T+1}\right)/\sigma_{Z.T+1}\right]}{\phi\left(\Phi^{-1}\left(u_{T+1}\right)\right)}dF_{T+1}\left(x_{T+1}\right).\end{aligned}$$

Applying the transformation $z=\left(\Phi^{-1}\left(u_{T+1}\right)-\mu_{Z.T+1}\right)/\sigma_{Z.T+1}$, we find that

$$\begin{aligned}E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right)&=\int_{-\infty}^{\infty}F_{T+1}^{-1}\left[\Phi\left(\mu_{Z.T+1}+\sigma_{Z.T+1}z\right)\right]\cdot\phi\left(z\right)dz\\&=E_Z\left\{F_{T+1}^{-1}\left[\Phi\left(\mu_{Z.T+1}+\sigma_{Z.T+1}Z\right)\right]\right\},\end{aligned}$$

which is the desired result. ■

We observe that the term $F_{T+1}^{-1}\left[\Phi\left(\mu_{Z.T+1}+\sigma_{Z.T+1}Z\right)\right]$ is a specific example of the Wang transformation when $F_{T+1}=\Phi$, with $\mu_{Z.T+1}$ as the market price of risk or risk aversion parameter and with $\sigma_{Z.T+1}$ as the parameter allowing for parameter uncertainty. Taking expectation of $F_{T+1}^{-1}\left[\Phi\left(\mu_{Z.T+1}+\sigma_{Z.T+1}Z\right)\right]$ will therefore give a risk-adjusted expected loss. As an example, consider the bivariate case with correlation parameter ρ. We then have $\mu_{Z.T+1}=\rho x_1$ and $\sigma_{Z.T+1}=\sqrt{1-\rho^2}$. It is interesting to note that the risk aversion parameter $\mu_{Z.T+1}$, depends on observed values of past claims experience. Indeed, the adjustment for risk increases for worse past claims experience, resulting in a higher credibility premium. This is a result expected of a credibility premium. For more on the Wang transformation, see Wang (2000, 2002).

Example 3.1.1: *The multivariate Normal distribution*
Consider the special case where $\mathbf{X}_{T+1}$ has a multivariate Normal distribution with common pairwise correlation of ρ so that we can write the correlation matrix as

$$\mathbf{\Sigma}_T=\begin{pmatrix}1&\rho&\cdots&\rho\\\rho&1&\cdots&\rho\\\vdots&\vdots&\ddots&\vdots\\\rho&\rho&\cdots&1\end{pmatrix}=(1-\rho)\left(\mathbf{I}_T+\frac{\rho}{1-\rho}\mathbf{e}_T\mathbf{e}_T'\right)$$

where $\mathbf{e}_T'=(1,1,\ldots,1)$ is a row-vector of dimension T and $\mathbf{I}_T$ is the $T\times T$ identity matrix. Then it can be shown that

$$\mathbf{\Sigma}_T^{-1}=(1-\rho)^{-1}\left(\mathbf{I}_T-\frac{\rho}{1-\rho+\rho T}\mathbf{e}_T\mathbf{e}_T'\right)$$

so that

$$\mu_{Z.T+1} = \frac{\rho}{1-\rho+\rho T}\mathbf{e}_T' \boldsymbol{\varsigma}_T = \frac{\rho}{1-\rho+\rho T}\sum_{i=1}^{T}\left(\frac{x_i-\mu}{\sigma}\right),$$

where we have assumed that all the marginals are also identically distributed with mean μ and variance σ^2. Thus, we see that the predicted claim becomes

$$E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right) = E_Z\left[F_{T+1}^{-1}\left(\Phi\left(\mu_{Z.T+1}+\sigma_{Z.T+1}Z\right)\right)\right] = \mu_{Z.T+1}\sigma+\mu,$$

and this expression can be written in the familiar credibility form

$$E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right) = \frac{\rho T}{1-\rho+\rho T}\overline{x} + \frac{1-\rho}{1-\rho+\rho T}\mu.$$

This result is as expected. A further special situation is when $T=1$, the bivariate case in which case, we would have

$$E\left(X_2|X_1=x_1\right) = \rho x_1 + (1-\rho)\mu.$$

This is the answer we would expect to arrive at.■

Example 3.1.2: *The Normal copula with Uniform marginals*
In the special case where X_{T+1} is believed to be Uniform on $(0,1)$ so that $F_{T+1}^{-1}(w) = w$, we have the following explicit form

$$E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right) = E_Z\left[\Phi\left(\sigma_{Z.T+1}Z+\mu_{Z.T+1}\right)\right] = \Phi\left(\frac{\mu_{Z.T+1}}{\sqrt{1+\sigma_{Z.T+1}}}\right).■$$

3.2 The case of the Student-t copula

The so-called Student-t copula with a correlation parameter $\boldsymbol{\Sigma}_T$ has the form

$$C_T\left(\mathbf{u}_T\right) = T_{\Sigma_T,r}\left(T_r^{-1}\left(u_1\right),\ldots,T_r^{-1}\left(u_T\right)\right) \tag{11}$$

where $T_{\Sigma_T,r}$ is a standardised T-dimensional Student-t distribution function and T_r^{-1} refers to the quantile function of the standard one-dimensional Student-t distribution with r degrees of freedom. It can be shown that the corresponding density of this copula can be expressed as

$$c_T\left(\mathbf{u}_T\right) = \frac{\Gamma\left(\frac{r+T}{2}\right)\left(1+\frac{1}{r}\boldsymbol{\varsigma}_T'\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T\right)^{-(r+T)/2}}{\Gamma\left(\frac{r}{2}\right)\sqrt{(r\pi)^T\left|\boldsymbol{\Sigma}_T\right|}\prod_{k=1}^{T}t_r\left(T_r^{-1}\left(u_k\right)\right)} \tag{12}$$

where $\boldsymbol{\varsigma}_T' = \left(T_r^{-1}\left(u_1\right),\ldots,T_r^{-1}\left(u_T\right)\right)$ and T_r refers to the distribution function of a univariate standardised Student-t distribution with r degrees of freedom.

Proposition 3 *Let $\mathbf{X}_T$ be a claims vector with a Student-t copula structure as specified in (11). The ratio of the densities of the Student-t copula can be expressed as*

$$\frac{c_{T+1}\left(\mathbf{u}_{T+1}\right)}{c_T\left(\mathbf{u}_T\right)} = \frac{1}{\sigma^*_{Z.T+1}} \cdot \frac{t_{r+T}\left(T_r^{-1}\left(\breve{u}_{T+1}\right)\right)}{t_r\left(T_r^{-1}\left(u_{T+1}\right)\right)},$$

where $T_r^{-1}\left(\breve{u}_{T+1}\right) = \left(T_r^{-1}\left(u_{T+1}\right) - \mu_{Z.T+1}\right)/\sigma^*_{Z.T+1}$, $\mu_{Z.T+1} = \boldsymbol{\rho}'_{T+1,T}\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T$ *and* $\sigma^{*2}_{Z.T+1} = r\sigma^2_{Z.T+1}\left(1+\frac{1}{r}\boldsymbol{\varsigma}'_T\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T\right)/\left(r+T\right)$.

Proof. From (11), we find the ratio of the densities is given by

$$\frac{c_{T+1}\left(\mathbf{u}_{T+1}\right)}{c_T\left(\mathbf{u}_T\right)} = \frac{\Gamma\left(\frac{r+T+1}{2}\right)\left(1+\frac{1}{r}\boldsymbol{\varsigma}'_{T+1}\boldsymbol{\Sigma}_{T+1}^{-1}\boldsymbol{\varsigma}_{T+1}\right)^{-(r+T+1)/2}}{\Gamma\left(\frac{r+T}{2}\right)\sqrt{r\pi\left|\boldsymbol{\Sigma}_{T+1}\right|/\left|\boldsymbol{\Sigma}_T\right|}\left(1+\frac{1}{r}\boldsymbol{\varsigma}'_T\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T\right)^{-(r+T)/2} t_r\left(T_r^{-1}\left(u_{T+1}\right)\right)},$$

From (28) in the appendix, we also have that

$$\begin{aligned}
\boldsymbol{\varsigma}'_{T+1}\boldsymbol{\Sigma}_{T+1}^{-1}\boldsymbol{\varsigma}_{T+1} &= \frac{\left(T_r^{-1}\left(u_{T+1}\right)-\mu_{Z.T+1}\right)^2}{1-\boldsymbol{\rho}'_{T+1,T}\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}} + \boldsymbol{\varsigma}'_T\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T \\
&= \left(\frac{T_r^{-1}\left(u_{T+1}\right)-\mu_{Z.T+1}}{\sigma_{Z.T+1}}\right)^2 + \boldsymbol{\varsigma}'_T\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T.
\end{aligned}$$

where $\mu_{Z.T+1} = \boldsymbol{\rho}'_{T+1,T}\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T$ and $\sigma^2_{Z.T+1} = 1-\boldsymbol{\rho}'_{T+1,T}\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}$. Now consider the term in the numerator of the ratio of the density copulas where we find

$$1+\frac{1}{r}\boldsymbol{\varsigma}'_{T+1}\boldsymbol{\Sigma}_{T+1}^{-1}\boldsymbol{\varsigma}_{T+1} = 1+\frac{1}{r}\left[\left(\frac{T_r^{-1}\left(u_{T+1}\right)-\mu_{Z.T+1}}{\sigma_{Z.T+1}}\right)^2 + \boldsymbol{\varsigma}'_T\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T\right].$$

