Pricing Non Tradable Assets: 
Duality Methods

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Abstract

A discrete two time period model is considered where there are both tradable and non-tradable assets. The optimal investment problem is solved by introducing a dual problem. The indifference bid and ask prices for the untradable asset are then determined for general utilities. The results generalize those of Musiela and Zariphopoulou.

Keywords: Optimal investment, utility, indifference price, discrete time, Arrow-Debreu security

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1 Introduction

We shall consider a simple one period model which includes both tradable and non-tradable assets. Our results generalize those of Musiela and Zariphopoulou, [3], [4].

When there are no non-tradable assets and sufficiently many tradable assets to span the market one can invest to replicate future risk. The initial value of the portfolio must then equal the arbitrage free price of the future position.

This method does not work when certain assets cannot be traded, giving rise to positions which cannot be hedged.

To price such assets and related claims, we proceed by investigating optimal investment policies for different utility functions. The price introduced, the indifference price, equates the difference in the initial values of an optimal investment in the two cases when one can, or cannot, obtain some untradable claim at the final time.

This concept of price has its origins in the work of Hodges and Neuberger [2], and more recently Fritelli [1] and Rouge and El Karoui [6]. These ideas have been applied in the context of real options by Smith and McCrade [7].

In Section 2 below we describe our model and define utility functions in Section 3. A dual optimization problem is discussed in Section 4 and the dual cost function treated in Section 5. The optimization of the dual cost is investigated in Section 6 and the original optimal cost determined in Section 7. The indifference asking price is defined in Section 8 and the indifference bid price in Section 9. Some examples are presented in Section 10. The properties of the indifference price are discussed in Section 11 and some numerical issues in Section 12.

Perhaps the most interesting results appear in Section 13, where approximate pricing formulae are derived. These relate the indifference prices we have obtained to risk neutral prices on the one hand and actuarial prices on the other. Finally, in Section 14, duality methods are used to obtain more explicit expressions for the indifference prices.
2 Model

Consider times $t = 0, t = 1$. At $t = 1$ our random variables will be defined on a finite probability space

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$$

$$\mathcal{F} = 2^\Omega.$$ 

The historical probability of $P$ is such that

$$p_i = P(\{\omega_i\}) > 0, \quad 1 \leq i \leq N.$$ 

Suppose there is a riskless asset, (a bank account), which charges and pays interest at the rate $r$. That is, this asset has value 1 at time 0 and value $R = 1 + r$ at time 1 in all states of the world. We shall usually assume that $r = 0$. Approximation formulae in the sequel remain true for $r \neq 0$.

There are $M$ tradable assets

$$S^1, S^2, \ldots, S^M.$$ 

$$S_0 = (S^1_0, \ldots, S^M_0) \text{ is known.}$$

$$S_1(\omega) = (S^1_1(\omega), S^2_1(\omega), \ldots, S^M_1(\omega)), \quad \omega \in \Omega, \text{ is random.}$$

We assume A1. There are sufficiently many assets $S^1, \ldots, S^M$ such that, together with the bank account, any $S$-contingent claim can be replicated by a portfolio of the form

$$\beta + \alpha \cdot S_1.$$ 

Here $\alpha \in R^M$, $\beta \in R$. Let $\mathcal{F}_S = \sigma\{S_1\}$ and suppose $F_1, \ldots, F_k$ are the atoms of $\mathcal{F}_S$.

For each atom $F_s$, $1 \leq s \leq k$, write $A_s = \{i \in \{1, \ldots, N\} | \omega_i \in F_s\}$. Suppose $a^s$ are Arrow-Debreu securities such that

$$a^s_i(\omega_i) = \begin{cases} 
1 & \text{if } i \in A_s \\
0 & \text{if } i \notin A_s.
\end{cases}$$

Write

$$a^s_0 = q_s, \quad s = 1, \ldots, k. \quad (1)$$
Now $\sum_{s=1}^{k} a_s^s$ is an asset which is 1 for every $\omega$. Therefore, at $t = 0$, the time 0 value of this asset must also be 1. That is,

$$\sum_{s=1}^{k} q_s = 1.$$ 

If $r \neq 0$ then

$$\sum_{s=1}^{k} q_s = 1/R.$$ 

The Arrow-Debreu prices $q_s$ are then no longer risk neutral probabilities, but the $\pi_s$ are, where

$$\pi_s = Rq_s, \quad s = 1, \ldots, k.$$ 

Suppose there are $L$ nontradable assets $Y^1, \ldots, Y^L$.

$$Y_0 = (Y^1_0, \ldots, Y^L_0) \quad \text{is known.}$$

$$Y_1 = (Y^1_1(\omega), \ldots, Y^L_1(\omega)), \quad \omega \in \Omega, \text{ is random.}$$

Initially we invest $x$, putting $\beta$ into the bank account and buying $\alpha$ units of $S^i$, $1 \leq i \leq M$. Write $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M)$. At $t = 0$ this investment is worth

$$X(0) = x = \beta + \alpha \cdot S_0. \quad (2)$$

At time $t = 1$ this becomes

$$X(1) = X^{x,\alpha} = \beta + \alpha \cdot S_1$$

$$= x + \alpha \cdot (S_1 - S_0). \quad (3)$$

As (3) is $\mathcal{F}_t$ measurable we can write

$$X(1) = \sum_{\ell=1}^{k} x_\ell a_1^\ell \quad \text{(4)}$$

so that

$$X(0) = \sum_{\ell=1}^{k} x_\ell \cdot q_\ell = x, \quad (5)$$

where we invest $x_\ell$ in the Arrow-Debreu security $a_\ell$, $\ell = 1, 2, \ldots, k$.

From Assumption A.1 there are numbers $\xi_{ij}$, $i = 1, \ldots, k$. 

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\( j = 1, \ldots, M \) such that
\[
a'_1 = q_i + \sum_{j=1}^{M} \xi_{ij}(S'_1 - S'_0) .
\] (6)

Therefore,
\[
X(1) = \sum_{\ell=1}^{k} x_{\ell}a'_{1}\]
\[
= \sum_{\ell=1}^{k} x_{\ell}(q_{\ell} + \sum_{j=1}^{M} \xi_{\ell j}(S'_1 - S'_0))
\] (7)
\[
= x + \sum_{j=1}^{M} \alpha_j(S'_1 - S'_0).
\]

If \( r \neq 0 \) this equation becomes
\[
x(1) = Rx + \sum_{j=1}^{M} \alpha_j(S'_1 - S'_0)
\]
where \( \alpha_j = \sum_{\ell=1}^{k} x_{\ell}\xi_{\ell j} \).

Further, if \( s = 1, 2, \ldots, k \) and \( i \in A_s \)
\[
x_s = x + \sum_{j=1}^{M} \alpha_j(S'_1(\omega_i) - S'_0).
\] (8)

Again, if \( r \neq 0 \) equation (8) becomes \( x_s = Rx + \sum_{j=1}^{M} (S'_1(\omega_i) - S'_0) \).

**Definition 2.1.** \( Q \) is an equivalent risk neutral measure if
\[
Q_i = Q(\omega_i) > 0, \ i = 1, \ldots, N, \sum_{i=1}^{N} Q_i = 1 \text{ and } E_Q[S_1] = \sum_{i} Q_i S_1(\omega_i) = S_0 \in R^M .
\] (Here \( S_0 \) is replaced by \( RS_0 \in R^M \) when \( r \neq 0 \).) Write \( \mathcal{M} \) for the set of equivalent risk neutral measures.

We quote the following important result from Pliska [5], (page 23).

**Theorem 2.2.** A random variable \( X \) on \( \Omega \) is of the form \( X^{x,\alpha} \) for some \( (x, \alpha) \) if and only if \( E_Q[X] = x \) for all \( Q \in \mathcal{M} \). (Here \( x \) is replaced by \( Rx \) if \( r \neq 0 \).)
We now characterize the risk neutral measures.

**Theorem 2.3.** Under Assumption A.1, \( Q \in \mathcal{M} \) if and only if there exists a random variable \( m \) on \( \Omega \) so that

\[
Q(\omega) = Q_\ell = q_se^{m_\ell}p_\ell
\]

for \( \ell \in A_s \) where

\[
\sum_{\ell \in A_s} p_\ell e^{m_\ell} = 1.
\]

(The right side of (10) is replaced by \( R \) when \( r \neq 0 \).)

