REAL OPTIONS

Version 3 - November 4th, 2003

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Real options deals with options over real assets (such as land, buildings, projects, and so on) as opposed to financial options which are options over financial assets (such as stocks, futures, bonds and so on).

Real options analysis (ROA) has brought about a small revolution in the way decisions and valuations are being made in corporate finance. The bibliography gives a good range of these developments and applications.

Traditionally, projects have been valued by the DCF (Discounted Cash Follows) using formulas like

\[ PV = \sum_{t=1}^{\infty} \frac{E[F_t]}{(1 + r)^t} \]  \hspace{1cm} (1)

that is, the present value (PV) is the sum of expected values of future cash flows \( F_t \) at \( t \) discounted appropriately with a risk adjusted interest rate \( r \) which is often estimated from the capital asset pricing model \( r = r_f + \beta(r_m - r_f) \) with \( r_f \) the risk free rate of interest, \( r_m \) the rate of return of a “market” and \( \beta \) is the beta of the project with respect to the market). The net present value (NPV) of a project is then given by

\[ NPV = PV - C_0 \] \hspace{1cm} (2)

the difference between the present value of the future cash flows and the set up costs \( C_0 \). Actually (1) is more flexible than the expression implies, as it is possible to use a different discount for each expected cash flow. An alternative to (1) is the valuation formula which assigns certainty equivalents to each cash flow \( CEF_t \) and discounts these certainty equivalents with the risk-free interest rate:

\[ PV = \sum_{t=1}^{\infty} \frac{CEF_t}{(1 + r_f)^t} \] \hspace{1cm} (3)

In fact formula (3) is basically a rearrangement of (1), and uses

\[ CEF_t = E[F_t] - \lambda \text{cov}(F_t, \tilde{r}_m) \] \hspace{1cm} (4)
where $\lambda$ is a market price of risk, and $\tilde{r}_m$ is the (random) return of the market.

The **NPV decision rule** suggest that the project could go ahead if $NPV > 0$ and not go ahead if $NPV < 0$.

The problem with this NPV approach is that it does not take into account managerial flexibility during the life of a project. Managers are often able to mitigate losses by scaling down operations, abandoning a project at a salvage value, or by dividing the project into stages, (starting with R&D for example), with decisions at the end of each stage being made whether to proceed further at a specified cost. Under favorable conditions, management is also able to expand operations. In any of these ways, management is able to increase the present value of the project. We can term these managerial flexibilities as **imbedded real options**. It is to be assumed that management is able to exercise these options optimally. It is clear that there is a strong connection between ROA and decision analysis for a project.

In this presentation we shall focus on some old and new observations on the valuation of these embedded options. We shall take the view that the additional value from flexibility (the value of the embedded real options) is given by

$$PV_f - PV_{nf}, \quad (5)$$

the difference between present value of future cash flows with flexibility $(f)$, using embedded options, and the present value of future cash flows with no flexibility $(nf)$. This difference in (5) is always positive and it is therefore possible that $NPV_f \equiv PV_f - C_0 > 0$ while $NPV_{nf} \equiv PV_{nf} - C_0 < 0$ which means that a project with flexibility is viable and without flexibility not so.

The first major studies on real options are Dixit and Pindyck (1994) and Trigeorgis (1996), but the ideas are also alluded to in Donaldson and Lorsch (1983). We leave it to the reader to consult these and other references.