By multiplying and dividing the term $1+\frac{1}{r}\boldsymbol{\varsigma}'_T\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T$, we can further re-write it as

$$\begin{aligned}
1+\frac{1}{r}\boldsymbol{\varsigma}'_{T+1}\boldsymbol{\Sigma}_{T+1}^{-1}\boldsymbol{\varsigma}_{T+1} &= \left(1+\frac{1}{r}\boldsymbol{\varsigma}'_T\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T\right)\left\{\frac{1+\frac{1}{r}\left[\left(\frac{T_r^{-1}\left(u_{T+1}\right)-\mu_{Z.T+1}}{\sigma_{Z.T+1}}\right)^2+\boldsymbol{\varsigma}'_T\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T\right]}{1+\frac{1}{r}\boldsymbol{\varsigma}'_T\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T}\right\} \\
&= \left(1+\frac{1}{r}\boldsymbol{\varsigma}'_T\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T\right)\left[1+\frac{1}{r+T}\left(T_r^{-1}\left(\breve{u}_{T+1}\right)\right)^2\right],
\end{aligned}$$

where we write $T_r^{-1}\left(\breve{u}_{T+1}\right) = \left(T_r^{-1}\left(u_{T+1}\right)-\mu_{Z.T+1}\right)/\sigma^*_{Z.T+1}$. Finally, by noting that the ratio of the determinants

$$\left|\boldsymbol{\Sigma}_{T+1}\right|/\left|\boldsymbol{\Sigma}_T\right| = 1-\boldsymbol{\rho}'_{T+1,T}\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T} = \sigma^2_{Z.T+1},$$

we write

$$\begin{aligned}
&\frac{\Gamma\left(\frac{r+T+1}{2}\right)\left(1+\frac{1}{r}\boldsymbol{\varsigma}'_{T+1}\boldsymbol{\Sigma}_{T+1}^{-1}\boldsymbol{\varsigma}_{T+1}\right)^{-(r+T+1)/2}}{\Gamma\left(\frac{r+T}{2}\right)\sqrt{r\pi\left|\boldsymbol{\Sigma}_{T+1}\right|/\left|\boldsymbol{\Sigma}_T\right|}\left(1+\frac{1}{r}\boldsymbol{\varsigma}'_T\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T\right)^{-(r+T)/2}} \\
=\ & \frac{\Gamma\left(\frac{r+T+1}{2}\right)}{\Gamma\left(\frac{r+T}{2}\right)\sqrt{\left(r+T\right)\pi}\sigma^*_{Z.T+1}}\left[1+\frac{1}{r+T}\left(T_r^{-1}\left(\breve{u}_{T+1}\right)\right)^2\right]^{-(r+T+1)/2} \\
=\ & t_{r+T}\left[T_r^{-1}\left(\breve{u}_{T+1}\right)\right]/\sigma^*_{Z.T+1}.
\end{aligned}$$

The result in the proposition then follows. ■

Using Proposition 3, the following result should now follow.

Corollary 2 *Let $\mathbf{X}_T$ be a claims vector with a Student-t copula structure as specified in (11). Then the next period's predicted claims can be expressed as*

$$E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right)=E_Z\left\{F_{T+1}^{-1}\left[T_r\left(\mu_{Z.T+1}+\sigma_{Z.T+1}^{*}Z\right)\right]\right\},$$

where the expectation on the right-hand side is computed for a standard univariate Student-t random variable Z with $(r+T)$ degrees of freedom.

Proof. From propositions 1 and 3, we find the conditional expectation as

$$\begin{aligned}&E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right)\\ =&\int_{-\infty}^{\infty}x_{T+1}\frac{t_{r+T}\left[\left(T_r^{-1}\left(u_{T+1}\right)-\mu_{t.T+1}\right)/\sigma_{Z.T+1}^{*}\right]}{t_r\left(T_r^{-1}\left(u_{T+1}\right)\right)}dF_{T+1}\left(x_{T+1}\right).\end{aligned}$$

Applying the transformation $z=\left(T_r^{-1}\left(u_{T+1}\right)-\mu_{t.T+1}\right)/\sigma_{Z.T+1}^{*}$,we find that

$$\begin{aligned}E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right)&=\int_{-\infty}^{\infty}F_{T+1}^{-1}\left[T_r\left(\mu_{Z.T+1}+\sigma_{Z.T+1}^{*}z\right)\right]\cdot t_{r+T}\left(z\right)dz\\ &=E_Z\left\{F_{T+1}^{-1}\left[T_r\left(\mu_{Z.T+1}+\sigma_{Z.T+1}^{*}Z\right)\right]\right\},\end{aligned}$$

which is the desired result. ■

The result above can further assist us in finding conditional expectations of a multivariate Student-t distributed random vector. This is usually not commonly found in standard textbooks on multivariate distributions.

Example 3.2.1: *The multivariate Student-t distribution*
Consider the special case where $\mathbf{X}_{T+1}$ has a multivariate Student-t distribution with common pairwise correlation of ρ and degrees of freedom r. This is similar to the Gaussian copula case in example 3.1.1. Thus we have

$$\boldsymbol{\Sigma}_T^{-1}=(1-\rho)^{-1}\left(\mathbf{I}_T-\frac{\rho}{1-\rho+\rho T}\mathbf{e}_T\mathbf{e}_T'\right)$$

so that

$$\mu_{Z.T+1}=\frac{\rho}{1-\rho+\rho T}\mathbf{e}_T'\varsigma_T=\frac{\rho}{1-\rho+\rho T}\sum_{i=1}^{T}\left(\frac{x_i-\mu}{\sigma}\right),$$

where we have assumed that all the marginals are also identically distributed as Student-t with location parameter μ and scale parameter σ. Note that the variance of the Student-t is not equal to square of the scale parameter. We see that the predicted claim (or conditional expectation) becomes

$$E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right)=E_Z\left[F_{T+1}^{-1}\left(T_r\left(\mu_{Z.T+1}+\sigma_{Z.T+1}^{*}Z\right)\right)\right]=\mu_{Z.T+1}\sigma+\mu,$$

and this expression can be written in the familiar credibility form

$$E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right)=\frac{\rho T}{1-\rho+\rho T}\overline{x}+\frac{1-\rho}{1-\rho+\rho T}\mu.$$

A further special situation is when $T=1$, the bivariate case in which case, we would have

$$E\left(X_2|X_1=x_1\right)=\rho x_1+\left(1-\rho\right)\mu.$$

Both these answers are what we expect them to be.■

4 Archimedean copulas

Consider the family of Archimedean copulas constructed by means of a single-valued generator function. This family of copulas has been widely used in the modelling of insurance and financial risks, primarily due to the numerous interesting properties they possess. One of these is the ease in which they are constructed, and a large variety of copulas belong to this family. Nelsen (1999) provides a discussion on constructing copulas belonging to the Archimedean family, as does Schweizer & Sklar (1983).

They are characterised by a generator, which is a single-valued function, thereby reducing the search for a high dimensional distribution function. Let ψ, an Archimedean generator, be a mapping from $[0,1]$ to $[0,1]$ satisfying the following three conditions: (1) $\psi\left(1\right)=0$; (2) ψ is monotonically decreasing; and (3) ψ is convex.

A copula C_T is called a T-dimensional Archimedean copula if there exists a generator function ψ such that

$$C_T\left(\mathbf{u}_T\right)=\psi^{-1}\left(\psi\left(u_1\right)+\cdots+\psi\left(u_n\right)\right), \tag{13}$$

where ψ^{-1} denotes the inverse function of the generator. Notice that if the first derivative, ψ', exists, then by condition (2), we must have $\psi'\leq 0$, and if the second derivative, ψ'', exists as well, then by condition (3), we must have $\psi''\leq 0$. For our purposes, we consider only Archimedean generators which are continuous and whose higher derivatives exist. Indeed, for C_T to be a proper copula, ψ^{-1} must be completely monotone so that if all derivatives exist, we must have

$$\left(-1\right)^k\frac{d^k\psi^{-1}\left(u\right)}{du^k}\geq 0 \text{ for } k=1,2,\ldots,T.$$

Examples of familiar Archimedean copulas and their respective generators are defined in Table 1 below.

The Archimedean family of copulas is a clear favourite for modelling dependent risks. The Frank copula, in particular, whose Archimedean generator has the form

$$\psi\left(u\right)=\log\left(\frac{\delta^u-1}{\delta-1}\right),\ \text{for } \delta\geq 0,$$

is a very popular choice for modelling dependent risks.