**Proof.** Suppose \( Q_\ell \) has the form (9) and \( j \in \{1, \ldots, M\} \). Then

\[
\sum_{\ell} Q_\ell S_j(\omega_\ell) = \sum_{s=1}^{k} q_s S_j^1(F_s)
\]

\[
= S_j^0.
\]

Conversely, suppose \( Q \in \mathcal{M} \) and \( \ell \in A_s \).

For each \( \ell \) define \( m_\ell = \ln\left[ \frac{Q_\ell}{p_\ell q_s}\right] \) so that \( Q_\ell = q_se^{m_\ell}p_\ell \). As the time 0 price of \( a^s \) is \( q_s \)

\[
\sum_{\ell \in A_s} Q_\ell = q_s.
\]

This implies that \( \sum_{\ell \in A_s} e^{m_\ell}p_\ell = 1. \)

### 3 Utility Functions

Our utility functions will be non decreasing functions \( U \) defined on \(( -\infty, \infty) \) with values in \([-\infty, \infty) \). Write \( \varepsilon = \inf \{ x : U(x) > -\infty \} \).

Note we allow \( \varepsilon \) to be \(-\infty\). On the subdomain \(( \varepsilon, \infty) \) we suppose \( U \) is strictly increasing, concave, continuous, continuously differentiable and satisfies

\[
\lim_{x \downarrow \varepsilon} U'(x) = \infty
\]

\[
\lim_{x \to \infty} U'(x) = 0.
\]

\( U' \) is defined on \(( \varepsilon, \infty) \). The continuous, strictly decreasing inverse of the function \( U' \) will be denoted by \( I : (0, \infty) \to (\varepsilon, \infty) \). Then

\[
I(0) := \lim_{y \downarrow 0} I(y) = \infty
\]

\[
I(\infty) := \lim_{y \to \infty} I(y) = \varepsilon.
\]
For $0 < y < \infty$ define
\[
\tilde{U}(y) := \max_{x>\varepsilon} [U(x) - xy] = U(I(y)) - yI(y).
\] (11)
Then $\tilde{U}'(y) = -I(y)$, $0 < y < \infty$. Also,
\[
\min_{y>0} [\tilde{U}(y) + xy] = \tilde{U}(U'(x)) + xU'(x)
= U(x), \text{ for all } x \in R.
\] (12)
From (11) and (12) we have
\[
U(I(y)) \geq U(x) + y[I(y) - x], \text{ for all } x \text{ and } y > 0
\]
\[
\tilde{U}(U'(x)) \leq \tilde{U}(y) - x[U'(x) - y] \text{ for } x > \varepsilon \text{ and } y > 0.
\]
As stated, we assume $r = 0$. However, our discussion immediately extends to the case $r \neq 0$ by considering a modified utility function:
\[
U_R(x) := U(Rx).
\]
Then $\tilde{U}_R(y) = \tilde{U}(y/R)$.

**Example 3.1.** In the sequel we shall usually consider the following utility functions:

- **U1:** $U(x) = -e^{-\gamma x}$, for $x \in \mathbb{R}$, $\gamma > 0$,
- **U2:** $U(x) = \frac{1}{\alpha} x^\alpha$, for $x > 0$, $\alpha < 1$,
- **U3:** $U(x) = \ln x$, $x > 0$.

For these cases with $I = (U')^{-1}$

- **U1:** $I(x) = -\frac{1}{\gamma} \ln \frac{x}{\gamma}$, $x > 0$
- **U2:** $I(x) = x^{1-\alpha}$, $x > 0$,
- **U3:** $I(x) = \frac{1}{x}$, $x > 0$.

In Example 10.1 we consider the quadratic utility
\[
U(x) = -(x - a)^2, \text{ for } x < a.
\]
Then $I(x) = a - \frac{x}{2}$ for $x \geq 0$.  

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Further, with
\[ \tilde{U}(y) = \sup_{x} [U(x) - xy] \]
\[ = U(I(y)) - yI(y) \]
we have in the three cases:

**U1:** \( \tilde{U}(y) = \frac{y}{\gamma} \ln \frac{y}{\gamma} - \frac{y}{\gamma}, \quad y > 0 \)

**U2:** \( \tilde{U}(y) = \frac{1-\alpha}{\alpha} y^{\frac{\alpha}{\alpha-1}}, \quad y > 0 \)

**U3:** \( \tilde{U}(y) = -1 - \ln y, \quad y > 0. \)

The relation (12)
\[ U(x) = \min_{y>0} [\tilde{U}(y) + xy] \]
\[ = \tilde{U}(U'(x)) + xU'(x) \]
can be verified in cases U1, U2 and U3.

## 4 Pricing Claims

Recall there are \( L \) nontradable assets

\[ Y^1, Y^2, \ldots, Y^L \]

\[ Y_0 = (Y_0^1, Y_0^2, \ldots, Y_0^L) \]

and \( Y_1(\omega) = (Y_1^1(\omega), Y_1^2(\omega), \ldots, Y_1^L(\omega)) \)

where \( \omega \in \Omega = \{\omega_1, \omega_2, \ldots, \omega_N\} \).

Suppose \( G(Y_1(\omega)) \) is some claim based on \( Y_1 \). Then \( G(Y_1(\omega)) \) takes the \( N \) values

\[ g_i = G(Y_1(\omega_i)), \quad i = 1, \ldots, N. \]
Definition 4.1. For $X = X^{x,\alpha} = x + \alpha \cdot (S_1 - S_0)$ define

$$V_G(x) = \sup_{\alpha} E_p[U(X^{x,\alpha} - G)]$$

$$= \sup_{\alpha} \left[ \sum_{i=1}^{N} p_i U(X^{x,\alpha}(\omega_i) - g_i) \right]$$

$$= \sup_{X, E \{X \} = x} E_p[U(X - G)]$$

$$= \sup_{\chi \in \mathbb{R}^k} E_p[U(X \cdot a_1 - G)]$$

where we recall that

$$a_1(\omega) = (a_1^1(\omega), a_1^2(\omega), \ldots, a_1^k(\omega))$$

is the vector of values of the Arrow-Debreu securities at time $t = 1$.

Notation 4.2. We shall wish to consider the special case when $G = 0$; we then write

$$V_0(x) = \sup_{\chi \in \mathbb{R}^k} E_p[U(X \cdot a_1)]$$

Remarks 4.3. We first discuss the maximization problem in the definition of $V_G(x)$ and show it is tractable only in the case of utility $U_1$.

Recall

$$V_G(x) = \sup_{X, E \{X \} = x} E_p[U(X \cdot a_1 - G)]$$

$$= \sup_{x_1, \ldots, x_k} \sum_{s=1}^{k} \left( \sum_{\ell \in A_s} U(x_s - g_\ell) p_\ell \right).$$

We can obtain equations for $x_1, \ldots, x_k$ from first order conditions using Lagrange multipliers. Write

$$F(x_1, \ldots, x_k) = \sum_{s=1}^{k} \left( \sum_{\ell \in A_s} U(x_s - g_\ell) p_\ell \right) + \lambda \left[ x - \sum_{s=1}^{k} x_s q_s \right].$$
Then $\frac{\partial F}{\partial x_s} = 0$ implies
\[ \sum_{\ell \in A_s} U'(x_s - g_\ell)p_\ell - \lambda q_s = 0, \text{ for } s = 1, \ldots, k. \]

For any $\lambda$ there is then a unique $x_s(\lambda)$, $s = 1, \ldots, k$. We then find $\lambda$ so that $\sum_{s=1}^{k} x_s(\lambda)q_s = x$. Unfortunately this approach gives explicit results only for utility $U_1$.

