Optimal Capital Allocation Principles

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The Capital Allocation Problem

- Consider a portfolio of $n$ individual losses $X_1, X_2, \ldots, X_n$ during some well-defined reference period.

- Assume these random losses have a dependency structure characterized by the joint distribution of the vector $X^T = (X_1, X_2, \ldots, X_n)$.

- The total portfolio loss is the random variable $S = \sum_{i=1}^{n} X_i$.

- Assume company holds capital-at-risk (or economic capital) $K$ determined from a risk measure $\rho$ such that $K = \rho(S) \in R$.

- Here economic capital is the smallest amount the company should set aside to withstand aggregated losses at an acceptable level.
The Capital Allocation Process

- What is the most sensible and efficient way of allocating the total capital to the various business units?

- Generally the procedure of allocating capital consists of:
  1. specifying the dependence structure for the random loss $X^T$
  2. selecting the risk measure $\rho$ such that $K = \rho(S)$
  3. calculating stand-alone risk capitals $\rho(X_i)$
  4. determining how to allocate the diversification benefit $\sum_{i=1}^{n} \rho(X_i) - \rho(S)$
Purpose of Paper

- The purpose of the paper is to re-cast the allocation problem as a minimum distance problem.

- Take into account some allocation criteria:
  - the manner in which the various segments interact - legal structure
  - purpose of allocating the capital

- Solution to minimizing distance measure leads to several existing allocation formulas.

- New allocation formulas also emerge.
The Allocation

- Denote the random vector of losses by $X^T = (X_1, X_2, \ldots, X_n)$.

- An allocation $A$ is a mapping $A : X^T \rightarrow R^n$ such that $A(X^T) = (K_1, K_2, \ldots, K_n)^T \in R^n$ where

$$\sum_{i=1}^{n} K_i = K$$

is the total capital.

- This requirement is referred to as the “full allocation”.

- We will see that this will be a constraint in the optimization problem.
Some Known Allocation Formulas

<table>
<thead>
<tr>
<th>Allocation Principle</th>
<th>Allocation Formula $K_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative</td>
<td>$\frac{\rho(X_i)}{\sum_{i=1}^{n} \rho(X_i)} \cdot K$</td>
</tr>
<tr>
<td>Covariance</td>
<td>$\frac{Cov(X_i, S)}{Var(S)} \cdot K$</td>
</tr>
<tr>
<td>Tail-VaR</td>
<td>$E(X_i</td>
</tr>
<tr>
<td>Quantile</td>
<td>$F_{X_i}^{-1}(F_{Sc}(K))$</td>
</tr>
</tbody>
</table>
**Allocation Criteria - Legal Structure**

- All risks are fully pooled.
  - there is subsidization of losses.
  - default of single portfolio can contribute to potential default of entire company.

- Firewalls are built.
  - no pooling of risks, no subsidization
  - each portfolio is allowed to consume only capital allocated to them.

- Risks are partially pooled.
  - losses from a single portfolio are shared by a pool of portfolios.
  - e.g. Lloyd’s of London
Allocation Criteria - Purpose of Allocation

- **Individual**
  - evaluate performance of individual portfolios as stand-alone entities.
  - e.g. calculation of performance bonuses of portfolio managers.

- **Collective**
  - examine individual portfolio’s contribution to the aggregate risks.
  - evaluate performance in the presence of the other portfolios.
  - e.g. evaluating new investment projects.

- **Market-relative**
  - evaluate portfolios in relation to performance in the market.
  - e.g. assessing market competitiveness.
Optimal Allocation by Minimizing Distances

- Optimality criterion is to have remaining risk as minimal as possible under a suitable measure of distance.

- We investigate distances of the form:

  $$D_\varepsilon(K_i; X_i, \zeta_i) = \mathbb{E}[\varepsilon(K_i; X_i) \zeta_i],$$

  where $\varepsilon(\cdot; \cdot)$ is a function of the risk $X_i$ and the allocated capital $K_i$ providing a measure of the distance, or closeness, between the two.

- $D_\varepsilon$ is sometimes called the decision loss function.

- $\zeta_i$ is a random variable satisfying $\mathbb{E}(\zeta_i) = 1$, reflecting both risk aversion of company and the purpose of the allocation.
- continued

• Company then wishes to minimize this decision loss function subject to the “full allocation” requirement.

• Find the allocation \((K_1, \ldots, K_n)\) by solving the following optimization problem:

\[
\min_{K_1, \ldots, K_n} \sum_{i=1}^{n} E [\varepsilon (K_i; X_i) \zeta_i] .
\]
Examples of Decision Loss Function

- **squared-error distance:** $\varepsilon(X_i, K_i) = (X_i - K_i)^2$.