Table 1: Some Examples of One-Parameter Archimedean Copulas

Family	Generator $\psi(u)$	Parameter constraint	Copula form $C_T(u_1, ..., u_T)$
Independence	$-\log(u)$	not applicable	$\prod_{t=1}^{T} u_t$
Cook-Johnson	$u^{-\delta} - 1$	$\delta > 1$	$\left(\sum_{t=1}^{T} u_t^{-\delta} - T + 1\right)^{-1/\delta}$
Gumbel-Hougaard	$(-\log u)^{\delta}$	$\delta \geq 1$	$\exp\left\{-\left[\sum_{t=1}^{T}(-\log u_t)^{\delta}\right]^{-1/\delta}\right\}$
Frank	$\log\left(\frac{\delta^u - 1}{\delta - 1}\right)$	$\delta \geq 0$	$\frac{1}{\log\delta}\log\left[1 + \frac{\prod_{t=1}^{T}(\delta^{u_t} - 1)}{(\delta - 1)^{T-1}}\right]$
Joe	$-\log\left[1 - (1-u)^{\delta}\right]$	$\delta \geq 1$	$1 - \left\{1 - \prod_{t=1}^{T}\left[1 - (1-u_t)^{\delta}\right]\right\}^{1/\delta}$
Gumbel-Barnett	$\log[1 - \delta(1-u)]$	$0 < \delta \leq 1$	$\exp\left\{-\frac{1}{\delta}\left[1 - \prod_{t=1}^{T}(1 - \delta\log u_t)\right]\right\}$

It allows for a wide range of dependence, one of which is it allows up to maximal positive dependence depending on the choice of parameter value. It has very few parameters, usually just one, to describe dependence. This simplifies the calibration process, but sometimes, this limits the modelling of more complex dependence structures unlike the Gaussian and Student-t copulas discussed in the previous section. This copula was derived by Frank (1979). Genest (1987) further examined many of the interesting properties of this family of copulas including explicit forms for evaluating the Kendall's and Spearman's correlation coefficients. Frees, et al. (1996) used the Frank copula to fit bivariate distributions of the lifetimes of a husband and wife.

The subsequent proposition gives an explicit representation of the conditional density copula necessary to evaluate the predicted claims for the family of Archimedean copulas.

Proposition 4 *Let $\mathbf{X}_T$ be a claims vector with an Archimedean copula structure as specified in (13). The ratio of the densities of the Archimedean copula has a representation*

$$\frac{c_{T+1}\left(\mathbf{u}_{T+1}\right)}{c_T\left(\mathbf{u}_T\right)} = \frac{\psi^{-1(T+1)}\left[\psi\left(C_{T+1}\left(\mathbf{u}_{T+1}\right)\right)\right]}{\psi^{-1(T)}\left[\psi\left(C_T\left(\mathbf{u}_T\right)\right)\right]}\psi\prime\left(u_{T+1}\right)$$

where $\psi^{-1(T+1)}$ and $\psi^{-1(T)}$ are the $(T+1)$-th and T-th derivatives of ψ^{-1} and $\psi\prime$ is the derivative of the generator function ψ.

Proof. It can be shown that indeed

$$c_T\left(\mathbf{u}_T\right) = \psi^{-1(T)}\left(\sum_{t=1}^{T}\psi\left(u_t\right)\right)\prod_{t=1}^{T}\psi\prime\left(u_t\right)$$

and the result above should be immediate. ■

For better appreciation of the use of the tilting induced by this conditional density copula, we present example 4.1 below that considers the specific case of the Cook-Johnson copula. This type of a copula is sometimes called the *Clayton* copula in the literature.

Example 4.1: *The Cook-Johnson copula*
Consider the multivariate Cook-Johnson copula with

$$C_T\left(\mathbf{u}_T\right) = \left(\sum\nolimits_{k=1}^{T} u_k^{-\delta} - T + 1\right)^{-1/\delta}.$$

It can be shown, for example, see p. 225 of Cherubini, et al. (2004), that the density of the Cook-Johnson copula can be expressed as

$$c_T\left(\mathbf{u}_T\right) = \frac{\delta^T\Gamma\left(T+1/\delta\right)}{\Gamma\left(1/\delta\right)} \times \left(\prod\nolimits_{k=1}^{T} u_k^{-\delta-1}\right)\left(\sum\nolimits_{k=1}^{T} u_k^{-\delta} - T + 1\right)^{-(1/\delta)-T}.$$

This can also be derived by first noting that its generator $\psi(u) = u^{-\delta} - 1$ so that $\psi'(u) = -\delta u^{-\delta-1}$ and that $\psi^{-1}(u) = (1+u)^{-1/\delta}$. The T-th derivative of the inverse can be shown to be

$$\begin{aligned}
\psi^{-1(T)}(u) &= (-1)^T (1+u)^{-(1/\delta)-T} \prod_{k=1}^{T} \left(\frac{1}{\delta} + k\right) \\
&= (-1)^T (1+u)^{-(1/\delta)-T} \frac{\Gamma(T + 1/\delta)}{\Gamma(1/\delta)}.
\end{aligned}$$

This therefore simplifies the ratio of the Cook-Johnson density copula as follows

$$\frac{c_{T+1}(\mathbf{u}_{T+1})}{c_T(\mathbf{u}_T)} = \frac{\delta(T + (1/\delta))}{\left(\sum_{k=1}^{T} u_k^{-\delta} - T + 1\right)^{-(1/\delta)-T}} \times u_{T+1}^{-\delta-1} \left(\sum_{k=1}^{T+1} u_k^{-\delta} - T\right)^{-(1/\delta)-T-1}$$

and by defining the constant

$$\xi = \sum_{k=1}^{T} u_k^{-\delta} - T$$

which depends only on past values of $x_1, \ldots, x_T$, we can further re-write this ratio as

$$\frac{c_{T+1}(\mathbf{u}_{T+1})}{c_T(\mathbf{u}_T)} = \delta(T + (1/\delta))(1+\xi)^{(1/\delta)+T} \times u_{T+1}^{-\delta-1} \left(u_{T+1}^{-\delta} + \xi\right)^{-(1/\delta)-T-1}.$$

Now, applying the transformation $y = u_{T+1}^{-\delta}$, we could write

$$E(X_{T+1}|\mathbf{X}_T = \mathbf{x}_T) = (T + (1/\delta))(1+\xi)^{(1/\delta)+T} \int_1^{\infty} F_{T+1}^{-1}\left(y^{\delta}\right) \times (y+\xi)^{-(1/\delta)-T-1} dy. \tag{14}$$

Now consider the random variable Y with density function expressed as

$$f_Y(y) = \frac{((1/\delta) + T)(1+\xi)^{(1/\delta)+T}}{(y+\xi)^{(1/\delta)+T+1}} \text{ for } y > 1. \tag{15}$$

Indeed, this is the density function of a translated Pareto random variable with parameters $(1/\delta) + T$ and $1 + \xi$, where the magnitude of the translation is $-B$. The expression in (14) becomes

$$\begin{aligned}
E(X_{T+1}|\mathbf{X}_T = \mathbf{x}_T) &= \int_1^{\infty} F_{T+1}^{-1}\left(y^{\delta}\right) \times \frac{((1/\delta) + T)(1+\xi)^{(1/\delta)+T}}{(y+\xi)^{(1/\delta)+T+1}} dy \\
&= E_Y\left[F_{T+1}^{-1}\left(Y^{\delta}\right)\right],
\end{aligned}$$

where the unconditional expectation is evaluated under the translated Pareto with density given in (15).■

5 The case of the F-G-M copula

We now consider the Farlie-Gumbel-Morgenstern (FGM) family of copulas whose members have the form

$$C_T(\mathbf{u}_T) = \left\{ 1 + \sum_{s=2}^{T} \sum_{1\leq t_1<\ldots<t_s\leq T} \alpha_{t_1,\ldots,t_s} \prod_{j=1}^{s} \left[1 - u_{t_j}\right] \right\} \prod_{t=1}^{T} u_t, \tag{16}$$

where the parameters $\alpha_{t_1,\ldots,t_s}$ must satisfy the following conditions:

$$1 + \sum_{s=2}^{m} \sum_{k_1\leq t_1<\ldots<t_s\leq k_m} \left(\alpha_{t_1,\ldots,t_s} \prod_{j=1}^{s} \zeta_{t_j} \right) \geq 0$$

where $\left|\zeta_{t_j}\right| \leq 1$ (in fact, each ζ_{t_j} is either a $+1$ or a -1) and $1 \leq k_1 < k_m \leq T$ successively for $m = 2, \ldots, T$. It is clear that the total number of parameters $\alpha_{t_1,\ldots,t_s}$ is $2^T - T - 1$. For further discussion of constraints on these parameters, we refer to Mari and Kotz (2001). We shall implicitly assume, in the foregoing discussion, that these parameter constraints hold. For a discussion of various properties of the FGM copula, see Johnson and Kotz (1975, 1977), Cambanis (1977), Mari and Kotz (2001) and Nelsen (1999).