**Case U1:** The first order condition is
\[ \sum_{\ell \in A_s} \gamma e^{-\gamma(x_s - g_\ell)} p_\ell = \lambda q_s \]
so
\[ \gamma e^{-\gamma x_s} \left( \sum_{\ell \in A_s} p_\ell e^{\gamma g_\ell} \right) = \lambda q_s \]
and
\[ \frac{\gamma}{\lambda q_s} \cdot \left( \sum_{\ell \in A_s} p_\ell e^{\gamma g_\ell} \right) = e^{\gamma x_s}. \]

Note then that $\lambda > 0$. Consequently,
\[ x_s = \frac{1}{\gamma} \ln \left[ \frac{\gamma}{q_s} \sum_{\ell \in A_s} p_\ell e^{\gamma g_\ell} \right] - \frac{1}{\gamma} \ln \lambda \]
and also
\[ x = -\frac{1}{\gamma} \ln \lambda + \frac{1}{\gamma} \sum_{s=1}^{k} q_s \ln \left[ \frac{\gamma}{q_s} \sum_{\ell \in A_s} p_\ell e^{\gamma g_\ell} \right]. \]

Therefore,
\[ x_s = x + \frac{1}{\gamma} \ln \left[ \frac{\gamma}{q_s} \sum_{\ell \in A_s} p_\ell e^{\gamma g_\ell} \right] \]
\[ - \frac{1}{\gamma} \sum_{r=1}^{k} \left( q_r \ln \left[ \frac{\gamma}{q_r} \sum_{\ell \in A_r} p_\ell e^{\gamma g_\ell} \right] \right) \]
and
\[ V_G(x) = -\sum_{s=1}^{k} \left( \sum_{\ell \in A_s} e^{-\gamma x_s} e^{\gamma g_\ell} p_\ell \right) \]
\[ = -\frac{1}{\gamma} e^{-\gamma x} \exp \left[ \sum_{r=1}^{k} q_r \ln \left[ \frac{\gamma}{q_r} \sum_{\ell \in A_r} p_\ell e^{\gamma g_\ell} \right] \right]. \]
Further, taking \( G = 0 \), (so \( g_\ell = 0 \), for all \( \ell \))

\[
V_0(x) = -\frac{1}{\gamma} e^{-\gamma x} \exp \left[ \sum_{r=1}^{k} q_r \ln \left( \frac{\tilde{p}_r}{q_r} \right) \right]
\]

where

\[
\tilde{p}_r = \sum_{\ell \in A_r} p_\ell.
\]

**Case U2:** The first order condition gives

\[
\sum_{\ell \in A_s} p_\ell (x_s - g_\ell)^{\alpha-1} = \lambda q_s.
\]

This has a unique solution \( x_s(\lambda) \) with

\[
\max_{\ell \in A_s} g_\ell < x_s(\lambda) < \infty.
\]

However, a closed form solution for \( x_s(\lambda) \) is not available, so \( \lambda \) and \( x_s \) cannot be found analytically.

**Case U3:** The first order condition gives

\[
\sum_{\ell \in A_s} \frac{p_\ell}{(x_s - g_\ell)} = \lambda q_s.
\]

Again, this has a unique solution \( x_s(\lambda) \) for any \( \lambda > 0 \) with

\[
\max_{\ell \in A_s} g_\ell < x_s < \infty.
\]

However, again a closed form solution is not available so \( \lambda \) and \( x_s \) cannot be found analytically.

**Remarks 4.4.** We now compute \( V_G(x) \) using duality theory.

We have defined for \( y > 0 \)

\[
\tilde{U}(y) = \max_{x \geq \epsilon} [U(x) - xy]
\]

\[
= U(I(y)) - yI(y).
\]

For \( y > 0 \) define, with \( U_i(x) = U(x - g_i) \),

\[
\tilde{U}_i(y) = \sup_x [U_i(x) - xy] \tag{13}
\]
This supremum occurs when \( x = g_i + I(y) \) so
\[
\tilde{U}_i(y) = U(I(y)) - (g_i + I(y))y
\]
\[
= \tilde{U}(y) - g_i y.
\]

**Definition 4.5.** For \( y > 0 \) define, (with \( Q_i = Q(\omega_i) \)),
\[
\tilde{V}_G(y) = \inf_{Q \in M} \sum_{i=1}^{N} p_i \tilde{U}_i\left(\frac{Q_i}{p_i} y\right)
\]
\[
= \inf_{Q \in M} \sum_{i=1}^{N} p_i \left[ \tilde{U}\left(\frac{Q_i}{p_i} y\right) - \frac{g_i Q_i}{p_i} y\right]
\]
\[
= \inf_{Q \in M} \left( E_P\left[ \tilde{U}\left(\frac{Q}{P} y\right) - y E_Q[G]\right]\right).
\]

**Remark 4.6.** Again, for \( G = 0 \) we define
\[
\tilde{V}_0(y) = \inf_{Q \in M} E_P\left[ \tilde{U}\left(\frac{Q}{P} y\right)\right].
\]

We now establish a key result.

**Theorem 4.7.**
\[
\tilde{V}_G(y) = \sup_x [V_G(x) - xy]
\]  
(15)
and for \( x \) in the domain of \( (U_i) \), \( i = 1, 2, 3 \),
\[
V_G(x) = \inf_{y > 0} [\tilde{V}_G(y) + xy].
\]  
(16)

**Proof.** From (13) for any \( y > 0 \)
\[
\tilde{U}_i(y) \geq U_i(x) - xy
\]
so \( U_i(x) \leq \tilde{U}_i(y) + xy \).

Then for any strategy \( \alpha \), any \( y > 0 \), any \( Q \in M \) and any \( x \) in the domain \( U \):
\[
U_i\left( X^{x,\alpha}(\omega_i) \right) \leq \tilde{U}\left(\frac{Q_i}{p_i} y\right) + X^{x,\alpha}(\omega_i)\left( y \frac{Q_i}{p_i} \right), \quad 1 \leq i \leq N.
\]
Therefore,

\[ E_p[U(X^{x,\alpha} - G)] = \sum_{i=1}^{N} p_i U_i(X^{x,\alpha}(\omega_i)) \]

\[ \leq \sum_{i=1}^{N} p_i \bar{U}(y \frac{Q_i}{p_i}) + yE_Q[X^{x,\alpha}] \]

\[ = \sum_{i=1}^{N} p_i \bar{U}(y \frac{Q_i}{p_i}) + yx \]

so

\[ V_G(x) = \sup_{\alpha} E_p[U(X^{x,\alpha} - G)] \]

\[ \leq \tilde{V}_G(y) + xy \]

and

\[ \tilde{V}_G(y) \geq \sup_x [V_G(x) - xy]. \] (17)

We now establish the converse inequality. For this we need two lemmas.

**Lemma 4.8.** For the \( \hat{Q} = \hat{Q}(G, y) \) in \( \mathcal{M} \), (which is compact), which attains the minimum of

\[ E_p\left[ \bar{U}\left(\frac{Q}{P} y\right) \right] - yE_Q[G] \] (18)

we have \( \hat{Q}(\omega_i) = \hat{Q}_i > 0 \) for \( i = 1, \ldots, N \).

**Proof.** We wish to minimize

\[ E_p\left[ \bar{U}\left(\frac{Q}{P} y\right) \right] - yE_Q[G] \]

subject to \( \sum_{i=1}^{N} Q_i = 1, Q_i > 0, i = 1, \ldots, N \) and \( \sum_{i=1}^{N} Q_i S_j^1(\omega_i) = S_j^0, \)

\( j = 1, \ldots, M. \)

The minimum exists as we are minimizing over a compact set. Using Lagrange multipliers consider

\[ F(Q) = \sum_{i=1}^{N} p_i \bar{U}\left(\frac{Q_i}{p_i} y\right) - y \sum_{i=1}^{N} Q_i g_i \]

\[ + \lambda_0 \left( \sum_{i=1}^{N} Q_i - 1 \right) + \sum_{j=1}^{M} \lambda_j \left( \sum_{i=1}^{N} Q_i S_j^1(\omega_i) - S_j^0 \right). \]
Then \( \frac{\partial F}{\partial q} = 0 \) implies

\[
y \tilde{U}' \left( \frac{Q_i}{p_i} y \right) - g_i y + \lambda_0 + \sum_{j=1}^{M} \lambda_j S_j^i(\omega_i) = 0.
\]

As \( \tilde{U}'(y) = -I(y) \) this gives

\[
\hat{Q}_i = \frac{p_i}{y} U' \left( \frac{1}{y} \left( \lambda_0 + \sum_{j=1}^{M} \lambda_j S_j^i(\omega_i) - g_i y \right) \right) > 0.
\]

Consequently, when normalized,

\[
\hat{Q}_i \in (0, 1), \quad i = 1, \ldots, N.
\]

