Optimal Investment for Managing Insurance Risks

Maritina T. Castillo
School of Actuarial Studies
Faculty of Commerce and Economics
University of New South Wales
E-mail: tina.castillo@unsw.edu.au

Abstract

We consider an insurance business with a Cramer-Lundberg risk process and an investment portfolio consisting of a risky and a non-risky asset. Investment strategies are determined to satisfy conditions on the profitability of the business. In particular, we determine investment strategies that minimize the probability of ruin subject to constraints on the amount available for investment. We also consider an investment strategy that maximizes the expected utility of the business surplus when Brownian motion with drift is used as an approximation to the Cramer-Lundberg process. The investment strategies are determined using dynamic programming methods.

Key words: Ruin theory, stochastic control theory, Hamilton-Jacobi-Bellman equation, dynamic programming

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1 Introduction

Insurance business surplus is often described as

\[
\text{Surplus} = \text{Initial capital} + \text{Income} - \text{Outflow}.
\]

The outflow process is determined by the claims incurred and the income process by the total premiums collected and the accrued investment income. Infinite time ruin probability associated with an insurance business with current surplus \( u \) is given by

\[
\psi(u) = \Pr\{U(t) < 0 \text{ for some } t \geq 0\}
\]  

(1)

where \( U(t) \) is the amount of business surplus at time \( t \). One would like to minimize this ruin probability subject to the dynamics of the surplus process. In practice, an actuary is often asked to make decisions regarding new or existing insurance business. Strategies that minimize the above ruin probabilities can be considered when business policies are determined.

Much interest is generated by the use of mathematical tools from stochastic control theory in addressing the problem of minimizing the infinite time ruin probability defined in (1). Control variables include investment, new business, reinsurance, and dividend payments. By means of a standard control tool such as the Hamilton-Jacobi-Bellman equation, optimal strategies can be characterized and computed, often numerically, and the smoothness of the value function can be shown. Recent applications of stochastic control tools in minimizing the probability of ruin include the optimization of reinsurance programs (see [7], [8], [9] and [10]), the issuance of new business (see [6]), optimal investment strategies (see [1], [4] and [5]), and simultaneous dynamic control of reinsurance and investment (see [12]). Here we focus on optimal investment strategies for managing the risks in an insurance business.

Browne [1] considers optimal investment policies for a firm faced with a random risk process and one investment opportunity, a risky stock. The risk process is modelled by a Brownian motion with drift and the price process of the risky stock is assumed to follow a geometric Brownian motion. Under the assumption that the investor is allowed to borrow an unlimited amount, and there is no risk-free interest rate, he arrives at a surprising result that the optimal policy that minimizes the probability of ruin invests a fixed constant amount, regardless of the level of wealth the company has. This policy is also optimal in maximizing exponential utility from terminal wealth and for the objective of maximizing the probability of reaching any given wealth level prior to hitting a given lower wealth level. In the presence of a positive interest rate however, the equivalence of the optimal policies maximizing exponential utility from terminal wealth, and minimizing the probability of ruin does not hold. In particular, the optimal strategy for maximizing terminal wealth is seen to be independent on the wealth level and minimizing ruin probability resulted in a strategy that depends on the wealth level (in a rather complicated way).
The question of finding the optimal investment amount for insurers that will minimize the probability of ruin for an insurance business is also addressed by Hipp and Plum [5] where they consider a risk process modelled as a compound Poisson process and an opportunity to invest in a risky asset whose price is modelled by a geometric Brownian motion. The ruin probability of this risk process is minimized by the choice of a suitable investment strategy for a capital market index. The optimal strategy is computed using the Bellman equation. The resulting strategy is different (and more intuitive) from the result obtained by Browne in that the optimal invested amount $A_t; t \geq 0$, at time $t$ is a function of the current surplus of the insurance business.

A follow-up paper by Hipp and Plum [4] considers an optimal control problem where a risky asset is used for investment and this investment is financed by initial wealth as well as by a state dependent income. The objective function is accumulated discounted expected utility of wealth, where the utility function is non-decreasing and bounded. This problem is investigated for constant as well as for stochastic discount rate, where the stochastic model is a time homogeneous finite state Markov process. Based on this the problem of optimal investment that would minimize infinite time ruin probability for an insurer with an insurance business modelled by a compound Poisson or a compound Cox process, under the presence of constant as well as (finite state space Markov) stochastic interest rate.

