Measuring capital requirement for operational risk using the compound Cox process with shot noise intensity: VaR and TCE

Ji-Wook Jang
Actuarial Studies
Faculty of Commerce and Economics
University of New South Wales
Sydney, AUSTRALIA

Presentation to UNSW Actuarial Studies Research Symposium

Friday 19 November 2004
Operational Risk

- Operational Risk: The risk of losses resulting from inadequate or failed internal processes, people and systems, or external events.

- Due to the new Basel Capital Accord (the third consultative paper: www.bis.org/bcbs/bcbscp3.htm), the financial institutions need to develop a risk management tool for losses arising from all types of operational risk before 2006.
<table>
<thead>
<tr>
<th>Table: A list of some typical types of operational risks</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) People risk:</td>
</tr>
<tr>
<td>· Incompetency</td>
</tr>
<tr>
<td>· Fraud</td>
</tr>
<tr>
<td>(ii) Process risk:</td>
</tr>
<tr>
<td>- Model risk</td>
</tr>
<tr>
<td>· Model/methodology error</td>
</tr>
<tr>
<td>· Mark-to-model error</td>
</tr>
<tr>
<td>- Transaction risk</td>
</tr>
<tr>
<td>· Execution error</td>
</tr>
<tr>
<td>· Product complexity</td>
</tr>
<tr>
<td>· Booking error</td>
</tr>
<tr>
<td>· Settlement error</td>
</tr>
<tr>
<td>· Documentation/contract risk</td>
</tr>
<tr>
<td>- Operational control risk</td>
</tr>
<tr>
<td>· Exceeding limits</td>
</tr>
<tr>
<td>· Security risks</td>
</tr>
<tr>
<td>· Volume risks</td>
</tr>
<tr>
<td>(iii) Technology risk:</td>
</tr>
<tr>
<td>· System failure</td>
</tr>
<tr>
<td>· Programming error</td>
</tr>
<tr>
<td>· Information risk</td>
</tr>
<tr>
<td>· Telecommunication failure</td>
</tr>
</tbody>
</table>
Facts on operational risk losses (Embrechts, Furrer, Kaufmann and Samorodnitsky)

- Loss amounts show extremes.
- Loss occurrence times are irregularly spaced in time.
- Non-stationary.
- Repetitive versus non-repetitive losses.
- Observations are not in line with standard modelling assumptions.
Suggestions by Embrechts, Furrer, Kaufmann and Samorodnitsky

- Heavy-tailed loss-sizes: Extreme Value Theory.

- Dependence matters: Copulae.

- Complicated loss frequencies so point processes matter: The Cox (or doubly-stochastic) processes.

- Full model analytically not tractable, hence rare event simulation.
Overview

- The Cox process with shot noise intensity as the loss arrival process
- The Laplace transform of the distribution of the aggregate losses
- The capital amount required for next $t$ years’ operational risk: the VaR and TCE
- Transform analysis and numerical illustrations of VaR and TCE
Illustration of shot noise process with a primary event
• $i$ is a primary event (i.e. fraud, settlement error, system failure etc.)

• $y_i$ is the jump size of primary event $i$ (i.e. magnitude of contribution of primary event $i$ to intensity)

• $s_i$ is the time at which primary event $i$ occurs

• $\delta$ is exponential decay.
Number of losses arising from primary event $i$

- The number of losses arising from primary event $i$ following the Poisson process is given by

$$N_t^{(i)} \sim \text{Poisson} \left(y_i e^{-\delta(t-s_i)}\right)$$

where $s_i < t < \infty$ and $E(y_i) < \infty$. 