Remark 4.9. This lemma justifies the differentiation below for any other \( Q \in \mathcal{M} \) and for \( \varepsilon \) in a small enough neighborhood of 0,

\[
(1 - \varepsilon) \hat{Q} + \varepsilon Q \in \mathcal{M}.
\]

Lemma 4.10. For \( \hat{Q} = \hat{Q}(G, y) \) as in Lemma 4.8 and for any other \( Q \in \mathcal{M} \)

\[
E_Q \left[ I \left( \frac{\hat{Q}}{P} y \right) + G \right] = E_{\hat{Q}} \left[ I \left( \frac{\hat{Q}}{P} y \right) + G \right].
\]

Proof. Write

\[
f(\varepsilon) = E_p \left[ \tilde{U} \left( y \cdot \left( 1 - \varepsilon \right) \hat{Q} + \varepsilon Q \right) - \left( (1 - \varepsilon) \hat{Q} + \varepsilon Q \right) \right] yG.
\]

Then

\[
f'(\varepsilon) = E_p \left[ y \frac{(Q - \hat{Q})}{P} \tilde{U}' \left( y \cdot \left( 1 - \varepsilon \right) \hat{Q} + \varepsilon Q \right) - \left( Q - \hat{Q} \right) \right] yG.
\]

However, \( \hat{Q} \) minimizes (18) so \( f'(0) = 0 \) and

\[
E_{\hat{Q}} \left[ y \tilde{U}' \left( \frac{\hat{Q}}{P} \right) - yG \right] = E_Q \left[ y \tilde{U}' \left( \frac{\hat{Q}}{P} \right) - yG \right].
\]

Recall \( \tilde{U}'(y) = -I(y) \) so

\[
E_{\hat{Q}} \left[ I \left( \frac{\hat{Q}}{P} \right) + G \right] = E_Q \left[ I \left( \frac{\hat{Q}}{P} \right) + G \right].
\]
We now conclude the proof of Theorem 4.7.

As \( I(y \hat{Q}) + G \) has the same expected value for all \( Q \in \mathcal{M} \) from Theorem 1.2 it is an attainable claim.

That is, with \( x = E_Q[I(y \hat{Q}) + G] \) there is an \( \hat{\alpha} \in R^N \) such that

\[
I(y \hat{Q}) + G = x + \hat{\alpha} \cdot (S_1 - S_0) = X^{x, \hat{\alpha}}.
\]

Now

\[
\tilde{U}(y \hat{Q}) = U(I(y \hat{Q})) - y \hat{Q} \frac{I(y \hat{Q})}{P}
\]

so

\[
\tilde{U}(y \hat{Q}) - y \frac{\hat{Q}}{P} G = U\left((I(y \hat{Q}) + G) - G\right) - y \frac{\hat{Q}}{P} \left(I(y \hat{Q}) + G\right).
\]

Therefore,

\[
\bar{V}_G(y) = \min_{Q \in \mathcal{M}} E_p\left[\tilde{U}(y \frac{Q}{P}) - y \frac{Q}{P} G\right]
\]

\[
= E_p\left[\tilde{U}(y \frac{Q}{P}) - y \frac{Q}{P} G\right]
\]

\[
= E_p\left[U\left((I(y \hat{Q}) + G) - G\right) - y \frac{\hat{Q}}{P} \left(I(y \hat{Q}) + G\right)\right]
\]

\[
= E_p\left[U(X^{x, \hat{\alpha}} - G) - yE_Q[X^{x, \hat{\alpha}}]\right]
\]

\[
= E_p[U(X^{x, \hat{\alpha}} - G)] - xy
\]

\[
\leq \max_{\alpha} E_p[U(X^{x, \alpha} - G)] - xy
\]

\[
= V_G(x) - xy.
\]

\[
\leq \sup_{x} [V_G(x) - xy].
\]

From (17) and (20)

\[
\bar{V}_G(y) = \sup_{x} [V_G(x) - xy].
\]

Define \( \nabla G(x) = \inf_{y > 0} [\bar{V}_G(y) + xy] \). From (19)

\[
\nabla G(x) \leq V_G(x).
\]
For any $\varepsilon < 0$ there is a $y > 0$ such that
\[
\overline{V}_G(x) > \breve{V}_G(y) + xy - \varepsilon = \sup_x [V_G(z) - zy] + xy - \varepsilon \\
\geq V_G(x) - xy + xy - \varepsilon \\
= V_G(x) - \varepsilon.
\]
Therefore,
\[
\overline{V}_G(x) \geq V_G(x). \tag{22}
\]
From (21) and (22)
\[
\overline{V}_G(x) = V_G(x) = \inf_{y > 0} [\breve{V}_G(y) + xy]
\]
and the proof is complete.

**Corollary 4.11.** Specializing to the case $G = 0$ we have
\[
\breve{V}_0(y) = \sup_x [V_0(x) - xy] \\
V_0(x) = \inf_{y > 0} [\breve{V}_0(y) + xy].
\]

## 5 The Dual Cost Function

In Section 4 we introduced the dual cost function $\breve{V}_G(y)$ and established a relation with $V_G(x)$.

We now investigate $\overline{V}_G(y)$ for the three utility functions $U_1$, $U_2$, $U_3$.

**U1:** For the exponential utility
\[
U(x) = -e^{-\gamma x} \\
U_i(x) = U(x - g_i) = -e^{-\gamma(x - g_i)}.
\]
For $y > 0$, $\breve{U}_i(y) = \sup_x [U_i(x) - xy]$. The maximizing $x$ is when $U_i'(x) = y$ so
\[
\hat{x} = -\frac{1}{\gamma} \log \frac{y}{\gamma} + g_i
\]
and
\[
\breve{U}_i(y) = U_i(\hat{x}) - \hat{x}y \\
= -\frac{y}{\gamma} - g_i y + \frac{y}{\gamma} \ln \frac{y}{\gamma}
\]
\[ \sum_{i=1}^{N} p_i \tilde{U}_i \left( y \frac{Q_i}{p_i} \right) = -\frac{y}{\gamma} - y E_Q[G] + \frac{y}{\gamma} E_Q \left[ \ln \frac{Q}{P} + \ln \frac{y}{\gamma} \right]. \]

Recalling the entropy is \( h(Q\|P) = E_Q[\ln \frac{Q}{P}] \) this is:

\[ -\frac{y}{\gamma} + \frac{y}{\gamma} \ln \frac{y}{\gamma} + \frac{y}{\gamma} \left[ h(Q\|P) - \gamma E_Q[G] \right]. \]

Therefore,

\[ \tilde{V}_G(y) = -\frac{y}{\gamma} + \frac{y}{\gamma} \ln \frac{y}{\gamma} + \frac{y}{\gamma} \inf_{Q \in M} h(Q\|P) - \gamma E_Q[G]. \]

Note the minimizing \( Q \) is independent of \( y \).

**U2:** For the power utility

\[ U(x) = \frac{1}{\alpha} x^\alpha, \quad \alpha < 1, \quad x > 0. \]

\[ U_i(x) = U(x - g_i) = \frac{1}{\alpha} (x - g_i)^\alpha \]

\[ \tilde{U}_i(y) = \sup_x \{ U_i(x) - xy \}, \quad y > 0. \]

The maximizing \( x \) occurs when

\[ U_i'(x) = y \]

so

\[ \hat{x} = y_{\alpha-1} + g_i \]

and

\[ \tilde{U}_i(y) = U_i(\hat{x}) - \hat{x}y \]

\[ = \left[ \frac{1 - \alpha}{\alpha} \right] y_{\alpha-1} - g_i y. \]

Then

\[ \sum_i p_i \tilde{U}_i \left( y \frac{Q_i}{p_i} \right) = \sum_i p_i \left[ \left( \frac{1 - \alpha}{\alpha} \right) y_{\alpha-1} \left( \frac{Q_i}{p_i} \right)^{\alpha-1} - g_i y \frac{Q_i}{p_i} \right] \]

\[ = y_{\alpha-1} \left( \frac{1 - \alpha}{\alpha} \right) \sum_i \left( \frac{Q_i}{p_i} \right) \left( \frac{Q_i}{p_i} \right)^{1-\alpha} - y E_Q[G] \]

\[ = y_{\alpha-1} \left( \frac{1 - \alpha}{\alpha} \right) E_Q \left[ \left( \frac{Q}{P} \right)^{\frac{1}{\alpha-1}} \right] - y E_Q[G] \]
and
\[ \tilde{V}_G(y) = \inf_{Q \in \mathcal{M}} \left[ y^{\frac{\alpha}{1-\alpha}} \left( \frac{1-\alpha}{\alpha} \right) E_Q \left[ \left( \frac{Q}{P} \right) ^{\frac{1}{\alpha}} \right] - yE_Q[G] \right]. \]

Here the optimizing \( Q \) depends on \( y \).