In a recent paper [3], we applied stochastic control tools in finding the investment strategy that would minimize the probability of ruin for an insurance business with a current surplus $u$ and a fixed amount $A$ available for investment, where $A$ is assumed to be independent of the business surplus. An investment portfolio consisting of one risky asset and one non-risky asset is available at each time $t \geq 0$ and a proportion $b(t) \in [0,1]$ of $A$ is invested in the risky asset with the remaining part to be invested in the non-risky asset. The strategy $b(t)$ is chosen predictable, i.e. it depends on all information available before time $t$. The existence of a solution to the Bellman equation is proved in the paper [3]. The verification lemma for the control problem is presented here.

In this paper, we determine optimal investment strategies for managing the risks in an insurance business for the specific models considered here. The investment strategy is optimal in the sense of minimizing the probability of ruin subject to constraints in the amount available for investment. An investment strategy that maximizes utility of the surplus at some pre-determined date is also considered.
2 Dynamics of the business surplus

We model the surplus process of an insurance business whose risk process \( \{R(t), t \geq 0\} \) follows a Cramér-Lundberg process

\[
dR(t) = cd t - dS(t), \quad R(0) = r
\]

where \( c \) is the loaded premium rate and \( \{S(t), t \geq 0\} \) is the random claims process consisting of a sum of independent, identically distributed claims \( X_i \). That is,

\[
S(t) = \sum_{i=1}^{N(t)} X_i
\]

and \( N(t) \) stands for the number of claims until time \( t \) and is modelled by a homogeneous Poisson process with intensity \( \lambda \). \( F(x) \) and \( f(x) \) are the distribution and density of the random claim amount \( X \), respectively.

The investment portfolio consists of a non-risky asset whose value \( B(t) \) follows

\[
\frac{dB(t)}{dt} = \rho B(t) dt, \quad \rho \geq 0,
\]

and a risky asset whose price \( Z(t) \) follows a geometric Brownian motion

\[
\frac{dZ(t)}{dt} = \mu Z(t) dt + \sigma Z(t) dW_1(t), \quad \mu \geq 0, \sigma > 0
\]

where \( \{W_1(t), t \geq 0\} \) is a standard Weiner process.

The insurance business has the following investment policy:

- A fixed amount \( A \), independent of the surplus, will be invested at time \( t \).
- A fraction \( b(t) \in [0, 1] \) of \( A \) will be invested at time \( t \) in the risky asset, the remaining part in the non-risky asset.
- The fraction \( b(t) \) may change through time depending on which combination of risky and non-risky asset minimizes the infinite time ruin probability.

The investment return process \( \{I(t), t \geq 0\} \) from the amount \( A \) is defined by

\[
\frac{dI(t)}{dt} = A[1 - b(t)] \rho \ dt + Ab(t) \mu \ dt + Ab(t) \sigma \ dW_1(t), \quad \text{(2)}
\]

and the surplus process \( \{U(t), t \geq 0\} \) for this business is then seen to be

\[
\frac{dU(t)}{dt} = cd t - dS(t) + A[1 - b(t)] \rho \ dt + Ab(t) \mu \ dt + Ab(t) \sigma \ dW_1(t) \quad \text{(3)}
\]

where \( U(0) = u \). Clearly, \( U(t) \) depends on the composition of the investment portfolio in which the fixed amount \( A \) is invested and is thus influenced by the investment strategy \( b(t) \).
The dynamics of the surplus process can be further described as follows:

- A claim of amount $X$ occurs with probability $\lambda dt + o(dt)$.
- No claim occurs with probability $1 - \lambda dt + o(dt)$.
- An amount $cdt + o(dt)$ is received as a premium income.
- An amount $A[1 - b(t)]\rho dt + o(dt)$ is received as an investment income from the non-risky asset.
- An amount $Ab(t)\mu dt + Ab(t)\sigma dW_1(t) + o(dt)$ is received as an investment income from the risky asset.