To illustrate the FGM copula, for example, consider the bivariate case where $T = 2$, then we have

$$C_2(\mathbf{u}_2) = u_1 u_2 \left[1 + \alpha_{12}(1 - u_1)(1 - u_2)\right],$$

where the sole dependence parameter α_{12} satisfies $|\alpha_{12}| < 1$. In the trivariate case where $T = 3$, we would have

$$C_3(\mathbf{u}_3) = u_1 u_2 u_3 \left[\begin{array}{l} 1 + \alpha_{12}(1 - u_1)(1 - u_2) + \alpha_{13}(1 - u_1)(1 - u_3) \\ +\alpha_{23}(1 - u_2)(1 - u_3) + \alpha_{123}(1 - u_1)(1 - u_2)(1 - u_3) \end{array} \right],$$

where the parameters α_{12}, α_{13}, α_{23} and α_{123} must satisfy

$$1 + \alpha_{12}\zeta_1\zeta_2 + \alpha_{13}\zeta_1\zeta_3 + \alpha_{23}\zeta_2\zeta_3 + \alpha_{123}\zeta_1\zeta_2\zeta_3 \geq 0$$

with $\zeta_k = +1$ or -1, $k = 1, 2, 3$.

An interesting representation of the FGM copula can be made by defining the polynomial term

$$P_T(\mathbf{u}_T) = \left\{ 1 + \sum_{s=2}^{T} \sum_{1\leq t_1<\ldots<t_s\leq T} \alpha_{t_1,\ldots,t_s} \prod_{j=1}^{s} \left[1 - u_{t_j}\right] \right\} \tag{17}$$

so that the FGM copula becomes

$$C_T(\mathbf{u}_T) = P_T(\mathbf{u}_T) \prod_{t=1}^{T} u_t.$$

This specification of the FGM copula allows us to produce a succinct representation of the density of the copula. By differentiating T times, the density of the FGM copula in (16) can be expressed as

$$c_T(\mathbf{u}_T) = 1 + \sum_{s=2}^{T} \sum_{1 \leq t_1 < \ldots < t_s \leq T} \alpha_{t_1,\ldots,t_s} \prod_{j=1}^{s} \left[1 - 2u_{t_j}\right] = P_T(2\mathbf{u}_T) \tag{18}$$

where $2\mathbf{u}_T = (2u_1, \ldots, 2u_T)$. For the additional dimension $T+1$ then, we can also write

$$c_{T+1}(\mathbf{u}_{T+1}) = P_{T+1}(2\mathbf{u}_{T+1}).$$

Because $C_T(\mathbf{u}_T) = C_{T+1}(\mathbf{u}_T, 1)$, it follows therefore that

$$P_T(2\mathbf{u}_T) = P_{T+1}(2\mathbf{u}_T, 1)$$

where the right hand side is evaluated at $u_{T+1} = 1$. Thus, we write the ratio as

$$\frac{c_{T+1}(\mathbf{u}_{T+1})}{c_T(\mathbf{u}_T)} = \frac{P_{T+1}(2\mathbf{u}_{T+1})}{P_T(2\mathbf{u}_T)} = \frac{P_{T+1}(2\mathbf{u}_{T+1})}{P_{T+1}(2\mathbf{u}_T, 1)}. \tag{19}$$

Another useful expression is to re-write

$$c_{T+1}(\mathbf{u}_{T+1}) = P_{T+1}(2\mathbf{u}_{T+1}) = P_T(2\mathbf{u}_T) + D_{T+1}(\mathbf{u}_T)(1 - 2u_{T+1}) \tag{20}$$

where

$$D_{T+1}(\mathbf{u}_T) = \sum_{s=1}^{T} \sum_{1 \leq t_1 < \ldots < t_s \leq T} \alpha_{t_1,\ldots,t_s,T+1} \prod_{j=1}^{s} \left[1 - 2u_{t_j}\right]$$

which involves only terms from the observed vector $\mathbf{u}_T$. Therefore, we have the following result giving a representation of the conditional expectation.

Proposition 5 *Let $\mathbf{X}_T$ be a claims vector with a Farlie-Gumbel-Morgenstern copula structure as specified in (16). Then the next period's predicted claims can be expressed as*

$$\begin{aligned} E(X_{T+1}|\mathbf{X}_T = \mathbf{x}_T) &= \int_{-\infty}^{\infty} x_{T+1} \cdot \frac{P_{T+1}(2\mathbf{u}_{T+1})}{P_{T+1}(2\mathbf{u}_T, 1)} dF_{T+1}(x_{T+1}) \\ &= \int_0^1 F_{T+1}^{-1}(u_{T+1}) \cdot \frac{P_{T+1}(2\mathbf{u}_T, 2u_{T+1})}{P_{T+1}(2\mathbf{u}_T, 1)} du_{T+1} \\ &= E_U\left[F_{T+1}^{-1}(U) \cdot \frac{P_{T+1}(2\mathbf{u}_T, 2U)}{P_{T+1}(2\mathbf{u}_T, 1)}\right] \end{aligned}$$

where U is a Uniform on $(0,1)$ random variable.

Proof. Using Proposition 1 and the expressions in (19) and 20, we obtain

$$E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right)=\int_{-\infty}^{\infty}x_{T+1}\cdot\frac{P_{T+1}\left(2\mathbf{u}_{T+1}\right)}{P_{T+1}\left(2\mathbf{u}_T,1\right)}dF_{T+1}\left(x_{T+1}\right).$$

Applying the transformation $u_{T+1}=F_{T+1}\left(x_{T+1}\right)$ gives the desired result. ■

By applying integration by parts, we can rewrite the predicted claims as

$$\begin{aligned}
& E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right)\\
= & \int_{-\infty}^{\infty}x_{T+1}\cdot\frac{P_{T+1}\left(2\mathbf{u}_{T+1}\right)}{P_{T+1}\left(2\mathbf{u}_T,1\right)}dF_{T+1}\left(x_{T+1}\right)\\
= & E\left(X_{T+1}\right)-\frac{D_{T+1}\left(\mathbf{u}_T\right)}{P_T\left(\mathbf{u}_T\right)}\int_{-\infty}^{\infty}F_{T+1}\left(x_{T+1}\right)\left[1-F_{T+1}\left(x_{T+1}\right)\right]dx_{T+1}. \qquad (21)
\end{aligned}$$

The expression above allows us to derive closed form expressions for the conditional expectations with various assumed marginal distributions. First, consider the bivariate case.

Example 5.1: *The bivariate FGM copula*
Consider the situation where $T=1$, then it becomes easy to show that

$$E\left(X_2|X_1=x_1\right)=E\left(X_2\right)+\alpha_{12}\left(2F_1\left(x_1\right)-1\right)\int_{-\infty}^{\infty}F_2\left(x_2\right)\left[1-F_2\left(x_2\right)\right]dx_2.$$

where α_{12} denotes the sole parameter describing the dependence between X_1 and X_2.■

Observe that in (21), although the terms $D_{T+1}\left(\mathbf{u}_T\right)$ and $P_T\left(\mathbf{u}_T\right)$ are cumbersome looking, they are simply polynomial in the powers of the elements of $\mathbf{u}_T$ and are straightforward to evaluate. In the illustrative examples below, we therefore emphasise evaluating the term $\int_{-\infty}^{\infty}F_{T+1}\left(x_{T+1}\right)\left[1-F_{T+1}\left(x_{T+1}\right)\right]dx_{T+1}$ in the proposition.

Example 5.2: *The exponential marginal*
In the case where X_{T+1} has exponential distribution with mean parameter $1/\upsilon$ so that its distribution function $F_{T+1}\left(x_{T+1}\right)=1-\exp\left(-\upsilon x_{T+1}\right)$, then it is easy to see that

$$\int_0^{\infty}F_{T+1}\left(x_{T+1}\right)\left[1-F_{T+1}\left(x_{T+1}\right)\right]dx_{T+1}=\frac{1}{2\upsilon}.$$

In this case, the predicted claim reduces to

$$E\left(X_{T+1}|\mathbf{X}_T=\mathbf{x}_T\right)=\frac{1}{\upsilon}\left[1-\frac{D_{T+1}\left(\mathbf{u}_T\right)}{2P_T\left(\mathbf{u}_T\right)}\right].■$$

Example 4.3: *The Weibull marginal*
In the case where X_{T+1} has a Weibull distribution with distribution function $F_{T+1}(x_{T+1}) = 1 - \exp\left(-x_{T+1}^{\kappa}\right)$, then it can be shown that

$$\int_0^\infty F_{T+1}(x_{T+1})\left[1 - F_{T+1}(x_{T+1})\right] dx_{T+1} = \left(1 - 2^{-1/\kappa}\right)\Gamma\left[1 + (1/\kappa)\right],$$

where $\Gamma(\cdot)$ denotes the gamma function. Thus, we obtain the formula

$$E\left(X_{T+1} | \mathbf{X}_T = \mathbf{x}_T\right) = \Gamma\left(1 + (1/\kappa)\right)\left[1 - \left(1 - 2^{-1/\kappa}\right)\frac{D_{T+1}(\mathbf{u}_T)}{P_T(\mathbf{u}_T)}\right]. \blacksquare$$