Major advances were made in the early 1970s on the valuation of financial options by Black and Scholes (1973) and Merton (1973). Fischer Black died in 1995, but Scholes and Merton received the Nobel Prize in Economics for this work in 1997. The basic idea was the following: replicate the payoff of an option by a portfolio in cash and the underlying asset (on which the option was written). The present value of the option is then argued to be the same as the present value of this hedge, (the replicating portfolio). We will review this argument in a simple setting. This procedure has been adapted to real option valuation. We will argue that there are problems with this approach and discuss some alternative approaches. However it is now recognised that option valuation theory, in some modified form, is the correct extension to the DCF approach for the valuation of real options. In fact the option pricing methodology is closest to the certainty equivalent approach of formula (3) where the $CEF_t$ is taken as the risk-neutral expectation of $F_t$, at least for financial options.
In order to proceed with the valuation of real options we assume that we have been supplied with a (decision) tree of the present value and future values and cash flows of a project up to some time horizon. We will assume for this discussion that this is a binomial tree with bifurcation at each decision point. As time evolves the number of possible outcomes magnify, consistent with the view that the further we are from the present the less certainty we have about outcomes. A binomial tree could look like

\[
a \rightarrow \begin{cases} 
  b \rightarrow \langle & d \rightarrow \\
  \quad \quad c \rightarrow \\
  \quad \quad \quad f \rightarrow \\
  \quad \quad \quad \quad g \rightarrow
\end{cases}
\]

As Cox, Ross and Rubinstein (1979) (CRR) pointed out, we only need to study a “single cell” in this binomial tree. If we can find the value of the project at b given the value and cash flows of the project at d and e, then we can repeat the same ideas for c, f, g and then a, b, c to find the value at a, representing the present. While trinomial and multinomial trees could also be discussed we will not do so here. In practice binomial trees (or an overlaying of binomial trees) suffice for many applications.

Let us be concrete and consider two times, a year apart, say. We write on the tree the values at the nodes, and we will assume that 170 and 65 are equally likely out comes:

\[
100 \rightarrow \langle 170 \\
\quad 65 \rangle \tag{6}
\]

Let us assume that the risk free interest rate is 8%, that is \( r_f = 0.08 \). So (6) describes a situation without flexibility. Of course, 170 (resp. 65) represents the value of the project plus any cash flows at this node in the tree, etc. while 100 represents only the value of the project at this node (this is a little like a dividend paying stock).

Let us consider the project which has an embedded abandonment for 115 option. We must then calculate \( \nu \) in

\[
\nu \rightarrow \langle 170 \\
\quad 115 \rangle \tag{7}
\]

the present value of the project with the abandonment option. We expect \( \nu > 100 \) and we are interested in the additional value \( \nu - 100 \).

Let us proceed as if we were pricing a financial option. Let us imagine that (6) describes the evolution of a stock. The CRR procedure is as follows: consider a portfolio \((x, y)\) of stocks and cash. The present value is \(100x + y\). In one year’s time the value of this portfolio is either \(170x + 1.08y\) or \(65x + 1.08y\), that is:

\[
100x + y \rightarrow \langle 170x + 1.08y \\
\quad 65x + 1.08y \rangle \tag{8}
\]

We now choose \( x \) and \( y \) so that \(170x + 1.08y = 170\) and \(65x + 1.08y = 115\), yielding \( x = 0.5238 \) and \( y = 74.9559 \) (4 decimal places), and hence \(100x + y = 127.3359\), which
implies that the abandonment option (abandonment put option) is worth 27.3359. This is the portfolio replication argument. In general, if we were to desire \( g_0 \) in terms of \( g_u \) and \( g_d \) in

\[
g_0 \rightarrow \frac{g_u}{g_d}
\]  

(9)

then setting \( 170x + 1.08y = g_u \) and \( 65x + 1.08y = g_d \) we would give on setting \( g_0 = 100x + y \), the expression

\[
g_0 = \frac{qg_u + (1 - q)g_d}{(1 + r_f)}
\]

\[
= \frac{0.4095g_u + 0.5905g_d}{(1 + 0.08)}.
\]

(10)

This is the **risk-neutral pricing formula**. The value of \( q \) in (10) is given by

\[
q = \frac{(1 + r_f)100 - 65}{170 - 65} = 0.4095..\]

(12)

which can be readily generalised to any node in the tree. These simple ideas are the basic ideas behind modern option pricing and were exploited by CRR to give a “simplified” derivation of the Black and Scholes formula.