3 The control problem for minimizing ruin with constraints on the investment amount

Our aim here is to minimize the ruin probability over all possible strategies $b(t)$. The control problem can be stated as follows:

\[
\begin{align*}
\text{minimize} & \quad \psi(u) = Pr\{U(t) < 0 \text{ for some } t \geq 0\} \\
\text{subject to} & \quad dU(t) = dR(t) + dI(t), \quad t \geq 0 \\
& \quad U(0) = u.
\end{align*}
\]

The solution to the control problem is found via the Hamilton-Jacobi-Bellman equation of the control problem. The solution to this equation determines the optimal proportion, $b_*(t)$, which is a function of the business surplus at time $t$. The optimal investment strategy is defined via the feedback equation

\[
\begin{align*}
b_*(t) &= b_*(U(t))
\end{align*}
\]

where $U(t)$ is the surplus at time $t$ resulting from the investment strategy $\{b_*(s), s < t\}$.

3.1 The Hamilton-Jacobi-Bellman equation

We define the probability of non-ruin, also known as the probability of survival, for a business with current surplus $u$ to be

\[
\delta(u) = 1 - \psi(u).
\]

Here we consider two distinct cases over the time interval $[0, dt]$: (1) that there is no claim during the period and, (2) that there is exactly one claim during the period. If there is
no claim, the surplus of the business grows to \( u + c dt + dI(t) \), where \( dI(t) \) is given in (2).

If there is a claim, the surplus of the company reduces to \( u + dI(t) - X \), where \( X \) is the random claim size. If there is a claim during the period \([0, dt]\), we assume that no premium is received for that period.

For an arbitrary strategy \( b(t) \), with current surplus level \( u \), the probability of non-ruin \( \delta(u) \) may now be determined. Taking expectations we have

\[
\delta(u) = \lambda dt \ E[\delta(u - X)] + (1 - \lambda dt) \ \delta(u + c dt + dI(t)).
\]

Applying Ito’s lemma, it then follows that

\[
\delta(u) = \delta(u) + \left\{ \frac{1}{2} \sigma^2 A^2 b^2 \delta''(u) + \left\{ c + A[1-b] \rho + A \mu \right\} \delta'(u) \right\} dt
\]

\[
+ \lambda E \left[ \delta(u - X) - \delta(u) \right]
\]

where the proportion invested in the risky asset depends only on the current surplus level at time \( t \). For convenience, we denote the proportion simply as \( b \). Equation (4) then leads to

\[
0 = \sup_b \left\{ \frac{1}{2} \sigma^2 A^2 b^2 \delta''(u) + \left\{ c + A[1-b] \rho + A \mu \right\} \delta'(u) \right\}
\]

where we have the natural conditions \( \delta'(u) \geq 0, \delta''(u) \leq 0 \) for \( u > 0 \), \( \delta(u) = 0 \) for \( u < 0 \) and \( \lim_{u \to -\infty} \delta(u) = 1 \). Furthermore, we assume that \( \delta(u) \) is continuous on \([0, \infty)\) and twice continuously differentiable on \((0, \infty)\). Equation (5) is called the Hamilton-Jacobi-Bellman (HJB) equation of the control problem.

### 3.2 The optimal investment strategy

Note that the quantity inside the braces in (5) is maximized by

\[
\hat{b} = -\frac{(\mu - \rho) \ \delta'(u)}{A \sigma^2 \delta''(u)}
\]

and is seen to be a function of the current surplus only. Substituting (6) in (5) results in the differential equation

\[
\lambda \ E \left[ \delta(u - X) - \delta(u) \right] = \frac{1}{2} \frac{(\rho - \mu)^2 \left[ \delta'(u) \right]^2}{\sigma^2 \delta''(u)} - (c + A \mu) \ \delta'(u).
\]

The solution \( \delta(u) \) to this equation determines the proportion required when the current surplus is \( u \).
3.3 The Verification Lemma

**Theorem 1** Suppose there exists a solution \( \delta^e_b(u) \) of the HJB equation (5), having maximizer defined as in (6), \( 0 \leq \tilde{b} \leq 1 \) and such that \( \delta^e_b(0) > 0 \), \( \delta^e_b(u) = 0 \) for \( u < 0 \), \( \lim_{u \to -\infty} \delta^e_b(u) = 1 \), and \( \delta^e_b(u) \) twice continuously differentiable on \( \{ u > 0 \} \). Then if \( b(t) \) is an arbitrary admissible investment strategy for which the corresponding surplus process \( \{ U_b(t), t \geq 0 \} \) is defined on \( 0 \leq t < \infty \), then the corresponding non-ruin probability \( \delta_b(u) \) at current surplus \( u \) satisfies

\[
\delta_b(u) \leq \delta^e_b(u), \quad u \geq 0.
\]