Example 4.4: *The Pareto marginal*
In the case where X_{T+1} has a Pareto distribution with distribution function $F_{T+1}(x_{T+1}) = 1 - \left(\dfrac{\lambda}{\lambda + x_{T+1}}\right)^{\beta}$, then it can be shown that

$$\int_0^\infty F_{T+1}(x_{T+1})\left[1 - F_{T+1}(x_{T+1})\right] dx_{T+1} = \frac{\beta}{\lambda(\beta+1)(2\beta+1)},$$

which is a straightforward integration. This then gives

$$E\left(X_{T+1} | \mathbf{X}_T = \mathbf{x}_T\right) = \frac{\lambda}{\beta - 1} - \frac{\beta D_{T+1}(\mathbf{u}_T)}{\lambda(\beta+1)(2\beta+1)P_T(\mathbf{u}_T)}. \blacksquare$$

The FGM family of copulas is suitable for modelling certain dependence structures. It is noted that most other families of multivariate copulas have only one or at most a few parameters. This essentially implies that the correlation structure is identical for all pairs of risks. This would not be realistic as one would expect risks that are "further apart" to exhibit weaker forms of dependence. As such, the suitability of these families for modelling risks where distances are involved, e.g. physical distance or time, is much reduced. The FGM copula is a notable exception to this. It can be easily observed from the form of its copula that it is possible to uniquely assign a dependence parameter for each pair or group of risks. This allows it to model more complicated dependence structures should the need arise. Another important advantage is that the conditional expectation can be solved analytically as we have demonstrated.

The downside of this feature is the calibration process would be much more tedious with the large number of parameters involved. Another disadvantage of the FGM copula is the limitation on its parameter values in order for the resulting density function to remain valid. This limits the dependence structure it can model to cases where there is only weak dependence.

Table 2: Relationship between Kendall's τ and Parameter of each Copula

Copula	Relationship to Kendall's τ
Gaussian	$\tau = \dfrac{2}{\pi} \arcsin \rho$
Student-t	$\tau = \dfrac{2}{\pi} \arcsin \rho$
Cook-Johnson	$\tau = \dfrac{\delta}{\delta + 2}$
FGM	$\tau = \dfrac{2\alpha}{9}$

6 Illustration

Here in this section, we consider some illustration on the effects of tilting induced by the conditional density copula. To do so, we provide visual illustrations of the effects of the copula density ratio because this ratio is the one affecting the change of the probability measure.

For ease of illustration, we consider the case whereby a single period's claims experience, X_1, has been observed. We are therefore looking at the copula density ratio, $c_2\left(u_2\right)/c_1\left(u_1\right)$. For our purposes, X_2 is assumed to have the Pareto distribution with density

$$f_{X_2}\left(x_2\right) = \frac{\beta\lambda^{\beta}}{\left(\lambda + x_2\right)^{\beta+1}}, \text{ for } x_2 > 0.$$

In the illustrations that follow, we assume the Pareto parameters take on the values of $\lambda = 1000$ and $\beta = 3$ so that $E\left(X_{T+1}\right) = \lambda/\left(\beta - 1\right) = 500$.

In Figure 1, we present four graphs of copula density ratios, pertaining to the Gaussian, Student-t, Cook-Johnson and the FGM copulas respectively. For the case of the Student-t copula, we have, in addition, assumed that the underlying Student-t distribution has 10 degrees of freedom. For each copula, we examined the effects of "high", "moderate" and "low" levels of dependence on the copula density ratio. Levels of dependence are measured in terms Kendall's τ. Table 2 summarises the relationship between the Kendall's τ and the dependence parameter of each copula considered. In particular, the "low", "moderate" and "high" levels relate to Kendall's τ of 0.2, 0.5 and 0.8 respectively. In the case of the FGM copula that only allows up to weak levels of dependence, we graph the cases of "very low" and "low" which corresponds to Kendall's τ of 0.1 and 0.2 respectively. The case of independence, in the figure, is represented by a horizonal line at 1. We have also assumed the observed claim $X_1 = 1000$.

Examining Figure 1, we easily observe that greater weightage is placed on past claims experience, X_1, with higher levels of positive dependence. This is observed for all the four copulas considered here. Where there is high positive dependence, a maximum point is observed near the observed value of X_1. This is especially true for the Gaussian and Student-t copulas. All these results are as expected.

In Figure 2, we present another four graphs of copula density ratios for the same copulas. Now this time, we assume a "moderate" level of dependence ("low" for the case of the FGM copula) and allow the value of past claims experience, X_1, to vary. We considered the case $X_1 = 200$ corresponding to a "low" observed amount, $X_1 = 500$ corresponding to an "average" observed amount, and $X_1 = 1000$ corresponding to a "high" observed amount. All other assumptions remain identical to that assumed in Figure 1.

Examining Figure 2, we observe a similar pattern of having a greater weightage placed on past experience, X_1. In addition, lower weightage is now placed on values that are "further away" from X_1. This occurs for all the different assumed values of X_1. Indeed, for the cases of the Gaussian and Student-t copulas, maximum points are even observed near these observed values.

Collectively, these results reaffirm those already observed from classical credibility models that gave linearised solutions. Greater weightage is placed upon past claims experience as the level of positive dependence between risks increases. Although in our case, the solutions will not be linear in general, this general observation still holds.

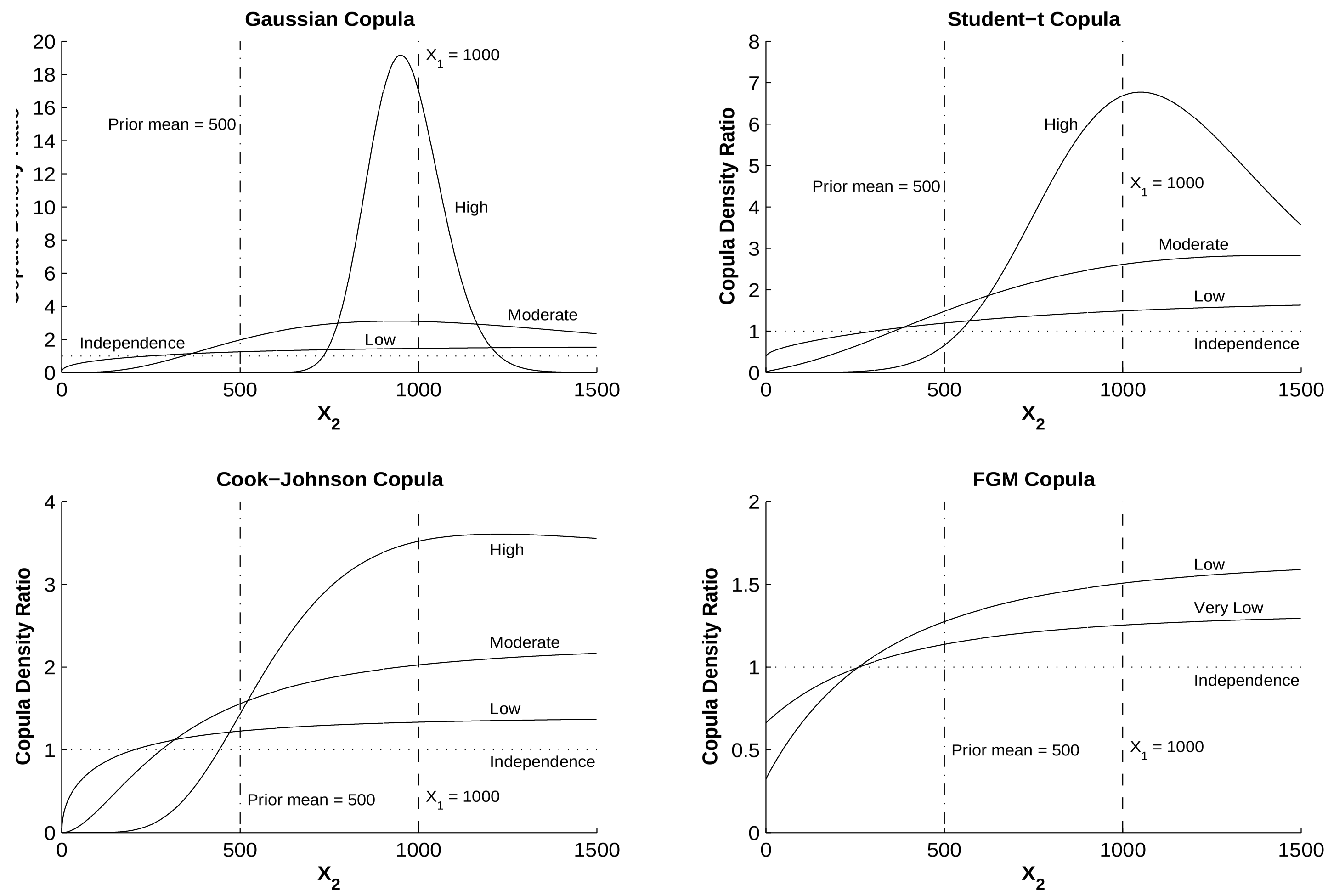

Figure 1: The resulting copula density ratios after observing $X_1 = 1000$, for different copulas and for different levels of Kendall's tau coefficients (low, moderate, high)