**U3:** For the logarithmic utility
\[ U(x) = \begin{cases} \ln x & \text{for } x > 0 \\ -\infty & \text{for } x \leq 0. \end{cases} \]
\[ U_i(x) = U(x - g_i) = \ln (x - g_i). \]

For \( y > 0 \), \( \tilde{U}_i(y) = \sup_x [U_i(x) - xy] \). The maximizing \( x \) occurs when
\[ U'(x) = y \]
so
\[ \hat{x} = \frac{1}{y} + g_i. \]

Then
\[ \tilde{U}_i(y) = U_i(\hat{x}) - \hat{x}y = -\ln y - 1 - yg_i. \]

\[ \sum_i p_i \tilde{U}_i(y \frac{Q_i}{p_i}) = -\sum_i p_i \ln \left( y \frac{Q_i}{p_i} \right) + p_i + yg_i \frac{Q_i}{p_i} \]
\[ = -\ln y - \sum_i p_i \ln \frac{p_i}{Q_i} - 1 - yE_Q[G] \]
\[ = -\ln y + h(P\|Q) - 1 - yE_Q[G]. \]

Consequently
\[ \tilde{V}_G(y) = \inf_{Q \in \mathcal{M}} \sum_i p_i \tilde{U}_i(y \frac{Q_i}{p_i}) \]
\[ = -\ln y - 1 + \inf_{Q \in \mathcal{M}} [h(P\|Q) - yE_Q[G]]. \]

Again, the optimizing \( Q \) depends on \( y \).

We now compute the minimizing \( \tilde{Q}_G \) and \( \tilde{Q}_0 = \hat{Q} \) for each case.

Recall from Theorem 1.3 that for \( Q \in \mathcal{M} \)
\[ Q(\omega_\ell) = Q_\ell = q_\ell e^{m_\ell p_\ell} \text{ for } \ell \in A_s. \]
Case U1: For the exponential utility

\[
\tilde{V}_G(y) = -\frac{y}{\gamma} + \frac{y}{\gamma} \ln \frac{y}{\gamma} + \frac{y}{\gamma} \inf_{Q \in M} \left[ h(Q\|P) - \gamma E_Q[G] \right].
\]

Recalling the form (23) for \( Q \) we are finding an infimum over the \( m_\ell \) subject to \( \sum_{\ell \in A_s} e^{m_\ell} p_\ell = 1 \).

Write

\[
J = h(Q\|P) - \gamma E_Q[G] + \sum_{s=1}^{k} \lambda_s \left( \sum_{\ell \in A_s} e^{m_\ell} p_\ell - 1 \right)
\]

\[
= \sum_{s=1}^{k} q_s \left( \sum_{\ell \in A_s} p_\ell e^{m_\ell} \ln (q_\ell e^{m_\ell}) \right)
\]

\[
- \gamma \sum_{s=1}^{k} q_s \left( \sum_{\ell \in A_s} e^{m_\ell} g_\ell \right) + \sum_{s=1}^{k} \lambda_s \left( \sum_{\ell \in A_s} e^{m_\ell} p_\ell - 1 \right).
\]

For \( \ell \in A_s \)

\[
\frac{\partial J}{\partial m_\ell} = q_s p_\ell e^{m_\ell} \ln (q_\ell e^{m_\ell}) + q_s p_\ell e^{m_\ell}
\]

\[- \gamma q_s p_\ell e^{m_\ell} g_\ell + \lambda_s e^{m_\ell} p_\ell \]

so \( \frac{\partial J}{\partial m_\ell} = 0 \) when

\[
q_s \ln (q_\ell e^{m_\ell}) + q_s - \gamma q_s g_\ell + \lambda_s = 0.
\]

That is,

\[
\ln q_s + m_\ell + 1 - \gamma g_\ell = -\frac{\lambda_s}{q_s}
\]

and

\[
m_\ell - \gamma g_\ell = -\frac{\lambda_s}{q_s} - 1 - \ln q_s
\]

so

\[
e^{m_\ell} = \frac{e^{\gamma g_\ell} e^{(-\frac{\lambda_s}{q_s} - 1)}}{q_s}.
\]

Now \( \frac{\partial J}{\partial \lambda_s} = 0 \) implies \( \sum_{\ell \in A_s} e^{m_\ell} p_\ell = 1 \). That is

\[
e^{(-\frac{\lambda_s}{q_s})} \left( \sum_{\ell \in A_s} e^{g_\ell} p_\ell \right) = 1
\]
so
\[
e^{-\frac{\lambda s}{q_s}} \frac{1}{q_s} = \left( \sum_{\ell \in A_s} e^{\gamma g_p \ell} \right)^{-1}
\]
and
\[
e^{m_\ell} = e^{\gamma g_p \left( \sum_{r \in A_s} e^{\gamma g_r} p_r \right)}^{-1}.
\]

The optimal \( \hat{Q}_G(\ell) \) is then
\[
\hat{Q}_G(\ell) = q_s e^{m_\ell} p_\ell = q_s e^{\gamma g_p \ell} \left( \sum_{r \in A_s} e^{\gamma g_r} p_r \right)^{-1}.
\]

When \( G = 0 \) the optimal \( \hat{Q}_\ell \) is
\[
\hat{Q}(\ell) = q_s p_\ell \left( \sum_{r \in A_s} p_r \right)^{-1} = \frac{q_s p_\ell}{\bar{p}_s},
\]
where \( \bar{p}_s = \sum_{r \in A_s} p_r \).

Case U2: For the power utility
\[
\bar{V}_G(y) = \inf_{Q \in \mathcal{M}} \left[ \frac{y}{\alpha} \left( \frac{1 - \alpha}{\alpha} \right) \bar{E}_Q \left[ \left( \frac{Q}{P} \right)^{\frac{1}{\alpha - 1}} \right] - y \bar{E}_Q[G] \right].
\]
Write
\[
J = \frac{y}{\alpha - 1} \left( \frac{1 - \alpha}{\alpha} \right) \bar{E}_Q \left[ \left( \frac{Q}{P} \right)^{\frac{1}{\alpha - 1}} \right] - y \bar{E}_Q[G]
\]
\[
+ \sum_{s=1}^{k} \lambda_s \left( \sum_{\ell \in A_s} e^{m_\ell} p_\ell - 1 \right)
\]
\[
= \frac{y}{\alpha - 1} \left( \frac{1 - \alpha}{\alpha} \right) \sum_{\ell=1}^{N} Q_\ell \left( \frac{Q_\ell}{P_\ell} \right)^{\frac{1}{\alpha - 1}} - y \sum_{\ell=1}^{N} Q_\ell g_\ell
\]
\[
+ \sum_{s=1}^{k} \lambda_s \left( \sum_{\ell \in A_s} e^{m_\ell} p_\ell - 1 \right).
\]

Recalling the form (23) for \( Q_\ell \) this is
\[
= \frac{y}{\alpha - 1} \left( \frac{1 - \alpha}{\alpha} \right) \sum_{s=1}^{k} q_s \left( \sum_{\ell \in A_s} \left( e^{m_\ell} \right)^{\alpha - 1} p_\ell \right) - y \sum_{s=1}^{k} q_s \left( \sum_{\ell \in A_s} p_\ell e^{m_\ell} g_\ell \right)
\]
\[
+ \sum_{s=1}^{k} \lambda_s \left( \sum_{\ell \in A_s} e^{m_\ell} p_\ell - 1 \right).
\]
Then for $\ell \in A_s$

$$\frac{\partial J}{\partial m_\ell} = p_\ell e^{m_\ell}\left\{ - y^{\frac{\alpha}{1-\alpha}} q_s^{\frac{\alpha}{1-\alpha}} (e^{m_\ell})^{\frac{1}{1-\alpha}} - yq_s g_\ell + \lambda_s \right\}$$

so $\frac{\partial J}{\partial m_\ell} = 0$ when $(yq_s)^{\frac{\alpha}{1-\alpha}}(e^{m_\ell})^{\frac{1}{1-\alpha}} = \lambda_s - yq_s g_\ell$. That is

$$e^{m_\ell} = (\lambda_s - yq_s g_\ell)^{\alpha-1}(yq_s)^{-\alpha}. \quad (24)$$