**Proof.** Consider an arbitrary strategy \( b(t) \) and the strategy \( \tilde{b}(t) \) having surplus processes \( \{ U_b(t), t \geq 0 \} \) and \( \{ U_{\tilde{b}}(t), t \geq 0 \} \), respectively, and ruin times \( \tau_b, \tau_{\tilde{b}} \) where

\[
dU_b(t) = c dt - dS(t) + A \left[ 1 - b(t) \right] \rho \, dt + Ab(t) \mu \, dt + Ab(t) \sigma \, dW_1(t),
U_b(0) = u
\]

and

\[
dU_{\tilde{b}}(t) = c dt - dS(t) + A \left[ 1 - \tilde{b}(t) \right] \rho \, dt + A\tilde{b}(t) \mu \, dt + A\tilde{b}(t) \sigma \, dW_1(t),
U_{\tilde{b}}(0) = u.
\]

For \( \varsigma > 0 \), let

\[
b_{\#}(t) = b(t) + \frac{\varsigma^2}{A}
\]

with ruin time \( \tau_{\#} \) and surplus process

\[
dU_{\#}(t) = c dt - dS(t) + A \left[ 1 - b(t) - \frac{\varsigma^2}{A} \right] \rho \, dt + A \left[ b(t) + \frac{\varsigma^2}{A} \right] \mu \, dt \\
+ A \left[ b(t) + \frac{\varsigma^2}{A} \right] \sigma \, dW_1(t),
U_{\#}(0) = u + \varsigma.
\]

Hence \( U_{\#}(t) = U_b(t) + \varsigma + \varsigma^2 (\mu - \rho) \, t + \varsigma^2 \sigma \, W_1(t) \) and

\[
\Pr\{ \tau_{\#} < \tau_b \} \leq \Pr\{ \varsigma + \varsigma^2 (\mu - \rho) \, t + \varsigma^2 \sigma \, W_1(t) < 0 \text{ for some } t \}
= \exp \left( -\frac{2(\mu - \rho)^2}{\sigma^2 \varsigma} \right)
\]
Define the non-ruin processes \( \delta_b(U_b(t \wedge \tau_b)), \delta_e(U_e(t \wedge \tau_e)) \) and \( \delta_\#(U_\#(t \wedge \tau_\#)) \). Note that \( \delta_b(U_b(t \wedge \tau_b)) \) is a martingale while \( \delta_e(U_e(t \wedge \tau_e)) \) and \( \delta_\#(U_\#(t \wedge \tau_\#)) \) are supermartingales. Thus

\[
E[\delta_b(U_b(t \wedge \tau_b))] = \delta_b(u) \\
\geq \lim_{t \to \infty} E[U_\#(t)1\{\tau_\# = \infty\}] \\
= \delta_\#(u)
\]

and

\[
\delta_\#(u + \varsigma) \geq \lim_{t \to \infty} E[U_\#(t)1\{\tau_\# = \infty, \tau_b = \infty\}] \\
\geq \Pr\{\tau_b = \infty \text{ and } \tau_\# = \infty\} \\
= \delta_b(u) - \exp\left(-\frac{2(\mu - \rho)^2}{\sigma^2\varsigma}\right).
\]

Letting \( \varsigma \to 0 \) gives the required result. ■

If \( \tilde{b} \in [0, 1] \), the above verification lemma proves that this is the optimal proportion when the current surplus is \( u \). However, note that the value of \( \tilde{b} \) is not necessarily inside the indicated interval. For values of \( \tilde{b} \) outside the interval, it will be necessary to consider the dynamics at the endpoints. Observe that if \( \tilde{b} \) is less than 0, then \( \rho > \mu \) and the non-risky asset is more advantageous than the risky asset. In this case, we do not invest in the risky asset. A different scenario is achieved when \( \tilde{b} \) is greater than 1. This time, the risky asset is more advantageous than the non-risky asset and the optimal strategy therefore is to invest the full amount \( A \) on the risky asset. Since (5) is quadratic in \( b \), the supremum value of the quantity inside the braces is therefore attained when \( \tilde{b} = 0, \tilde{b} = 1, \) or \( \tilde{b} = \tilde{b} \). More specifically, the optimal strategy \( b_\bullet(t) \) at when the surplus is \( U(t) \) is determined as follows:

\[
b_\bullet(t) = \begin{cases} 
0 & \text{if } \tilde{b} < 0 \\
-\frac{(\mu - \rho) \delta'(U(t))}{\lambda \delta(U(t))} & \text{if } 0 \leq \tilde{b} \leq 1 \\
1 & \text{if } \tilde{b} > 1. 
\end{cases}
\]  