- **weighted squared-error:** $\varepsilon(X_i, K_i) = \left[X_i - \left(S/K\right) K_i\right]^2$.

- **normalization constant:** $\varepsilon(X_i, K_i) = \frac{(X_i - K_i)^2}{\nu_i}$.

- **absolute deviation:** $\varepsilon(X_i, K_i) = |X_i - K_i|$.

- **stop-loss:** $\varepsilon(X_i, K_i) = (X_i - K_i)_+$.
Solving the Optimization

• Distance formula can be re-expressed as:

\[ D_{\varepsilon}(K_i; X_i, \zeta_i) = Cov[\varepsilon(K_i; X_i), \zeta_i] + E[\varepsilon(K_i; X_i)] \]

• The Lagrange function is given by

\[ L(K_1, \ldots, K_n) = \sum_{i=1}^{n} E[\varepsilon(K_i; X_i)\zeta_i] - \lambda \left( \sum_{i=1}^{n} K_i - K \right). \]

• For an optimum then, the first-order conditions are given by

\[ E\left[ \frac{\partial \varepsilon(K_i; X_i)}{\partial K_i} \zeta_i \right] = Cov\left[ \frac{\partial \varepsilon(K_i; X_i)}{\partial K_i}, \zeta_i \right] + E\left[ \frac{\partial \varepsilon(K_i; X_i)}{\partial K_i} \right] = \lambda. \]
The Role of $\zeta$

- The random variable $\zeta$ provides an effect similar to a change of measure.

- This effect can be further viewed as inducing a stress scenario for which the risk management of the company operates.

- It is a useful reflection therefore of the risk aversion behavior of the company, and can be influenced by the purpose to which the allocation of the capital is being made.

- General form: $\hat{\zeta_i} = \frac{h(Z)}{E[h(Z)]}$. 
Examples of Change of Measure

<table>
<thead>
<tr>
<th>Description</th>
<th>Functional form $h(Z)$</th>
<th>Resulting $\zeta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Esscher transform</td>
<td>$\exp(aZ)$</td>
<td>$\frac{\exp(aZ)}{E[\exp(aZ)][$}\right$</td>
</tr>
<tr>
<td>Variance principle</td>
<td>$1 + aZ$</td>
<td>$\frac{1 + aZ}{E[1 + aZ]][$}$</td>
</tr>
<tr>
<td>Tail-VaR</td>
<td>$I(Z &gt; F_Z^{-1}(q))$</td>
<td>$\frac{I(Z &gt; F_Z^{-1}(q))}{1 - q}[$$</td>
</tr>
</tbody>
</table>


The Random Variable $Z$

- The random variable $Z$ reflects the purpose of the allocation: individual, collective, or market-relative.

- In the case of individual assessment, we choose $Z = X_i$.

- In the case of collective assessment, we choose $Z = \sum_{i=1}^{n} X_i$.

- In the case of market-relative assessment, we choose the market portfolio for $Z$. 
or alternatively, $\zeta_i = \zeta$ (i.e. same for all portfolios) provides a market pricing measure for which it can be estimated based on market premium data.
Distortion Form

- Distortion form:
  \[ \zeta_i = g \left( F_Z (Z) \right) \]
  for some distortion function \( g \).

- Wang (1996) defines family of risk measures using concept of distortion function based on Yaari’s dual theory of choice:
  \[ \rho_g [X] = - \int_{-\infty}^{0} \left[ 1 - g \left( F_X (x) \right) \right] dx + \int_{0}^{\infty} g \left( F_X (x) \right) dx. \]

- \( g \) is a non-decreasing function \( g : [0, 1] \to [0, 1] \) satisfying \( g (0) = 0 \) and \( g (1) = 1 \).
The Expected Value Allocation Principle

- Consider the decision loss function:

\[ E[\varepsilon(K_i; X_i) \zeta_i] = E\left[\frac{(X_i - K_i)^2}{v_i} \zeta_i\right]. \]

- For this case, we find the following allocation formula:

\[ K_i = E[X_i \zeta_i] + \frac{v_i}{\sum_{j=1}^{n} v_j} \left( K - \sum_{j=1}^{n} E[X_j \zeta_i] \right) \]

which we call the expected value allocation formula.

- This general case of the EV allocation principle leads to several other well-known interesting capital allocation methods.
- continued

- On one hand, in an individual risk assessment, using the formulas for $\zeta_i$ suggested, with therefore $Z = X_i$, the capital allocation results in formulas based on:
  - the Esscher premium principle, the variance premium principle, the tail-VaR premium principle, and the distortion premium principle.