Illustration of shot noise process with a primary events over a period of time

where \( \rho \) is the rate of primary event arrival in time period \( t \).
Number of losses arising from primary events in time period $t$

- The number of losses arising from primary events following the Poisson process is given by

$$N_t \sim \text{Poisson} \left( \lambda_0 e^{-\delta t} + \sum_{\text{all } i \atop s_i \leq t} y_i e^{-\delta (t-s_i)} \right).$$

- Let $\lambda_t = \lambda_0 e^{-\delta t} + \sum_{\text{all } i \atop s_i \leq t} y_i e^{-\delta (t-s_i)}$. Then $N_t \sim \text{Poisson} (\lambda_t)$ which can be illustrated as below:
The Cox process with shot noise intensity
Aggregate losses process: Compound Cox process

- Aggregate losses is $L_t = \sum_{i=1}^{N_t} X_i$ where $X_i, i = 1, 2, \cdots$, is the loss amount which are assumed to be independent and identically distributed with the distribution function $H(x) (x > 0)$, and $N_t$ is the number of losses up to time $t$, which follows the Cox process with shot noise intensity $\lambda_t$. 
Compound Cox process
• We use the Value at Risk (VaR) (or the $q$-quantile) denoted by $l_q$ that is the smallest value satisfying ‘Pr($L_t \geq l_q$) = 1 – $q$’ as it was discussed as a risk measure to decide the capital amount required for next $t$ years’ operational risk in the consultative document on the New Basel Capital Accord.

• We also employ the tail conditional expectation (TCE) (also known as TailVaR) defined by ‘$E(L_t \mid L_t \geq l_q)$’ to obtain the capital amount required for next $t$ years’ credit risk as a coherent risk measure that satisfies the following definition (Artzner et al., 1999).
A coherent risk measure

Definition  A coherent risk measure is a function $\rho : L_t \to \mathbb{R}$ such that

(i) Subadditivity - For all random losses $L^1_t$ and $L^2_t$,

$$\rho(L^1_t + L^2_t) \leq \rho(L^1_t) + \rho(L^2_t);$$

(ii) Monotonicity - If $L^1_t \leq L^2_t$ almost surely, then $\rho(L^1_t) \leq \rho(L^2_t);$

(iii) Positive Homogeneity - For all $a \geq 0$, $\rho(aL^1_t) = a \rho(L^1_t);$

(iv) Translation Invariance - For any constant $b$, $\rho(L^1_t + b) = \rho(L^1_t) + b.$
The generator of the process \((\Lambda_t, \lambda_t, t)\)

- The generator of the process \((\Lambda_t, \lambda_t, t)\) acting on a function \(f(\Lambda, \lambda, t)\) belonging to its domain is given by

\[
Af(\Lambda, \lambda, t) = \lim_{dt \downarrow 0} \frac{E[f(\Lambda_{t+dt}, \lambda_{t+dt}, t + dt) \mid \Lambda_t = a, \lambda_t = b] - f(a, b, t)}{dt}
\]

\[
= \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial \Lambda} + \lambda [f(\Lambda, \lambda, t) - f(\Lambda, \lambda, t)] - \delta(t)\lambda \frac{\partial f}{\partial \lambda}
\]

\[
+ \rho(t) \left[ \int_0^\infty f(\Lambda, \lambda + y, t) \, dG(y; t) - f(\Lambda, \lambda, t) \right]
\]

where \(\Lambda_t = \int_0^t \lambda_s \, ds\).
A suitable martingale

- Considering constants $k$ and $\nu$ such that $k \geq 0$ and $\nu \geq 0$,

\[
\exp \left( -\nu \Lambda_t \right) \exp \left[ - \left\{ ke^{\Delta(t)} - \nu e^{\Delta(t)} \int_0^t e^{-\Delta(r)} dr \right\} \lambda_t \right] \\
\times \exp \left[ \int_0^t \rho(s) \left[ 1 - \tilde{g} \left\{ ke^{\Delta(s)} - \nu e^{\Delta(t)} \int_0^s e^{-\Delta(r)} dr; s \right\} \right] ds \right]
\]

is a martingale where $\tilde{g}(u; s) = \int_0^\infty e^{-uy} dG(y; s)$ and $\Delta(t) = \int_0^t \delta(s) ds$. 
The Laplace transform of the distribution of $\lambda_t$ and $\Lambda_t$