(7)

As a special case, if the current surplus is zero we do not invest in the risky asset. This is evident from the fact that if the current surplus is 0 and most of the amount \( A \) is invested in the risky asset then there will be greater chance of shortage of surplus to pay out possible early claims. It will follow from (5) that

\[
0 = (c + A\rho) \delta'(0) + \lambda E[\delta(0 - X) - \delta(0)] \\
= (c + A\rho) \delta'(0) - \lambda \delta(0),
\]

and

\[
\delta'(0) = \frac{\lambda \delta(0)}{c + A\rho}.
\]  

(8)
3.4 The optimal non-ruin probabilities

We determine the non-ruin probabilities \( \delta(u) \) for cases \( b_*(t) = 0, b_*(t) = 1, \) and \( b_*(t) = \tilde{b}, \) denoted by \( \delta_0(u), \delta_1(u), \) and \( \delta_{e_3}(u), \) respectively.

3.4.1 Case 1: \( b_*(t) = 0 \)

The classical risk process for insurance business with premium rate \( c + A\rho \) results when we have \( b_*(t) = 0. \) Here it follows that

\[
dU(t) = (c + A\rho) \, dt - dS(t), \quad U(0) = u.
\]

and HJB equation (5) simplifies to

\[
0 = (c + A\rho) \delta'_0(u) + \lambda \, E[\delta_0(u - X) - \delta_0(u)] .
\]

When solved for \( \delta'_0(u), \) the preceding equation is transformed to the form

\[
\delta'_0(u) = \frac{\lambda}{c + A\rho} \, E[\delta_0(u) - \delta_0(u - X)] . \tag{9}
\]

Generally, the differential equation (9) is solved numerically.

3.4.2 Case 2: \( b_*(t) = 1 \)

A similar procedure is used to derive an expression for \( \delta'_1(u). \) If \( b \) is replaced by 1, the HJB equation (5) simplifies to

\[
0 = \frac{1}{2} \sigma^2 A^2 \delta''_1(u) + (c + A\mu) \delta'_1(u) + \lambda \, E[\delta'_1(u - X) - \delta_1(u)].
\]

Equivalently,

\[
\delta''_1(u) = \frac{2}{\sigma^2 A^2} \left\{ \lambda \, E[\delta_1(u) - \delta_1(u - X)] - (c + A\mu) \delta'_1(u) \right\} . \tag{10}
\]

To simplify (10), we integrate both sides of the equation from \( u_1 \) to \( u, \) where \( u_1 \) is taken to be the least capital such that \( \tilde{b} = 1. \) The following differential equation results:

\[
\delta'_1(u) = \frac{2}{\sigma^2 A^2} \int_{u_1}^{u} \left\{ \lambda \, E[\delta_1(t) - \delta_1(t - X)] - (c + A\mu) \delta'_1(t) \right\} \, dt + \delta'_1(u_1) . \tag{11}
\]
3.4.3 Case 3: $b_s(t) = \tilde{b}$

Assuming that $b = \tilde{b}$ where $\tilde{b}$ is given in (6), equation (5) can be equivalently written as

$$-\frac{\delta''_b(u)}{\delta'_b(u)^2} = \frac{(\rho - \mu)^2}{2\sigma^2} \left\{ \lambda E[\delta_b(u) - \delta_b(u - X)] - (c + A\rho) \delta'_b(u) \right\}. \quad (12)$$

Integrating from 0 to $u$,

$$\frac{1}{\delta'_b(u)} = \frac{(\rho - \mu)^2}{2\sigma^2} \int_0^u \frac{1}{\lambda E[\delta_b(t) - \delta_b(t - X)] - (c + A\rho)\delta'_b(t)} \, dt + \frac{1}{\delta'_b(0)}. \quad (13)$$