- On the other hand, in a collective risk assessment, we can use $Z = \sum_{j=1}^{n} X_j$ which yields to the allocation formulas based on:
  - Wang’s exponential tilting (Wang, 2002),
  - the tail-VaR allocation suggested by Panjer (2002),
  - allocation suggested by Tsanakas (2003), and even
  - the covariance-based allocation formulas.

- If $Z$ is the market portfolio in the change of measure $\zeta_i$, then we derive allocations similar to equilibrium-based pricing formulas. Buhlmann (1980).
The Individual Distortion Allocation Principle

- Assuming the $\zeta_i$ are given and that the $g_i$ are defined such that $X_i\zeta_i \overset{d}{=} F_{X_i}^{-1}(1 - U) g_i'(U)$, we have $E[X_i\zeta_i] = \rho g_i [X_i]$.

- This leads us to the capital allocation formula:

$$K_i = \rho g_i [X_i] + \frac{v_i}{\sum_{j=1}^{n} v_j} \left( K - \sum_{j=1}^{n} \rho g_j [X_j] \right).$$

- As a special case, consider $\zeta_i = \frac{I(X_i > F_{X_i}^{-1}(q))}{\Pr[X_i > F_{X_i}^{-1}(q)]}$. We therefore have the capital allocation

$$K_i = E \left[ X_i \mid X_i > F_{X_i}^{-1}(q) \right] + \frac{v_i}{\sum_{j=1}^{n} v_j} \left( K - \sum_{j=1}^{n} E \left[ X_j \mid X_j > F_{X_j}^{-1}(q) \right] \right).$$
The Collective Distortion Allocation Principle

- Assuming all $\zeta_i$ are all equal to $\zeta$ and defining the distortion functions $g$ and $g_i$ by
  \[ X_i \zeta \overset{d}{=} F_{X_i}^{-1}(1 - U) \ g'_i(U) \] \[ S \zeta \overset{d}{=} F_S^{-1}(1 - U) \ g'(U). \]

- The allocation formula becomes
  \[ K_i = \rho g_i [X_i] + \frac{v_i}{\sum_{j=1}^{n} v_j} \left( K - \sum_{j=1}^{n} \rho g_j [X_j] \right). \]

- Take as a special case: $\zeta = \frac{I(S > F_S^{-1}(q))}{\Pr[S > F_S^{-1}(q)]}$.

- Then we find that
  \[ K_i = \mathbb{E}[X_i \mid S > F_S^{-1}(q)] + \frac{v_i}{\sum_{j=1}^{n} v_j} \left( K - \mathbb{E}[S \mid S > F_S^{-1}(q)] \right). \]
The Quantile Allocation Principle

- The following allocation formula

\[ K_i = F_{X_i,Q}^{-1} \left( F_{S_c,Q}^{-1}(K) \right). \]

- We call the stop-loss (or quantile) allocation formula.

- In the special case where \( K = F_{S_c,Q}^{-1}(q) \), then this reduces to

\[ K_i = F_{X_i,Q}^{-1}(q). \]

- We find that this allocation principle can interestingly be derived from a variety of ways: (1) a stop-loss distance minimizing criterion, (2) a cost function minimizing criterion, and (3) an absolute distance minimizing criterion.
can be derived from . . .

- A stop-loss distance minimizing criterion using the decision function given by

\[
E \left[ \varepsilon_i (K_i; X_i) \zeta \right] = E \left[ (X_i - K_i)_+ \zeta \right],
\]

or equivalently

\[
E^Q \left[ \varepsilon_i (K_i; X_i) \right] = E^Q \left[ (X_i - K_i)_+ \right].
\]

- An absolute distance minimizing criterion using the decision function given by

\[
E \left[ \varepsilon_i (K_i; X_i) \zeta \right] = E \left[ |X_i - K_i| \zeta \right],
\]

or equivalently

\[
E^Q \left[ \varepsilon_i (K_i; X_i) \right] = E^Q \left[ |X_i - K_i|_+ \right].
\]
Concluding Remarks

- We re-examine existing allocation formulas that are in use and have appeared in the literature, by re-expressing the allocation issue as optimization problem.

- No allocation formula may serve multiple purposes, hence, by expressing the allocation problem as an optimization problem, this can further provide us insights.

- The primary input to the optimization problem is the development of the decision loss function.

- We find that there can be a wide variety of functions suitable for different purposes.

- Additional allocation methodologies have also emerged as a result.