- Assuming that $\delta(t) = \delta$ and using an exponential jump size distribution, i.e. $g(y; t) = \left(\alpha + \gamma e^{\delta t}\right) e^{-\left(\alpha + \gamma e^{\delta t}\right)y}$, $y > 0$, $-\alpha e^{-\delta t} < \gamma \leq 0$ with $\rho(t) = \rho \frac{\alpha}{\alpha + \gamma e^{\delta t}}$, if $\lambda_t$ is $' - \infty'$ asymptotic, then the Laplace transform of the distribution of $\Lambda_t$ is given by

$$E\left\{ e^{-\nu(\Lambda_{t_2} - \Lambda_{t_1})} \right\} = \left( \frac{\gamma e^{\delta t_1} + \alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1} + \alpha + \frac{\nu}{\delta} \left(1 - e^{-\delta(t_2-t_1)}\right)} \right)^{\rho \frac{\alpha}{\delta \alpha + \nu}} \times \left( \frac{\gamma e^{\delta t_1} + \alpha + \frac{\nu}{\delta} \left(1 - e^{-\delta(t_2-t_1)}\right)}{\gamma e^{\delta t_1} + \alpha e^{-\delta(t_2-t_1)}} \right)^{\frac{\alpha \rho}{\delta \alpha + \nu}}.$$
The generator of the process \((N_t, L_t, t)\)

- The generator of the process \((N_t, L_t, t)\) acting on a function \(f(n, l, t)\) belonging to its domain is given by

\[
A f(l, n, t) = \lim_{dt \downarrow 0} \frac{E[f(L_{t+dt}, N_{t+dt}, t+dt)|L_t=l, N_t=n] - f(l,n,t)}{dt}
\]

\[
= \frac{\partial f}{\partial t} + \lambda [f(n + 1, l + x, t)dH(x) - f(n, l, t)].
\]
A suitable martingale

- Considering constants $\theta$ and $\nu$ such that $0 \leq \theta \leq 1$ and $\nu \geq 0$,

\[
\theta^N \exp (-\nu L_t) \exp \left[ - \left\{ \theta \hat{h} (\nu) - 1 \right\} \Lambda_t \right]
\]

is a martingale where $\hat{h} (\nu) = \int_0^\infty e^{-\nu x} dH (x)$.
The Laplace transform of the distribution of $L_t$

- The Laplace transform of the distribution of $L_t$ is given by

$$E \left( e^{-\nu L_t | \lambda_0} \right) = E \left[ \exp \left\{ - \left( 1 - \hat{h}(\nu) \right) \Lambda_t \right\} \mid \lambda_0 \right]$$

and if the loss size distribution is exponential, i.e. $h(x) = \beta e^{-\beta x}$, $x > 0$, $\beta > 0$ with $g(y; t) = (\alpha + \gamma e^{\delta t}) e^{-(\alpha + \gamma e^{\delta t}) y}$, $y > 0$, $-\alpha e^{-\delta t} < \gamma \leq 0$, $\rho(t) = \rho \frac{\alpha}{\alpha + \gamma e^{\delta t}}$ and $\lambda_t$ is $`-\infty$' asymptotic, it is given by

$$\left( \frac{\gamma + \alpha e^{-\delta t}}{\gamma + \alpha + \frac{\nu}{\delta(\beta + \nu)} (1 - e^{-\delta t})} \right)^{\frac{\rho}{\delta}} \left( \frac{\gamma + \alpha + \frac{\nu}{\delta(\beta + \nu)} (1 - e^{-\delta t})}{\gamma + \alpha e^{-\delta t}} \right)^{\frac{\alpha \rho}{\delta}} \frac{\alpha \rho}{\beta + \nu}.$$
• The VaR can be expressed as

\[
\text{VaR}(q, L_t) = \inf \{l_q \in \mathbb{R} : \Pr(L_t > l_q) \leq 1 - q\}.
\]