Solving for $\delta'_b(u)$, the following differential equation results

$$\delta'_b(u) = \left\{ \frac{(\rho - \mu)^2}{2\sigma^2} \int_0^u \frac{1}{\lambda E[\delta_b(t) - \delta_b(t - X)] - (c + A\rho)\delta'_b(t)} \, dt + \frac{1}{\delta'_b(0)} \right\}^{-1}. \quad (13)$$

If $\mu > \rho$, then $\tilde{b} > 0$. As $u$ approaches 0, $\tilde{b}$ approaches 0. Since $\delta'_b(0) > 0$ is finite, it follows that

$$\delta''_b(0^+) = -\infty.$$ 

The left hand side of (12) is infinite and it follows that the denominator of the right hand of the equation must be zero. Thus

$$\lambda E[\delta_b(0) - \delta_b(-X)] = (c + A\rho) \delta'_b(0^+).$$

Note that $\delta_b(-X) = 0$ and

$$\delta'_b(0^+) = \frac{\lambda \delta_b(0)}{c + A\rho}$$

which is similar to (8). Equation (5) can then be written as

$$\delta'_b(u) = \left\{ \frac{(\rho - \mu)^2}{2\sigma^2} \int_0^u \frac{1}{\lambda E[\delta_b(t) - \delta_b(t - X)] - (c + A\rho)\delta'_b(t)} \, dt + \frac{c + A\rho}{\lambda \delta_b(0)} \right\}^{-1}. \quad (13)$$

Equation (13) determines the optimal strategy $b^*$ since once the solution to this equation is characterized, $\tilde{b}$ is solved which in turn gives the value of $b_s(u)$. 

10
The control problem for maximizing expected exponential utility of surplus

In this section we use for our objective function the exponential utility of the surplus at some fixed time $T$. A similar result can be found in [1]. We assume that the insurance business has an exponential utility function

$$h(u) = \kappa - \frac{\pi}{\varphi} e^{-\varphi u}, \quad u \geq 0$$

with constant risk parameter $-h''(u) = \varphi$. Furthermore, we use the Brownian motion approximation for the Cramer-Lundberg process

$$dR(t) = \eta dt + \xi dW_2(t)$$

where

$$\eta = c - \lambda E[X],$$

$$\xi^2 = \lambda E[X^2]$$

and $\{W_2(t), t \geq 0\}$ is a standard Brownian motion. Hence the surplus process in (3) may be written

$$dU(t) = \eta dt + \xi dW_2(t) + A[1 - b(t)]\rho dt + Ab(t)\mu dt + Ab(t)\sigma dW_1(t)$$

$$= \{\eta + A[1 - b(t)]\rho + Ab(t)\mu\} dt + Ab(t)\sigma dW_1(t) + \xi dW_2(t).$$

If we allow the two Brownian motions to be correlated with correlation $\nu$, the quadratic variation of the surplus process is

$$d\langle U \rangle (t) = \{A^2 b(t)^2 \sigma^2 + \xi^2 + 2\nu Ab(t)\sigma\xi\} dt.$$

4.1 The Hamilton-Jacobi-Bellman equation

To find the investment strategy that maximizes the exponential utility of the surplus at the fixed time $T$ with current surplus level $u$, we define the value function

$$V(t, u) = \sup_{b(t)} E_t^{t,u}[h(U(T))]$$

and the Hamilton-Jacobi-Bellman equation for the control problem given by

$$\sup_{b(t)} \left\{ V_t + \{\eta + A[1 - b(t)]\rho + Ab(t)\mu\} V_u + \frac{1}{2} \left\{ A^2 b(t)^2 \sigma^2 + \xi^2 + 2\nu Ab(t)\sigma\xi\right\} V_{uu} \right\} = 0$$

$$V(T, u) = h(u).$$
Note that
\[ V_t + \{ \eta + A[1 - b(t)]\rho + Ab(t)\mu \} V_u + \frac{1}{2} \left\{ A^2 b(t)^2 \sigma^2 + \xi^2 + 2\nu Ab(t)\sigma \xi \right\} V_{uu} \]

is a quadratic equation in \( b(t) \) and that the coefficient \( \frac{1}{2}A^2\sigma^2V_{uu} \) is negative. Hence the maximizer is given by
\[ b(t) = -\frac{1}{A} \left\{ \left( \frac{\mu - \rho}{\sigma^2} \right) \frac{V_u}{V_{uu}} - \frac{\nu \varepsilon}{\sigma} \right\} . \]  