We claimed above that the \( \nu = 100x + y \). This cannot be deduced mathematically but comes about from two ideas. There are no arbitrage opportunities in an efficient market, and the underlying asset is tradeable. We shall now see why both these notions are crucial.

Let us suppose that \( \nu > 127.3359 \). Using the fact that the underlying asset is tradeable we suppose we can make a portfolio of this underlying asset and cash in any quantity we please. So let us proceed as follows. Let us suppose that \( \nu = 130 \). Short sell the option (borrow and sell) for 130, and invest in the portfolio \((0.5238, 74.9559)\) which costs 127.3359. We have 2.6641 left over, which we pocket. In one year, we unwind the position and cash settle the option (which we borrowed). We will have no unfunded liabilities, as the portfolio will have the same value as the option in one years time (in each state of the world). We have carried out this transaction without using any personal funds, so the 2.6641 in our pocket is riskless profit with which we can buy a “free lunch” (snack, depending on units). What we have described is called an **arbitrage opportunity**. In efficient markets these should not exist, (or not for too long), and so we have derived a **financial contradiction** from the assumption that \( \nu > 127.3359 \). We can write down a similar argument leading to an arbitrage opportunity if we start with \( \nu < 127.3359 \). We then conclude this: if **there are no arbitrage opportunities AND the underlying asset is tradeable** then \( \nu = 100x + y \) as claimed.

If we go back to the **real option pricing** situation we can see where the problem is: the **underlying asset may not be tradeable**, so there may still be no arbitrage opportunities with \( \nu > 127.3359 \) as we may not be able to trade to capture the arbitrage. There seem to be two solutions to this problem:
(a) Assume that the non-tradeable underlying asset can be replicated (hedged) by a portfolio of tradeable assets. We can then synthetically trade the non-tradeable asset by using the tradeable assets which replicate it. This situation is described by saying there is a twin security which is tradeable (or a portfolio of tradeable assets). Under this assumption all the ideas from financial options carry over to real options.

(b) Develop some new principles for valuing options on non-tradeable assets. We will present some recent ideas stimulated by the work of Henderson, Musiela and Zariphopolou, which has roots in the actuarial sciences. These ideas were developed through collaboration with Professor Robert Elliott in 2002/3 during visits to the University of Calgary.

Let us recall that actuaries are faced with a similar dilemma when valuing life insurance premiums. Let us price the real option using actuarial principles. We need to introduce some choice theory. Basically we want to replace the uncertain future cash position with a certain cash position based on choice theory (certainty equivalent), and then discount the certainty equivalent. In this sense we have an extension of (3). However, we shall compute the certainty equivalent using actuarial principals and not the Capital Asset Pricing Model (CAPM) or its variants.

Von Neumann and Morgenstern (1947) showed that if an investor has a way of ranking uncertain outcomes, which occur in one year’s time, say, which satisfy some “natural” properties, then we can assign a utility function $u$ to our investor so that random $X$ is preferred to random $Y$ if and only if $E[u(X)] > E[u(Y)]$. This $u$ satisfies some other “natural” properties. As investors prefer more to less, $u$ should be an increasing function, and as investors always prefer $E[X]$ to $X$, this implies that (and is implied by) $u$ is concave (up). This last condition says (for example) that we prefer $10 to a gamble on a game which pays $0 if tails and $20 if heads. There are many examples of utility functions, but a popular one is the exponential utility. This is of the form

$$u(x) = -\exp(-\gamma \cdot x)$$

for some $\gamma > 0$. The number $\gamma$ measures the level of risk aversion of the investor. The larger $\gamma$ the more is the investor averse to (avoiding) risk. We shall modify the actuarial principle to meet out needs. In our case we are interested in bid prices, (the maximum an investor is prepared to pay for an asset), while an insurance firm is interested in ask prices so that the premiums collected do meet the expected claims in some sense. At wealth $\omega$ of the investor, the certainty equivalent $\bar{G}$ of an uncertain position $G$ in one year’s time is determined from