Again $\frac{\partial J}{\partial \lambda_s} = 0$ implies $\sum_{\ell \in A_s} p_\ell e^{m_\ell} = 1$ so $\lambda_s$ is determined by the condition

$$\sum_{\ell \in A_s} p_\ell (\lambda_s - yq_s g_\ell)^{\alpha-1} = (yq_s)^\alpha.$$ 

For $y > 0$ write $\phi_s(\lambda) = \sum_{\ell \in A_s} p_\ell (\lambda - yq_s g_\ell)^{\alpha-1}$ and recall $\alpha < 1$. Then

$$\phi_s(\lambda) \neq 0$$

and

$$\phi_s(\lambda) = +\infty$$

where

$$\lambda_s = (\max_{\ell \in A_s} g_\ell) yq_s.$$ 

Also,

$$\phi'_s(\lambda) = -(1 - \alpha) \sum_{\ell \in A_s} \frac{p_\ell}{(\lambda - yq_s g_\ell)^{2-\alpha}}$$

so $\phi'_s(\lambda) < 0$ for $\lambda > \lambda_s$. Therefore, there is a unique $\lambda_s \in (\lambda_s, \infty)$ such that $\phi_s(\lambda_s) = (yq_s)^\alpha$. For this $\lambda_s$ the minimizing $Q \in \mathcal{M}$ is given by:

$$\hat{Q}_G(\ell) = \frac{q_s p_\ell}{(yq_s)^\alpha (\lambda_s - yq_s g_\ell)^{1-\alpha}}.$$ 

When $G = 0$ we see from (24) that

$$\lambda_s^{\alpha-1} = (yq_s)^\alpha \cdot e^{m_\ell}.$$ 

As $\sum_{\ell \in A_s} p_\ell e^{m_\ell} = 1$ and $\hat{p}_s = \sum_{\ell \in A_s} p_\ell$ we have $\hat{p}_s \lambda_s^{\alpha-1} = (yq_s)^\alpha$ giving $e^{m_\ell} = 1/\hat{p}_s$. Then

$$\hat{Q}_G = q_s p_\ell e^{m_\ell} = \frac{q_s p_\ell}{\hat{p}_s}.$$ 

**Case U3:** For the logarithmic utility

$$\tilde{V}_G(y) = -\ln y - 1 + \inf_{\hat{Q} \in \mathcal{M}} \left[ h(P\|Q) - y E_Q[G] \right].$$
Write

\[ J = h(P||Q) - yE_Q[G] + \sum_{s=1}^{k} \lambda_s \left( \sum_{\ell \in A_s} e^{m_\ell} p_\ell - 1 \right) \]

\[ = \sum_{\ell=1}^{N} p_\ell \ell \ln \left( \frac{p_\ell}{Q_\ell} \right) - y \sum_{\ell=1}^{N} Q_\ell g_\ell + \sum_{s=1}^{k} \lambda_s \left( \sum_{\ell \in A_s} e^{m_\ell} p_\ell - 1 \right) \]

\[ = \sum_{\ell} p_\ell \ell \ln p_\ell - \sum_{\ell} p_\ell \ell \ln Q_\ell - y \sum_{\ell} Q_\ell g_\ell + \sum_{s=1}^{k} \lambda_s \left( \sum_{\ell \in A_s} e^{m_\ell} p_\ell - 1 \right). \]

Writing \( Q_\ell = q_s p_\ell e^{m_\ell} \) this is

\[ = \sum_{\ell} p_\ell \ell \ln p_\ell - \sum_{\ell} p_\ell \ell [\ln q_s + m_\ell + \ell n p_\ell] - y \sum_{\ell} q_s p_\ell e^{m_\ell} g_\ell \]

\[ + \sum_{s=1}^{k} \lambda_s \left( \sum_{\ell \in A_s} e^{m_\ell} p_\ell - 1 \right) \]

\[ = - \sum_{\ell} p_\ell [\ln q_s + m_\ell] - y \sum_{\ell} q_s p_\ell e^{m_\ell} g_\ell \]

\[ + \sum_{s=1}^{k} \lambda_s \left( \sum_{\ell \in A_s} e^{m_\ell} p_\ell - 1 \right). \]

For \( \ell \in A_s \) we have

\[ \frac{\partial J}{\partial m_\ell} = p_\ell e^{m_\ell} \left[ -e^{-m_\ell} - yq_s g_\ell + \lambda_s \right] \]

and \( \frac{\partial J}{\partial m_\ell} = 0 \) when \( \frac{1}{e^{m_\ell}} = (\lambda_s - yq_s g_\ell) \) and

\[ e^{m_\ell} = \frac{1}{\lambda_s - yq_s g_\ell} \quad \text{for} \quad \ell \in A_s. \]

The condition \( \frac{\partial J}{\partial A_s} = 0 \) implies that

\[ \sum_{\ell \in A_s} e^{m_\ell} p_\ell = 1 = \sum_{\ell \in A_s} \frac{p_\ell}{(\lambda_s - yq_s g_\ell)}. \]

With \( \lambda_s = (\max_{\ell \in A_s} g_\ell) yq_s \) the same argument as in Case U2 shows there is a unique solution \( \lambda_s \) with \( \lambda_s < \lambda_s < \infty \). Once found we then have

\[ \hat{Q}_G(\ell) = \frac{q_s p_\ell}{(\lambda_s - yq_s g_\ell)}. \]
As in the previous case, when $G = 0$

$$\hat{Q}_\ell = \frac{q_s p_\ell}{p_s}. $$

Of course, case U3 is really the limiting version of U2 as $\alpha$ tends to 0.

**Case U4:** The general utility function.

Write

$$J = E_p[\bar{U}(y \frac{Q}{P})] - yE_Q[G] + \sum_{s=1}^{k} \lambda_s \left( \sum_{\ell \in A_s} e^{m_\ell} p_\ell - 1 \right)$$

$$= \sum_{\ell} p_\ell \bar{U}(y q_se^{m_\ell}) - y \sum_{\ell} q_s p_\ell e^{m_\ell} g_\ell + \sum_{s=1}^{k} \lambda_s \left( \sum_{\ell \in A_s} e^{m_\ell} p_\ell - 1 \right).$$

For $\ell \in A_s$

$$\frac{\partial J}{\partial m_\ell} = p_\ell \bar{U}'(y q_se^{m_\ell}) y q_s e^{m_\ell} - y q_s p_\ell e^{m_\ell} g_\ell + \lambda_s e^{m_\ell} p_\ell$$

$$= p_\ell e^{m_\ell} \left[ \bar{U}'(y q_se^{m_\ell}) y q_s - y q_s g_\ell + \lambda_s \right]$$

and $\frac{\partial J}{\partial m_\ell} = 0$ when

$$\bar{U}'(y q_se^{m_\ell}) y q_s - y q_s g_\ell + \lambda_s = 0.$$ 

Now $\bar{U}' = -I$ where $I$ is the inverse function of $U'$. Therefore,

$$\lambda_s = y q_s g_\ell + y q_s I(y q_se^{m_\ell})$$

so

$$y q_se^{m_\ell} = U'(\frac{\lambda_s - y q_s g_\ell}{y q_s}).$$

The condition $\frac{\partial J}{\partial \lambda_s} = 0$ implies that $\sum_{\ell \in A_s} e^{m_\ell} p_\ell = 1$ so

$$\sum_{\ell \in A_s} \frac{p_\ell}{y q_s} U'(\frac{\lambda_s - y q_s g_\ell}{y q_s}) = 1.$$ 

Again for each $s = 1, \ldots, k$ this has a unique solution $\lambda_s$ with $\lambda_s < \infty$ and then

$$\hat{Q}_G(\ell) = \frac{p_\ell}{y} U'(\frac{\lambda_s - y q_s g_\ell}{y q_s}).$$
The case $G = 0$ can be treated as before giving

\[ yq_s e^{m_\ell} = U' \left( \frac{\lambda_s}{yq_s} \right). \]

Then

\[ \sum_{\ell \in A_s} e^{m_\ell} p_\ell = \sum_{\ell \in A_s} \frac{1}{yq_s} U' \left( \frac{\lambda_s}{yq_s} \right) \cdot p_\ell \]

\[ = 1 = \frac{\bar{p}_s}{yq_s} U' \left( \frac{\lambda_s}{yq_s} \right) \]

\[ = \frac{\bar{p}_s}{yq_s} U'(\theta^0_s(y)), \]

where

\[ \theta^0_s(y) = \frac{\lambda_s}{yq_s}, \]

so

\[ \lambda_s = yq_s I \left( \frac{yq_s}{p_s} \right). \]

This gives, as in the previous cases, that

\[ \hat{Q}_\ell = \frac{q_s p_\ell}{p_s}. \]

The value of $\tilde{V}_G(y)$ will now be computed for $y > 0$.