Substituting (16) into (15) results in the following partial differential equation
\[ V_t + \left\{ \eta + A\rho - \frac{\nu \xi (\mu - \rho)}{\sigma} \right\} V_u - \frac{1}{2} \left( \frac{\mu - \rho}{\sigma} \right)^2 \frac{V_u^2}{V_{uu}} + \frac{1}{2} \xi^2 (1 - \nu^2) V_{uu} = 0, \]  

(17)
\[ V(T, u) = h(u). \]

### 4.2 The optimal investment strategy

For the case \( h(u) = \kappa - \frac{\pi}{\phi} e^{-\phi u} \) that we consider here, we try to find a solution to (17) of the form
\[ V(t, u) = \kappa - \frac{\pi}{\phi} \exp \left\{ -\phi u + p(T - t) \right\}, \quad u \geq 0 \]  

(18)
for a suitable function \( p(.) \). Note that the initial condition \( V(T, u) = h(u) \) implies \( p(0) = 0 \). Now substituting (18) into (17) we have
\[ 0 = -p'(T - t) + \frac{1}{2} \xi^2 (1 - \nu^2) \phi^2 - \left( \eta + A\rho - \frac{\nu \xi (\mu - \rho)}{\sigma} \right) \phi - \frac{1}{2} \left( \frac{\mu - \rho}{\sigma} \right)^2. \]

Integrating and setting \( p(0) = 0 \) gives the the value function
\[ V(t, u) = \kappa - \frac{\pi}{\phi} \exp \left\{ -\phi u + (T - t) \cdot Q(\phi) \right\} \]  

(19)
where
\[ Q(\phi) = \frac{1}{2} \phi^2 \xi (1 - \nu^2) - \phi \left( \eta + A\rho - \frac{\nu \xi (\mu - \rho)}{\sigma} \right) + \frac{1}{2} \left( \frac{\mu - \rho}{\sigma} \right)^2. \]

Finally, substituting (19) into (16) gives the surplus independent proportion
\[ b(t) = \frac{1}{A} \left\{ \left( \frac{\mu - \rho}{\sigma^2 \phi} \right) - \frac{\nu \varepsilon}{\sigma} \right\} . \]  

(20)

Note that under the policy defined by (20) the resulting surplus process \( \{ U_b(t), t \geq 0 \} \) is a Brownian motion with drift defined by
\[ dU_b(t) = \left\{ \frac{(\mu - \rho)^2}{\sigma^2 \phi} + \left( \eta + A\rho - \frac{\nu \xi (\mu - \rho)}{\sigma} \right) \right\} dt + \sqrt{\left( \frac{\mu - \rho}{\sigma \phi} \right)^2 + \xi^2 (1 - \nu^2)} dW(t) \]
where \{W(t), t \geq 0\} is a standard Weiner process. Thus, we can see that for \(s \leq T - t\),

\[ E[\exp\{-\phi(U_b(t+s) - U_b(t))\}] = \exp\{s \cdot Q(\phi)\} \]

and we can see that the value function given in (19) is a martingale.

The preceding discussion proves the following result

**Theorem 2** The optimal investment policy that would maximize the expected exponential utility of the surplus at a fixed date \(T\) is to invest a constant proportion

\[ b(t) = \frac{1}{A} \left\{ \frac{(\mu - \rho)}{\sigma^2 \phi} - \frac{\nu \varepsilon}{\sigma} \right\}. \]

of the amount \(A\) available for investment to the risky asset and the maximal expected exponential utility value for such policy is given by

\[ V(t, u) = \kappa - \frac{\pi}{\phi} \exp\{-\phi u + (T - t) \cdot Q(\phi)\} \]

where

\[ Q(\phi) = \frac{1}{2} \phi^2 \xi (1 - \nu^2) - \phi \left( \eta + A \rho - \nu \frac{\varepsilon (\mu - \rho)}{\sigma} \right) - \frac{1}{2} \left( \frac{\mu - \rho}{\sigma} \right)^2. \]
References