$$u(\omega + \bar{G}) = E[u(\omega + G)]$$  \hspace{0.5cm} (13)

and with the exponential utility $\bar{G}$ is independent of $\omega$ and is given by the approximate formula

$$\bar{G} \approx E[G] - \frac{\gamma}{2} \text{var}(G)$$  \hspace{0.5cm} (14)
where $\text{var}(G)$ denotes the variance of $G$. We note that for a risky asset $\text{var}(G) > 0$ and so an investor with larger $\gamma$ assigns a smaller certainty equivalent value to $G$. For the binomial situation (13) is equivalent to

$$
\tilde{G} = -\frac{1}{\gamma} \ln \left[ p_u \exp(-\gamma G_u) + p_d \exp(-\gamma G_d) \right]
$$

\hspace{1cm}

(15)

$$
= -\frac{1}{\gamma} \ln \left[ \frac{1}{2} \exp(-\gamma G_u) + \frac{1}{2} \exp(-\gamma G_d) \right]
$$

\hspace{1cm}

(16)

when the “up” and “down” probabilities are the same, $(G_u$ and $G_d$ are equally likely to occur).

Let us apply approximation (14) to get a feel for things. We can write it in the form (with equal up and down probabilities)

$$
\tilde{G} = \frac{G_u + G_d}{2} - \frac{\gamma}{8} (G_u - G_d)^2
$$

\hspace{1cm}

(17)

Using $\tilde{G} = 108 = 100 \times (1.08)$, $G_u = 170$ and $G_d = 65$ this would yield $\gamma = 0.00689342\ldots$, and with this $\gamma$ and $G_u = 170$ and $G_d = 115$ we get $\tilde{G} = 139.8934\ldots$ which has present value $\tilde{G}/1.08 = 129.5309\ldots$, yielding 29.5389 as the value of the abandonment option. This compares with 27.3359 from the risk neutral approach. Carrying this out more accurately using (16) gives $\gamma = 0.00704866\ldots$ and the value of the abandonment put as 29.4919, so the approximation (14) is not too bad.

Let us consider two other real options: (a) the option to expand and (b) the option to contract.

In the first we choose $\tilde{G} = \max [G, 1.1 \times G - 10]$ meaning that we have an embedded option to expand by 10% for an additional 10, and we only exercise if this gives a better outcome (hence the maximum). So we have

$$
\nu \rightarrow \langle \frac{177}{65} \rangle
$$

\hspace{1cm}

(18)

The risk neutral valuation of $\nu$ is 102.6543 implying the embedded option to expand is worth 2.6543. With the “actuarial” approach we get 102.058625 and the embedded option is worth 2.0586 which is less than the risk neutral price. Note that the project with the option to expand is very volatile as there is larger spread in the outcome than the project without embedded option.

We get a different result in the contraction case. Here we take $\tilde{G} = \max [G, 0.7 \times G + 30]$ meaning that we have the option to scale down to 70% and receive (from sale or leasing out) 30 which we activate only if optimal to do so. We then obtain

$$
\nu \rightarrow \langle \frac{170}{75.5} = 0.7 \times 65 + 30 \rangle
$$

\hspace{1cm}

(19)
which is less volatile than the expansion case. Here the risk neutral price \( \nu \) is 105.7407 and the embedded option is 5.7407. The “actuarial” approach gives 106.50287 and 6.502785.

Let us note that the actuarial pricing is non-linear, but it is convex. The average of two assets has less risk than the average of the risks. However, were the assets tradeable then the pricing must be linear to avoid arbitrage.