• The TCE can be expressed by

\[
\text{TCE}(q, L_t) = \mathbb{E}\{L_t \mid L_t \geq \text{VaR}(q, L_t)\} = \frac{\mathbb{E}[L_t \mathbb{I}\{L_t \geq \text{VaR}(q, L_t)\}]}{1-q}.
\]

where \( \mathbb{I}(\cdot) \) is the indicator function. The TCE tells us how great the losses are as it takes an average over the worst cases and therefore takes into account the tail distribution of the losses. However, the VaR only looks at a quantile and it does not tell us how great losses are.
Inverting the Laplace transform of the distribution of $L_t$

- By analogy with option pricing problems in finance we can first notice that the Compound Cox process is ‘affine’. This suggests that transform analysis techniques developed by Heston (1993) and Duffie et al. (2000) might prove useful. We highlight their methodology as applied to our problem below.
We know the Laplace transform $\xi(-\nu)$ of $L_t$:

$$
\xi(-\nu) = E\left( e^{-\nu L_t} \right)
= \left( \frac{\gamma + \alpha e^{-\delta t}}{\gamma + \alpha + \frac{\nu}{\delta(\beta+\nu)} (1 - e^{-\delta t})} \right) \left( \frac{\gamma + \alpha + \frac{\nu}{\delta(\beta+\nu)} (1 - e^{-\delta t})}{\gamma + \alpha e^{-\delta t}} \right)^{\frac{\rho}{\delta}} \frac{\alpha \rho}{\delta(\beta+\nu)}.
$$

So consider the function

$$
\hat{\Psi}(z) = \int_0^\infty e^{izl} d\left( \int_0^l dF_{L_t}(x) \right) = \int_0^\infty e^{izl} dF_{L_t}(l) = E[e^{izL_t}] = \xi(i z)
$$

and the standard Lévy inversion formula gives

$$
E\{I(L_t < l)\} = P(L_t < l) = \frac{\hat{\Psi}(0)}{2} - \frac{1}{\pi} \int_0^\infty \frac{1}{z} \text{Im} \left( e^{-izl} \hat{\Psi}(z) \right) dz.
$$

from which we can easily obtain $P(L_t \geq l) = 1 - P(L_t < l)$. 
Consider the another function

$$\tilde{\Theta} (z) = \int_{0}^{\infty} e^{izl} \left( \int_{0}^{l} x dF_{L_{t}} (x) \right).$$

Assume \( \int |\Theta (y)| \, dy < \infty \), and we find that

$$\tilde{\Theta} (z) = \int_{0}^{\infty} e^{izl} \, dF_{L_{t}} (l) = E \left( L_{t} \, e^{izL_{t}} \right),$$

which can be calculated as follows.
• Differentiating $\xi(-\nu)$ with respect to $-\nu$ gives

$$
-\frac{\partial}{\partial \nu} \xi(-\nu) = E \left( L_t \ e^{-\nu L_t} \right) = \rho \left( \frac{\beta}{\beta+\nu} \right)^2 \left( \frac{\gamma+\alpha e^{-\delta t}}{\gamma+\alpha+\delta(\beta+\nu)(1-e^{-\delta t})} \right)^{\frac{\rho}{\delta}}
$$

$$
\times \left( \frac{\gamma+\alpha+\frac{\nu}{\delta(\beta+\nu)}(1-e^{-\delta t})}{\gamma+\alpha e^{-\delta t}} \right)^{\frac{\alpha \rho}{\delta \alpha+\left( \frac{\nu}{\beta+\nu} \right)}}
$$

$$
\times \left[ \begin{array}{c}
\left\{ \frac{1-e^{-\delta t}}{\delta \left\{ \gamma+\alpha+\frac{\nu}{\delta(\beta+\nu)}(1-e^{-\delta t}) \right\} } \right\} \left( \frac{1}{\delta} - \frac{\alpha}{\delta \alpha+\frac{\nu}{\beta+\nu}} \right) \\
+ \left\{ \frac{\alpha}{(\delta \alpha+\frac{\nu}{\beta+\nu})^2} \right\} \ln \left\{ \frac{\gamma+\alpha+\frac{\nu}{\delta(\beta+\nu)}(1-e^{-\delta t})}{\gamma+\alpha e^{-\delta t}} \right\}
\end{array} \right]
$$