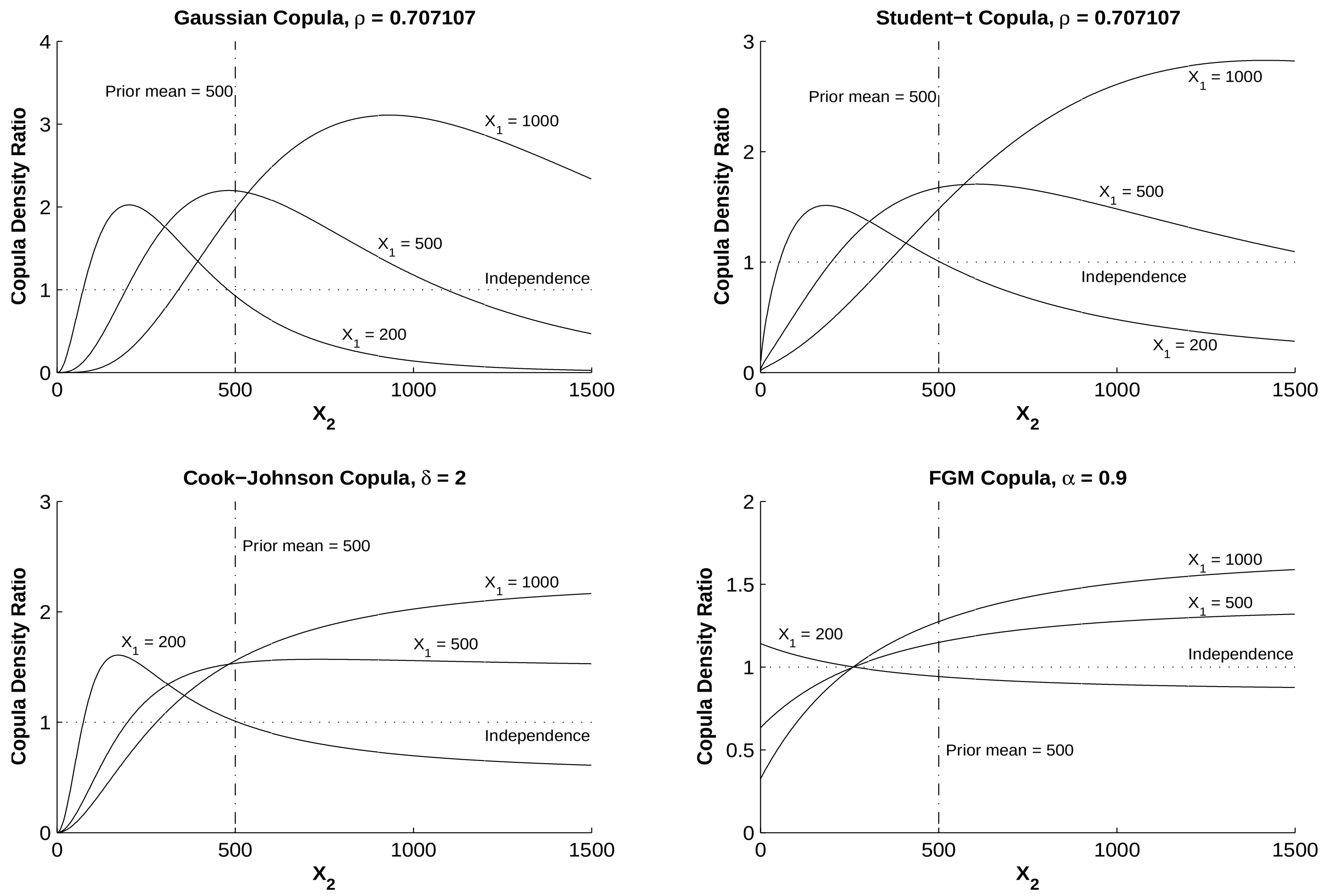

Figure 2: The resulting copula density ratios for a fix Kendall's tau coefficient of 0.5, for different copulas and for varying past claims X_1

7 Extension

In a previous work, Yeo and Valdez (2004) addressed a simultaneous dependence of claims across individuals for a fixed time period and across time periods for every individual. This was accomplished by introducing the notion of a common effect affecting all individuals and another common effect affecting each individual over time. Specifically, a random variable Λ was used to describe the common dependence across the insured individuals, and for a fixed individual j, the random variable Θ_j was used to describe the common dependence across the time periods. In statistics, such dependence has sometimes been called "common effects", "latent or unobservable variables", and the term "frailty" variables is more often used in the biostatistics and survival models literature. See, for example, Vaupel, et al. (1979) and Oakes (1989).

In the subsequent subsections, we summarise the results of this work and show how the two-level common effects specification, together with copulas, help to extend the idea of simultaneous dependence among individual risks and across time period.

7.1 Credibility and Common Effects

We named our model the two-level common effects model. Our primary interest is again to predict the next period's claim for each individual based on all the observed claims $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_J$, where $\mathbf{X}_j = (X_{j,1}, X_{j,2}, \ldots, X_{j,T})'$. This is given by $E(X_{j,T+1}|\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_I)$ since it is well known to be the best predictor of $X_{j,T+1}$, based on all the observed claims $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_J$, in the sense of the "mean squared prediction error". With some reasonable assumptions made, we showed the following general result holds:

$$\begin{aligned} & f_{X_{j,T+1}|\mathbf{X}}(x_{j,T+1}|\mathbf{x}) \\ = & \int\int\cdots\int f_{X_{j,T+1}|\Theta_j,\Lambda}(x_{j,T+1}|\theta_j,\lambda)\, f_{\boldsymbol{\Theta},\Lambda|\mathbf{X}}(\boldsymbol{\theta},\lambda|\mathbf{x}) \left(\prod_{i=1}^{J} d\theta_i\right) d\lambda, \end{aligned}$$

where $\mathbf{X} = (\mathbf{X}'_1, \ldots, \mathbf{X}'_J)'$ and $\Theta = (\Theta_1, \Theta_2, \ldots, \Theta_J)'$. With the conditional density, $X_{j,T+1}|\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_J$ now known, we can compute $E(X_{j,T+1}|\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_J)$.

With this general result, we then considered the credibility premium, i.e. the conditional expectation $E(X_{j,T+1}|\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_J)$, for the case where the common effects are Normally distributed. Specifically, we assumed that

$$\begin{aligned} X_{j,t}|\theta_j,\lambda &\sim \text{Normal}\left(\theta_j+\lambda, \sigma_x^2\right), \text{ for } j=1,2,\ldots,J,\ t=1,2,\ldots,T, \\ \Theta_j &\sim \text{Normal}\left(\mu_\theta, \sigma_\theta^2\right), \text{ for } j=1,2,\ldots,J \text{ and} \\ \Lambda &\sim \text{Normal}\left(\mu_\lambda, \sigma_\lambda^2\right). \end{aligned}$$

Without loss of generality, we considered $j = 1$. Based on these assumptions, we were then able to derive an explicit expression for the credibility premium and found that it takes the form

$$E(X_{1,T+1}|\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_I) = w_1\overline{X}_1 + w_{j\neq 1}\overline{\overline{X}}_{j\neq 1} + w_{\theta,\lambda}(\mu_\theta + \mu_\lambda),$$

where $\overline{X}_1 = \frac{1}{T}\sum_t X_{1,t}$ is the observed sample mean of the individual 1 and $\overline{\overline{X}}_{j\neq 1} = \frac{1}{(J-1)T}\sum_{j\neq 1}\sum_t X_{j,t}$ is the observed sample mean of the rest of the individuals. The credibility premium can be seen to be the weighted average of these observed sample means together with the aggregated means of the common effects. The weights have the following expressions:

W1. the weight attached to individual's own experience:

$$w_1 = \frac{T\left[\left(\sigma_\lambda^2 J + \sigma_\theta^2\right)\sigma_\theta^2 T + \sigma_x^2\left(\sigma_\lambda^2 + \sigma_\theta^2\right)\right]}{\left[\left(\sigma_\lambda^2 J + \sigma_\theta^2\right)T + \sigma_x^2\right]\left(\sigma_\theta^2 T + \sigma_x^2\right)};$$

W2. the weight attached to the rest of the group's experience:

$$w_{j\neq 1} = \frac{T\left(J-1\right)\sigma_\lambda^2\sigma_x^2}{\left[\left(\sigma_\lambda^2 J + \sigma_\theta^2\right)T + \sigma_x^2\right]\left(\sigma_\theta^2 T + \sigma_x^2\right)};\text{ and}$$

W3. the weight attached to prior beliefs:

$$w_{\theta,\lambda} = \frac{\sigma_x^2\left(\sigma_\theta^2 T + \sigma_x^2\right)}{\left[\left(\sigma_\lambda^2 J + \sigma_\theta^2\right)T + \sigma_x^2\right]\left(\sigma_\theta^2 T + \sigma_x^2\right)}.$$

These credibility premium and weights were found to possess properties expected of a realistic credibility factor, such as unbiasedness and $w_1 + w_{j\neq 1} + w_{\theta,\lambda} = 1$. The numerous asymptotic properties of this credibility premium formula were also found to be intuitively appealing.