We believe that this actuarial approach is new, but it has some defects which we address briefly. We could regard the risk neutral pricing and the actuarial pricing as two extremes. The first when options are on tradeable assets, the second when no hedging is possible. We, (Elliott and van der Hoek), have studied an approach which basically collapses to these two extremes when relevant. This overcomes the fact that the actuarial approach does not price derivatives on tradeable assets correctly and risk neutral pricing cannot price unhedgeable claims. Rather we proceed as follows: Let

\[
V_0(x) = \max E[u(X(1))]
\]

where the maximum is taken over all portfolios in tradeable assets with the property that their initial value \( X(0) = x \), and they have uncertain value \( X(1) \) in one year’s time. We also let

\[
V_G(x) = \max E[u(X(1) + G)]
\]

where again the maximum is taken over all portfolios in tradeable assets with the property that their initial value \( X(0) = x \). We then define \( \nu^b(G) \), the bid price of \( G \), as the solution of

\[
V_G(x - \nu) = V_0(x)
\]

In words, this statement says that in the presence of optimal investment in tradeable assets at a given level of wealth \( x \) we are indifferent between \( \nu \) now and \( G \) in one year’s time. The expression for \( \nu^b(G) \) can be computed explicitly for exponential and quadratic utilities, and numerically for a wide range of other utility functions.

In a similar way we can define \( \nu^a(G) \), the asking price for \( G \), and it can be shown that

\[
\nu^a(G) = -\nu^b(-G).
\]

We can then deduce properties of the asking price from those of the bid price using this identity. We will continue to focus on the bid price of \( G \).

We now give some explicit and approximate formulas. For the exponential utility \( u(x) = \exp(-\gamma x) \), where \( \gamma > 0 \) for the two time scenario:

\[
S_0 \rightarrow \begin{pmatrix} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} \\ \begin{bmatrix} S_u \\ S_d \end{bmatrix} \end{pmatrix}
\]
Let
\[ q = \frac{S_0 \cdot (1 + r_f) - S_d}{S_u - S_d} \]  
which generalises (12). Define the (actuarial) certainty equivalents:
\[ CEG_u = -\frac{1}{\gamma} \ln \left[ \frac{p_1 \exp(-\gamma G_1) + p_2 \exp(-\gamma G_2)}{p_1 + p_2} \right] \]  
\[ CEG_d = -\frac{1}{\gamma} \ln \left[ \frac{p_3 \exp(-\gamma G_3) + p_4 \exp(-\gamma G_4)}{p_3 + p_4} \right] \]
and then
\[ \nu^b(G) = \frac{q \cdot CEG_u + (1 - q) \cdot CEG_d}{1 + r_f} \]  
Let us now see what (25) says in two extreme situations. First, $G$ is attainable (hedgeable) if and only if $G_1 = G_2 = G_u$ and $G_3 = G_4 = G_d$, say. Then
\[ \nu^b(G) = rnp(G) = \frac{q \cdot G_u + (1 - q) \cdot G_d}{1 + r_f} \]
the risk neutral price for $G$. Second, if there are no hedging instruments then this is the same as setting $q = 1$, and (25) reduces to the actuarial pricing formula (15) with $p_u = p_1$, $p_d = p_2$, and $p_u + p_d = 1$ which we discussed earlier.

Let us note that the bid and ask prices do not depend on the wealth $x$ in (22). This is a consequence of the exponential utility function. We now discuss approximate formulas for general utilities. The general results are presented in Elliott and van der Hoek, but we give them now for the setup above.