**Case U1**: With $\tilde{Q}_G(\ell) = q_s e^{\gamma g_\ell} p_\ell \cdot \left( \sum_{r \in A_s} e^{\gamma g_r} p_r \right)^{-1}$ and writing $\gamma_s = \sum_{r \in A_s} e^{\gamma g_r} p_r$

\[ \tilde{V}_G(y) = -\frac{y}{\gamma} + \frac{y}{\gamma} \ln \frac{y}{\gamma} + \frac{y}{\gamma} \left[ h(\tilde{Q}_G || P) - \gamma E \tilde{Q}_G[G] \right] \]

\[ = -\frac{y}{\gamma} + \frac{y}{\gamma} \ln \frac{y}{\gamma} + \frac{y}{\gamma} \sum_{\ell} \left[ \frac{q_s e^{\gamma g_\ell}}{\gamma_s} \ln \frac{q_s e^{\gamma g_\ell}}{\gamma_s} \right] \]

\[ - \gamma \cdot \frac{y}{\gamma} \sum_{\ell} \left( \frac{q_s p_\ell e^{\gamma g_\ell}}{\gamma_s} g_\ell \right) \]

\[ = -\frac{y}{\gamma} + \frac{y}{\gamma} \ln \frac{y}{\gamma} + \frac{y}{\gamma} \sum_{s=1}^k \left( q_s \ln \frac{q_s}{\gamma_s} \right). \]
When $G = 0$ we saw $\hat{Q}_\ell = \frac{q_\ell p_\ell}{\tilde{p}_s}$ so

$$\tilde{V}_0(y) = -\frac{y}{\gamma} + \frac{y}{\gamma} \ln \frac{y}{\gamma} + \frac{y}{\gamma} h(\hat{Q}) P$$

$$= -\frac{y}{\gamma} + \frac{y}{\gamma} \ln \frac{y}{\gamma} + \frac{y}{\gamma} \sum_{\ell} q_\ell p_\ell \ln \left( \frac{q_\ell}{\tilde{p}_s} \right)$$

$$= -\frac{y}{\gamma} + \frac{y}{\gamma} \ln \frac{y}{\gamma} + \frac{y}{\gamma} \sum_{s=1}^k q_\ell \ln \left( \frac{q_\ell}{\tilde{p}_s} \right),$$

where $\tilde{p}_s = \sum_{\ell \in A_s} p_\ell$.

**Case U2:** With $\hat{Q}_G(\ell) = \frac{q_\ell p_\ell}{(yq_s)^\alpha (\lambda_s - yq_s g_\ell)^{1-\alpha}}$

$$\tilde{V}_G(y) = y^{\frac{\alpha}{\alpha-1}} (1 - \frac{1}{\alpha}) E_{Q_G} \left[ \left( \frac{\hat{Q}_G}{P} \right)^{\frac{1}{\alpha}} \right] - y E_{Q_G}[G]$$

$$= \frac{1}{\alpha} \sum_{\ell} \frac{p_\ell}{(yq_s)^\alpha (\lambda_s - yq_s g_\ell)^{1-\alpha}} - \sum_{s=1}^k \lambda_s$$

where $\lambda_s = \lambda_s^G(y)$ is the unique solution of

$$\sum_{\ell \in A_s} \frac{p_\ell}{(yq_s)^\alpha (\lambda_s - yq_s g_\ell)^{1-\alpha}} = 1$$

for $s = 1, \ldots, k$.

We shall write

$$\theta_s^G(y) = \frac{\lambda_s^G(y)}{yq_s}$$

so

$$\sum_{\ell \in A_s} \frac{p_\ell}{yq_s \left( \theta_s^G(y) - g_\ell \right)^{(1-\alpha)}} = 1.$$

When $G = 0$

$$\tilde{V}_0(y) = y^{\frac{\alpha}{\alpha-1}} (1 - \frac{1}{\alpha}) E_{Q_G} \left[ \left( \frac{\hat{Q}_G}{P} \right)^{\frac{1}{\alpha}} \right].$$

The minimizing $Q$ is of the form

$$\hat{Q}_\ell = \frac{q_\ell p_\ell}{\tilde{p}_s}$$

so

$$\tilde{V}_0(y) = \frac{1 - \alpha}{\alpha} \sum_{s=1}^k \lambda_s^0(y)$$

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where \( \lambda^0_s(y) = y^{-\alpha} q_s^{-\alpha} \frac{1}{p_s^{1-\alpha}}. \)

With \( G = 0 \)
\[
\theta^0_s(y) = \left( \frac{y q_s}{p_s} \right)^{1-\alpha}.
\]

**Case U3:** Here
\[
\tilde{V}_G(y) = -\ln y - 1 + h(P\|\hat{Q}_G) - y E_{\hat{Q}_G}(G)
\]
\[
= -\ln y - 1 - \sum_{\ell} p_{\ell} \ln \left[ \frac{q_s}{\lambda_s - y q_s g_{\ell}} \right] - y \sum_{\ell} \frac{q_s p_{\ell} g_{\ell}}{(\lambda_s - y q_s g_{\ell})}
\]

where \( \lambda_s = \lambda^G_s(y) \) is the unique solution of
\[
\sum_{\ell \in A_s} \frac{p_{\ell}}{(\lambda_s - y q_s g_{\ell})} = 1.
\]

Then
\[
\tilde{V}_G(y) = -\ln y - 1 - \sum_{\ell} p_{\ell} \ln \left[ \frac{\lambda_s - y q_s g_{\ell}}{q_s} \right]
\]
\[
+ \sum_{\ell} \frac{p_{\ell}}{(\lambda_s - y q_s g_{\ell})} \cdot (\lambda_s - y q_s g_{\ell}) - \sum_{\ell} \frac{\lambda_s p_{\ell}}{(\lambda_s - y q_s g_{\ell})}
\]
\[
= \sum_{\ell} p_{\ell} \ln \left[ \frac{\lambda_s - y q_s g_{\ell}}{y q_s} \right] - \sum_{s=1}^k \lambda^G_s(y).
\]

Writing
\[
\theta^G_s(y) = \frac{\lambda^G_s(y)}{y q_s}
\]

we have
\[
\tilde{V}_G(y) = \sum_{\ell} p_{\ell} \ln \left( \theta^G_s(y) - g_{\ell} \right) - y \sum_{s=1}^k q_s \theta^G_s(y).
\]

When \( G = 0 \)
\[
\tilde{V}_0(y) = -\ln y - 1 + h(P\|\hat{Q})
\]
\[
= -\ln y - 1 + \sum_{\ell} p_{\ell} \ln \left( \frac{p_{\ell}}{Q_{\ell}} \right)
\]
\[
= -\ln y - 1 - \sum_{\ell} p_{\ell} \ln \left( \frac{q_s}{p_s} \right)
\]
\[
= -\ln y - 1 - \sum_{s=1}^k \tilde{p}_s \ln \left( \frac{q_s}{p_s} \right).
\]
Here $\lambda_s^0(y) = \bar{p}_s$, so with $\theta_s^0(y) = \frac{\bar{p}_s}{yq_s}$ we have

$$\bar{V}_0(y) = -1 + \sum_{s=1}^k \bar{p}_s \ln \theta_s^0(y).$$

**Case 4:** For a general utility

$$\bar{V}_G(y) = E_p \left[ \bar{U} \left( y \frac{\hat{Q}_G}{P} \right) \right] - y E_{\hat{Q}_G}[G]$$

$$= \sum_{\ell} p_{\ell} \bar{U} \left[ U' \left( \frac{\lambda_s - yq_s g_{\ell}}{yq_s} \right) \right] - y \sum_{\ell} \frac{p_{\ell}}{y} U' \left( \frac{\lambda_s - yq_s g_{\ell}}{yq_s} \right) g_{\ell}$$

$$= \sum_{\ell} p_{\ell} \bar{U} \left[ U' \left( \frac{\lambda_s - yq_s g_{\ell}}{yq_s} \right) \right] - \sum_{\ell} U' \left( \frac{\lambda_s - yq_s g_{\ell}}{yq_s} \right) p_{\ell} g_{\ell}.$$