Finally, we compared the credibility premium based on our model and that of the ordinary Bayesian Normal model using simulated observations that assumed the two-level common effects model assumptions held in reality. We found that the ordinary Bayesian Normal model tended to overstate the credibility premium from its true value.

The two-level common effects model does have its shortcomings. For one, it is not tractable in general due to some of the complicated integration operations involved. See the appendix of Yeo and Valdez (2004) for an example. As such, explicit solutions may not be available in many cases. In addition, the types of dependence structures allowed for via common effects may also be limited. There is therefore a need to consider other constructions that allow for a broader spectrum of dependence structures. To address these considerations, the alternative construction using copulas was considered in this work.

7.2 Model with simultaneous dependence

Consider a portfolio of J insurance contracts where each individual contract has observable claims vector of T periods denoted by

$$\mathbf{X}_j' = \left(X_{j,1}, X_{j,2}, \ldots, X_{j,T}\right) \text{ for } j = 1, 2, \ldots, J$$

and we denote the observable claims vector for all the contracts by

$$\mathbf{X}' = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_J).$$

One can also view $\mathbf{X}$ as a random matrix with a total of $J \times T$ single valued random variables. Similar to before, small letters will refer to the observed claim amounts. In the previous sections, we investigated the case where for a single individual risk j, there is dependence of claims across time periods. This section suggests how to extend these constructions when there is also dependence across the individuals.

We model the dependence structure across individuals using common effects similar to the construction suggested in Yeo and Valdez (2004) and briefly outlined in the previous section. Conditional on Θ, we shall assume that $\mathbf{X}_j$'s are independent so that in effect, we have

$$f_{\mathbf{X}}(\mathbf{x}) = \int_{-\infty}^{\infty} f_{\Theta}(\theta) f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta)\, d\theta = \int_{-\infty}^{\infty} f_{\Theta}(\theta) \prod_{j=1}^{J} f_{\mathbf{X}_j|\Theta}(\mathbf{x}_j|\theta)\, d\theta. \tag{22}$$

We also model the dependence structure over time periods, $t = 1, \ldots, T$, for the same individual j with a copula function denoted C. This is similar in construction as discussed in section 2. For convenience, we may assume the same copula C applies to all the individuals so that we would have

$$F_{\mathbf{X}_j|\Theta}(\mathbf{x}_j|\theta) = C\left(\mathbf{u}_{j|\theta}\right), \text{ for } j = 1, \ldots, J,$$

where we have denoted $\mathbf{u}_{j|\theta} = \left(u_{j,1|\theta}, u_{j,2|\theta}, \ldots, u_{j,T|\theta}\right)$ and $u_{j,t|\theta} = F_{X_{j,t}|\Theta}(x_{j,t}|\theta)$.

Now taking partial derivatives with respect to $x_{j,1}, x_{j,2}, \ldots, x_{j,T}$, we obtain the joint conditional density

$$f_{\mathbf{X}_j|\Theta}(\mathbf{x}_j|\theta) = c\left(\mathbf{u}_{j|\theta}\right) \prod_{t=1}^{T} f_{X_{j,t}|\Theta}(x_{j,t}|\theta). \tag{23}$$

Substituting (23) back to (22), we then obtain

$$f_{\mathbf{X}}(\mathbf{x}) = \int_{-\infty}^{\infty} f_{\Theta}(\theta) \prod_{j=1}^{J} \left(c\left(\mathbf{u}_{j|\theta}\right) \prod_{t=1}^{T} f_{X_{j,t}|\Theta}(x_{j,t}|\theta) \right) d\theta. \tag{24}$$

We are interested in computing the predicted claim $E\left(X_{j,T+1}|\mathbf{X} = \mathbf{x}\right)$ for each contract $j = 1, \ldots, J$. Denote the random vector

$$\mathbf{X}_j^{(T+1)\prime} = \left(X_{j,1}, X_{j,2}, \ldots, X_{j,T}, X_{j,T+1}\right) \text{ for } j = 1, 2, \ldots, J,$$

augmented by the next period's observable claim, and the corresponding overall random vector

$$\mathbf{X}^{(j,T+1)\prime} = \left(\mathbf{X}_1, \ldots, \mathbf{X}_j^{(T+1)}, \ldots, \mathbf{X}_J\right) \text{ for } j = 1, 2, \ldots, J.$$

Then, following similar steps to that one described in developing formula (24), we find that

$$\begin{aligned} f_{\mathbf{X}^{(j,T+1)}}\left(\mathbf{x}^{(j,T+1)}\right) &= \int_{-\infty}^{\infty} f_{\Theta}(\theta)\, c\left(\mathbf{u}_{j|\theta}\right) \prod_{t=1}^{T+1} f_{X_{j,t}|\Theta}\left(x_{i,t}|\theta\right) \\ &\quad \times \prod_{\substack{i=1 \\ i\neq j}}^{J} \left(c\left(\mathbf{u}_{i|\theta}\right) \prod_{t=1}^{T} f_{X_{i,t}|\Theta}\left(x_{i,t}|\theta\right) \right) d\theta. \end{aligned} \tag{25}$$

From the above intermediate results, the predictive claims density can then be expressed as the ratio of (25) and (24)

$$f_{X_{j,T+1}|\mathbf{X}}\left(x_{j,T+1}|\mathbf{x}\right) = \frac{f_{\mathbf{X}^{(j,T+1)}}\left(\mathbf{x}^{(j,T+1)}\right)}{f_{\mathbf{X}}(\mathbf{x})}$$

so that our predicted claims can be evaluated using the conditional expectation

$$\begin{aligned} E\left(X_{j,T+1}|\mathbf{X}=\mathbf{x}\right) &= \int_{-\infty}^{\infty} x_{j,T+1} f_{X_{j,T+1}|\mathbf{X}}\left(x_{j,T+1}|\mathbf{x}\right) dx_{j,T+1} \\ &= \int_{-\infty}^{\infty} x_{j,T+1} \frac{f_{\mathbf{X}^{(j,T+1)}}\left(\mathbf{x}^{(j,T+1)}\right)}{f_{\mathbf{X}}(\mathbf{x})} dx_{j,T+1}. \end{aligned} \tag{26}$$

Because of the integration, it is no longer possible to isolate the marginal density of $X_{j,T+1}$ in order to be able to write this as an unconditional expectation. It will be an interesting challenge in the future to explore how one might be able to evaluate this conditional expectation and be able to construct explicit results. We leave this work for future research.

8 Concluding statements

Credibility theory has long been considered a milestone in the actuarial and insurance literature. Even today, many insurance companies adopt the methodology in evaluating experience-rated premium. The fundamental idea is to predict the next period's claims, given the history of all available claims. This history could include the previous claims records of the individual risks (or a group of considered homogeneous risks) and those of everyone else in the insurance portfolio. Credibility theory then is all about deriving the optimal mix of the claims experience of the individual and the collective group.

Classical credibility models provide for predictive claims in linear form. For example, the Bühlmann and the Bühlmann-Straub credibility models express the next period's claims as a weighted average of historical claims arising from each group's own experience and the entire portfolio's experience, with the weight that is attached to the own experience reflecting a credibility factor. These classical models typically assume claim independence, or at best conditional independence.

This paper provides further extension to the notion of predicting the next period's claims by relaxing these independence assumptions. Claims dependence structure is specified according to copula models which, in the recent past, have been a widely accepted statistical tool for handling non-independence. This paper extends the models offered by Frees and Wang (2005) and Yeo and Valdez (2004), which respectively used the frameworks of copula models and common effects. Here we find that the predictive claim can be expressed as an expectation under a new probability measure (in effect, a change of measure) that reflects the ratio of the densities of the copulas relating to the historical claims. This ratio of densities can also be interpreted as a "re-weighting" of the marginal density function after observing one or more of the claims. From the various figures shown in this paper, we observe that heavier weightage is placed on the prior mean for cases of weak positive dependence. For cases of strong positive dependence, heavier weightage is placed on the claims history.

We examine this expectation for different families of copulas and in several instances, we find we are able to explicitly express the predictive claim in terms of historical claims, and as we suspect, these predictive claims are in general no longer linear in form. There are certain cases where the usual linear form prevails, as is shown via several examples. However, these cases require some additional assumptions to be made.

Finally, we also suggest a further extension to the use of copula models. In particular, it may possible to use it to model the dependence structure described in the two-level common effects model, i.e. dependence between individual risks as well as dependence over time period. We leave this important work for future research.