**Approximation 1** is the following:
\[ \nu^b(G) \approx \frac{1}{1 + r_f} [q \cdot L_1 + (1 - q) \cdot M_1] \]  
where
\[ L_1 = \mathbb{E}[G|u] + \frac{1}{2} \frac{u''(\theta_u)}{u'(\theta_u)} \text{var}[G|u] \]  
\[ M_1 = \mathbb{E}[G|d] + \frac{1}{2} \frac{u''(\theta_d)}{u'(\theta_d)} \text{var}[G|d] \]  
\[ V_0(x) = (p_1 + p_2)u(\theta_u) + (p_3 + p_4)u(\theta_d) \]  
\[ (1 + r_f)x = q\theta_u + (1 - q)\theta_d \]  
and
\[ \mathbb{E}[G|u] = \frac{1}{p_1 + p_2} [p_1 G_1 + p_2 G_2] \]  
\[ \text{var}[G|u] = \mathbb{E}[(G - \mathbb{E}[G|u])^2|u] \]
an so on. Note that $G$ is attainable if and only if $\text{var}[G|u] = \text{var}[G|d] = 0$ in which case (27) reduces to risk neutral pricing. The bid price $\nu^b(G)$ now will depend on $x$ through $\theta_u$ and $\theta_d$, and the Arrow-Pratt risk-aversion parameter

$$A(z) = \frac{u''(z)}{u'(z)} \quad (33)$$

which will not be constant for general utilities. A full discussion of results for power utilities ($u(x) = \frac{1}{\alpha} x^\alpha$ with $\alpha < 1, x > 0$) and log and quadratic utilities are discussed in Elliott and van der Hoek.

The second term (for the asking price) can be regarded as a riskloading when pricing a life contingency. So even if these contingencies cannot be hedged in financial markets, the value of the risk loading term is affected by the optimal investment of an insurance company in the financial markets, as this investment strategy has an impact on the company’s attitude to risk.

**Approximation 2** is the following:

$$\nu^b(G) \approx \frac{1}{1 + r_f} [q \cdot L_2 + (1 - q) \cdot M_2] \quad (34)$$

where

$$L_2 = CE[\theta_u + G|u] - \theta_u = u^{-1} [E[u(\theta_u + G)|u]] - \theta_u \quad (35u)$$

$$M_2 = CE[\theta_d + G|d] - \theta_d = u^{-1} [E[u(\theta_d + G)|d]] - \theta_d \quad (35d)$$

where as above $\theta$ is the payoff of the optimal portfolio for $V_0(x)$.

This provides further economic understanding of the formula for $\nu^b(G)$.

**Example**

Consider the example with $S_0 = 20, S_u = 25, S_d = 16$.

$G_1 = 8, G_2 = 6, G_3 = 8$ and $G_4 = 6.$

$p_1 = 0.375, p_2 = 0.125, p_3 = 0.2$ and $p_4 = 0.3$

If $r = 0, \text{ then } q = 0.4444$. Let $x = 200.$

**part (a)** Exponential Utility with $\gamma = 0.01.$

Then $\theta_u = 212.396864, \theta_d = 190.0825088, CEG_u = 7.496238, CEG_d = 6.795206$.

The exact and approximate values for $\nu^b(G)$ are $7.106776, 7.106778, 7.106776$. In fact Approximation 2 is exact for this utility function.
part (b) Power Utility with $\alpha = 0.5$.

Then $\theta_u = 250$, $\theta_d = 160$, $CEG_u = 7.4993$, $CEG_d = 6.7986$.

The exact and approximate values for $v^b(G)$ are 7.109988, 7.110017, 7.109988.

Properties

This pricing functional has some interesting properties including: If $G = G_1 + G_2$ where $G_1$ is hedgeable, then

$$v^b(G_1 + G_2) = rnp(G_1) + v^b(G_2).$$  \hspace{1cm} (23)

This implies the claims made earlier. It generalises both rnp (risk-neutral pricing) and actuarial pricing into one theory. As the pricing functional is non-linear we cannot price the real option separately as we could with financial options, but we calculate instead the additional value that the embedded option imparts to a project. The NPV of a project could then be the difference between the bid price, (the maximum price of a willing buyer), minus the establishment cost, ($C_0$ used earlier).

It is clear that the same precision is not available for real option valuation as for the valuation of financial options. A topic of ongoing study is the investigation of the spread of the valuations that are achieved when using a broad range of utility classes. It would be conjectured that such valuations should not vary appreciably.

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Website: www.puc-rio.br/marco.ind/ro-links.html has links to many real options sites.