$$
= \eta(-\nu) \text{ and hence } \hat{\Theta}(z) = \eta(iz).
$$
Since we now have a closed form formula for $\hat{\Theta}(z)$ the inversion lemma gives

$$
\int_0^q l \, dF_{L_t}(l) = E[L_t \, I\{L_t < \text{VaR}(q, L_t)\}] = \Theta(l)
$$

$$
= \frac{\hat{\Theta}(0)}{2} - \frac{1}{\pi} \int_0^\infty \frac{1}{z} \text{Im}(e^{-izl} \hat{\Theta}(z)) \, dz
$$

with $\hat{\Theta}(0) = \eta(0) = E(L_t)$, which allows us to calculate the numerator of $\text{TCE}(q, L_t)$ as

$$
E[L_t \, I\{L_t \geq \text{VaR}(q, L_t)\}] = E(L_t) - E[L_t \, I\{L_t < \text{VaR}(q, L_t)\}]
$$

$$
= \int_0^\infty l \, dF_{L_t}(l) - \int_0^q l \, dF_{L_t}(l).
$$
Example 1

- The parameter values used to calculate the VaR and TCE are $\alpha = 0.1$, $\beta = 0.01$, $\delta = 0.3$, $\gamma = -0.01$, $\rho = 4$, $t = 1$ and

$$E(L_t) = \frac{\rho}{\alpha \beta \delta^2} \ln \left( \frac{\gamma + \alpha}{\gamma + \alpha e^{-\delta t}} \right) = 15,096$$

which is the mean of the total loss in a unit period of time. Using Matlab, the calculation of VaR and TCE are shown in Table 1.
Table 1.

<table>
<thead>
<tr>
<th>$q$</th>
<th>VaR</th>
<th>TCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>99.9%</td>
<td>$31,488</td>
<td>$33,219</td>
</tr>
<tr>
<td>99.75%</td>
<td>$29,629</td>
<td>$31,541</td>
</tr>
<tr>
<td>95%</td>
<td>$22,707</td>
<td>$25,121</td>
</tr>
<tr>
<td>90%</td>
<td>$20,766</td>
<td>$23,381</td>
</tr>
<tr>
<td>50%</td>
<td>$14,730</td>
<td>$18,480</td>
</tr>
</tbody>
</table>
Example 2

- In order to include losses from non-quantifiable operational risk, let us levy a security loading factor $\zeta > 0$, which is well-known for premium calculation in actuarial science, on the original VaR and TCE calculated only for quantifiable operational risk., i.e.

$$ \text{VaR}^{\text{all}}(q, L_t) = (1 + \zeta) \times \text{VaR}(q, L_t) $$

and

$$ \text{TCE}^{\text{all}}(q, L_t) = (1 + \zeta) \times \text{TCE}(q, L_t). $$
Assuming that $\zeta = 0.3$, where the banks’ attitude towards non-quantifiable operational risk determines $\zeta$ unless the Basel Committee for Banking Supervision predetermines, the VaR and TCE from all types of operational risk are shown in Table 2.

Table 2.

<table>
<thead>
<tr>
<th>$q$</th>
<th>VaR$^{all}$</th>
<th>TCE$^{all}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>99.9%</td>
<td>$40,934$</td>
<td>$43,185$</td>
</tr>
<tr>
<td>99.75%</td>
<td>$38,518$</td>
<td>$41,003$</td>
</tr>
<tr>
<td>95%</td>
<td>$29,519$</td>
<td>$32,657$</td>
</tr>
<tr>
<td>90%</td>
<td>$26,996$</td>
<td>$30,395$</td>
</tr>
<tr>
<td>50%</td>
<td>$19,149$</td>
<td>$24,024$</td>
</tr>
</tbody>
</table>