Appendix A: Inverse of the partitioned correlation matrix

Here in this appendix, we show how we can simplify the inverse of the partitioned correlation matrix $\mathbf{\Sigma}_{T+1}$ used in Section 3 for the case of the Gaussian and t copulas. First, notice that we can express this correlation as the partition matrix as

$$\mathbf{\Sigma}_{T+1} = \begin{pmatrix} \mathbf{\Sigma}_T & \boldsymbol{\rho}_{T+1,T} \\ \boldsymbol{\rho}'_{T+1,T} & 1 \end{pmatrix}$$

where $\boldsymbol{\rho}'_{T+1,T} = \left(\rho_{1,T+1}, \rho_{2,T+1}, \ldots, \rho_{T,T+1}\right)$ which refers to the vector of correlations of X_{T+1} with each element of $\mathbf{X}_T$. Using the inverse of partitioned matrix, we find that its inverse can be written as

$$\mathbf{\Sigma}_{T+1}^{-1} = \begin{pmatrix} \boldsymbol{\varepsilon}_{T,T} & \boldsymbol{\varepsilon}_{T+1,T} \\ \boldsymbol{\varepsilon}'_{T+1,T} & \varepsilon_{T+1,T+1} \end{pmatrix}$$

where

$$\begin{aligned}
\boldsymbol{\varepsilon}_{T,T} &= \mathbf{\Sigma}_T^{-1} + \mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}\left(1-\boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}\right)^{-1}\boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}, \\
\varepsilon_{T+1,T+1} &= \left(1-\boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}\right)^{-1}, \\
\boldsymbol{\varepsilon}_{T+1,T} &= -\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}\left(1-\boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}\right)^{-1} \text{ and} \\
\boldsymbol{\varepsilon}'_{T+1,T} &= -\left(1-\boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}\right)^{-1}\boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}.
\end{aligned}$$

Therefore, using these results, we can write for the case of the Gaussian copula,

$$\begin{aligned}
& \boldsymbol{\varsigma}'_{T+1}\mathbf{\Sigma}_{T+1}^{-1}\boldsymbol{\varsigma}_{T+1} \\
= & \left(\boldsymbol{\varsigma}'_T, \Phi^{-1}\left(u_{T+1}\right)\right)\begin{pmatrix} \boldsymbol{\varepsilon}_{T,T} & \boldsymbol{\varepsilon}_{T+1,T} \\ \boldsymbol{\varepsilon}'_{T+1,T} & \varepsilon_{T+1,T+1} \end{pmatrix}\begin{pmatrix} \boldsymbol{\varsigma}_T \\ \Phi^{-1}\left(u_{T+1}\right) \end{pmatrix} \\
= & \left(\boldsymbol{\varsigma}'_T\boldsymbol{\varepsilon}_{T,T} + \Phi^{-1}\left(u_{T+1}\right)\boldsymbol{\varepsilon}'_{T+1,T}, \boldsymbol{\varsigma}'_T\boldsymbol{\varepsilon}_{T+1,T} + \Phi^{-1}\left(u_{T+1}\right)\varepsilon_{T+1,T+1}\right)\begin{pmatrix} \boldsymbol{\varsigma}_T \\ \Phi^{-1}\left(u_{T+1}\right) \end{pmatrix} \\
= & \boldsymbol{\varsigma}'_T\boldsymbol{\varepsilon}_{T,T}\boldsymbol{\varsigma}_T + \Phi^{-1}\left(u_{T+1}\right)\boldsymbol{\varepsilon}'_{T+1,T}\boldsymbol{\varsigma}_T + \boldsymbol{\varsigma}'_T\boldsymbol{\varepsilon}_{T+1,T}\Phi^{-1}\left(u_{T+1}\right) + \Phi^{-1}\left(u_{T+1}\right)\varepsilon_{T+1,T+1}\Phi^{-1}\left(u_{T+1}\right).
\end{aligned}$$

Substituting back $\boldsymbol{\varepsilon}_{T,T}$, $\varepsilon_{T+1,T+1}$, $\boldsymbol{\varepsilon}_{T+1,T}$ and $\boldsymbol{\varepsilon}'_{T+1,T}$, we obtain

$$\begin{aligned}
& \boldsymbol{\varsigma}'_{T+1}\mathbf{\Sigma}_{T+1}^{-1}\boldsymbol{\varsigma}_{T+1} \\
= & \boldsymbol{\varsigma}'_T\mathbf{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T + \boldsymbol{\varsigma}'_T\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}\left(1-\boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}\right)^{-1}\boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T \\
& -\Phi^{-1}\left(u_{T+1}\right)\left(1-\boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}\right)^{-1}\boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T \\
& -\boldsymbol{\varsigma}'_T\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}\left(1-\boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}\right)^{-1}\Phi^{-1}\left(u_{T+1}\right) \\
& +\Phi^{-1}\left(u_{T+1}\right)\left(1-\boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}\right)^{-1}\Phi^{-1}\left(u_{T+1}\right) \\
= & \boldsymbol{\varsigma}'_T\mathbf{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T + \boldsymbol{\varsigma}'_T\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}\left(1-\boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}\right)^{-1}\boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T \\
& -2\Phi^{-1}\left(u_{T+1}\right)\left(1-\boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}\right)^{-1}\boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\varsigma}_T \\
& +\Phi^{-1}\left(u_{T+1}\right)\left(1-\boldsymbol{\rho}'_{T+1,T}\mathbf{\Sigma}_T^{-1}\boldsymbol{\rho}_{T+1,T}\right)^{-1}\Phi^{-1}\left(u_{T+1}\right),
\end{aligned}$$

where by symmetry, we have applied the relationship

$$
\begin{aligned}
& \Phi^{-1}\left(u_{T+1}\right)\left(1-\boldsymbol{\rho}_{T+1,T}^{\prime}\boldsymbol{\Sigma}_{T}^{-1}\boldsymbol{\rho}_{T+1,T}\right)^{-1}\boldsymbol{\rho}_{T+1,T}^{\prime}\boldsymbol{\Sigma}_{T}^{-1}\boldsymbol{\varsigma}_{T} \\
= \; & \boldsymbol{\varsigma}_{T}^{\prime}\boldsymbol{\Sigma}_{T}^{-1}\boldsymbol{\rho}_{T+1,T}\left(1-\boldsymbol{\rho}_{T+1,T}^{\prime}\boldsymbol{\Sigma}_{T}^{-1}\boldsymbol{\rho}_{T+1,T}\right)^{-1}\Phi^{-1}\left(u_{T+1}\right).
\end{aligned}
$$

Factorising, we obtain

$$
\begin{aligned}
& \boldsymbol{\varsigma}_{T+1}^{\prime}\boldsymbol{\Sigma}_{T+1}^{-1}\boldsymbol{\varsigma}_{T+1} \\
= \; & \boldsymbol{\varsigma}_{T}^{\prime}\boldsymbol{\Sigma}_{T}^{-1}\boldsymbol{\varsigma}_{T}+\left(1-\boldsymbol{\rho}_{T+1,T}^{\prime}\boldsymbol{\Sigma}_{T}^{-1}\boldsymbol{\rho}_{T+1,T}\right)^{-1}\left(\Phi^{-1}\left(u_{T+1}\right)-\boldsymbol{\rho}_{T+1,T}^{\prime}\boldsymbol{\Sigma}_{T}^{-1}\boldsymbol{\varsigma}_{T}\right)^{2}.
\end{aligned}
$$

Thus, it becomes clear that

$$
\begin{aligned}
& \boldsymbol{\varsigma}_{T+1}^{\prime}\boldsymbol{\Sigma}_{T+1}^{-1}\boldsymbol{\varsigma}_{T+1}-\boldsymbol{\varsigma}_{T}^{\prime}\boldsymbol{\Sigma}_{T}^{-1}\boldsymbol{\varsigma}_{T} \\
= \; & \left(1-\boldsymbol{\rho}_{T+1,T}^{\prime}\boldsymbol{\Sigma}_{T}^{-1}\boldsymbol{\rho}_{T+1,T}\right)^{-1}\left(\Phi^{-1}\left(u_{T+1}\right)-\boldsymbol{\rho}_{T+1,T}^{\prime}\boldsymbol{\Sigma}_{T}^{-1}\boldsymbol{\varsigma}_{T}\right)^{2}. \qquad (27)
\end{aligned}
$$

In a similar manner, the inverse of the augmented correlation matrix $\boldsymbol{\Sigma}_{T+1}$ for the case of the Student-t copula can be shown to be

$$
\begin{aligned}
& \boldsymbol{\varsigma}_{T+1}^{\prime}\boldsymbol{\Sigma}_{T+1}^{-1}\boldsymbol{\varsigma}_{T+1}-\boldsymbol{\varsigma}_{T}^{\prime}\boldsymbol{\Sigma}_{T}^{-1}\boldsymbol{\varsigma}_{T} \\
= \; & \left(1-\boldsymbol{\rho}_{T+1,T}^{\prime}\boldsymbol{\Sigma}_{T}^{-1}\boldsymbol{\rho}_{T+1,T}\right)^{-1}\left(T_{r}^{-1}\left(u_{T+1}\right)-\boldsymbol{\rho}_{T+1,T}^{\prime}\boldsymbol{\Sigma}_{T}^{-1}\boldsymbol{\varsigma}_{T}\right)^{2}. \qquad (28)
\end{aligned}
$$

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