Here $\lambda_s = \lambda_s^G(y)$ is the unique solution of

$$\sum_{\ell \in A_s} \frac{p_{\ell}}{yq_s} U' \left( \frac{\lambda_s - yq_s g_{\ell}}{yq_s} \right) = 1.$$

In terms of $\theta_s^G(y) = \frac{\lambda_s^G(y)}{yq_s}$ that is

$$\sum_{\ell \in A_s} \frac{p_{\ell}}{yq_s} U' \left( \theta_s^G(y) - g_{\ell} \right) = 1.$$

When $G = 0$

$$\bar{V}_0(y) = E_p \left[ \bar{U} \left( y \frac{\hat{Q}}{P} \right) \right]$$

$$= \sum_{\ell} p_{\ell} \bar{U} \left( y \frac{q_{\ell}}{p_{\ell}} \right)$$

$$= \sum_{s=1}^k \bar{p}_s \bar{U} \left( y \frac{q_{s}}{p_{s}} \right).$$

In terms of $\theta_s^0(y) = \frac{\bar{p}_s}{yq_s}$ that is

$$\bar{V}_0(y) = \sum_{s=1}^k \bar{p}_s \bar{U} \left( \frac{1}{\theta_s^0(y)} \right).$$
Example: Note for Case U3, the logarithmic utility, $\tilde{U}(z) = -1 - \elln z$, $z > 0$. Then

$$
\sum_{s=1}^{k} \tilde{p}_s \left( -1 - \elln \left( y \frac{q_s}{p_s} \right) \right) = -1 - \elln y - \sum_{s=1}^{k} \tilde{p}_s \elln \left( \frac{q_s}{p_s} \right) \\
= -1 + \sum_{s=1}^{k} \tilde{p}_s \ln \theta_s^0(y).
$$

agreeing with the previous result.

6 The Minimum of $\tilde{V}_G(y)$ and $\tilde{V}_0(y)$

We now calculate $V_G(x)$ and $V_0(x)$. Recall

$$
V_G(x) = \inf_{y>0} [\tilde{V}_G(y) + xy]
= \tilde{V}_G(\hat{y}) + x\hat{y}
$$

where $x + \tilde{V}_G'(\hat{y}) = 0$.

Case 1: We have in this case

$$
\tilde{V}_G(y) = -\frac{y}{\gamma} + \frac{y}{\gamma} \elln \frac{y}{\gamma} + \frac{y}{\gamma} \sum_{s=1}^{k} \left( q_s \elln \left( \frac{q_s}{\gamma_s} \right) \right),
$$

with

$$
\gamma_s = \sum_{r \in A_s} p_r e^{\gamma r}.
$$

Then

$$
\tilde{V}_G'(y) = \frac{1}{\gamma} \elln \frac{y}{\gamma} + \frac{1}{\gamma} \sum_{s=1}^{k} q_s \elln \left( \frac{q_s}{\gamma_s} \right).
$$

The $\lambda^G_s(y)$ quantities, $s = 1, \ldots, k$, which arise in Cases U2 and U3, do not appear in Case U1. However, write

$$
\lambda^G_s(y) = yq_s g_e - yq_s \tilde{U}'(yq_s e^{m*})
= yq_s g_e + yq_s I(yq_s e^{m*}),
$$

so

$$
\theta^G_s(y) = \frac{\lambda^G_s(y)}{yq_s} = g_e + I(yq_s e^{m*}).
$$
For the exponential utility $I(z) = -\frac{1}{\gamma} \ln \frac{z}{\gamma}$, so $z > 0$, so

$$\lambda^G_s(y) = yq_s g_e - \frac{yq_s}{\gamma} \ln \left( \frac{yq_se^{m_e}}{\gamma} \right)$$

and

$$\theta^G_s(y) = g_e - \frac{1}{\gamma} \ln \left( \frac{yq_se^{m_e}}{\gamma} \right).$$

As $e^{m_e} = \frac{e^{\gamma g_e}}{\gamma}$,

$$\lambda^G_s(y) = yq_s g_e - \frac{yq_s}{\gamma} \left[ \ln \frac{y}{\gamma} + \ln \frac{q_s}{\gamma} + \gamma g_e \right]$$

$$= -q_s \frac{y}{\gamma} \ln \frac{y}{\gamma} - q_s \frac{y}{\gamma} \ln \frac{q_s}{\gamma}$$

so

$$-\frac{1}{y} \sum_{s=1}^{k} \lambda^G_s(y) = \frac{1}{\gamma} \ln \frac{y}{\gamma} + \frac{1}{\gamma} \sum_{s=1}^{k} q_s \ln \left( \frac{q_s}{\gamma} \right)$$

$$= \tilde{V}'_G(y)$$

as above.

We shall see this result holds in general. When $G = 0$ we saw

$$\tilde{V}_0(y) = -\frac{y}{\gamma} + \frac{y}{\gamma} \ln \frac{y}{\gamma} + \frac{y}{\gamma} \sum_{s=1}^{k} q_s \ln \left( \frac{q_s}{p_s} \right)$$

so

$$\tilde{V}'_0(y) = \frac{1}{\gamma} \ln \frac{y}{\gamma} + \frac{1}{\gamma} \sum_{s=1}^{k} q_s \ln \left( \frac{q_s}{p_s} \right).$$

Here the appropriate $\lambda^0_s(y)$ is

$$\lambda^0_s(y) = yq_s I \left( \frac{yq_s}{p_s} \right)$$

$$= -\frac{yq_s}{\gamma} \ln \left( \frac{y}{\gamma} \frac{q_s}{p_s} \right)$$

$$= -\frac{yq_s}{\gamma} \left[ \ln \frac{y}{\gamma} + \ln \frac{q_s}{p_s} \right].$$

That is,

$$\theta^0_s(y) = \lambda^0_s(y)$$

$$= \frac{1}{\gamma} \left[ \ln \frac{y}{\gamma} + \ln \frac{q_s}{p_s} \right].$$
Again we see
\[
\tilde{V}_0'(y) = -\frac{1}{y} \sum_{s=1}^{k} \lambda_0^s(y)
\]
\[= -\sum_{s=1}^{k} q_s \theta_s^0(y).
\]

**Case U2:** For the power utility we have seen that
\[
\tilde{V}_G(y) = \frac{1}{\alpha} \sum_{\ell} \frac{p_{\ell}}{(yq_s)^\alpha (\lambda_s - yq_s g_{\ell})^{-\alpha}} - \sum_{s=1}^{k} \lambda_s
\]
so
\[
\tilde{V}_G'(y) = -\frac{1}{y} \sum_{\ell} \frac{p_{\ell}}{(yq_s)^\alpha (\lambda_s - yq_s g_{\ell})^{-\alpha}} - \frac{1}{y} \sum_{\ell} \frac{y p_{\ell} q_s g_{\ell}}{(yq_s)^\alpha (\lambda_s - yq_s g_{\ell})^{1-\alpha}} = -\frac{1}{y} \sum_{\ell} \lambda_s^G(y) = -\sum_{s=1}^{k} q_s \theta_s^G(y),
\]
where we recall \(\lambda_s = \lambda_s^G(y)\) is the unique solution of
\[
\sum_{\ell \in A_s} \frac{p_{\ell}}{(yq_s)^\alpha (\lambda_s - yq_s g_{\ell})^{1-\alpha}} = 1
\]
for each \(s = 1, \ldots, k\).

When \(G = 0\)
\[
\tilde{V}_0(y) = \frac{1 - \alpha}{\alpha} \sum_{s=1}^{k} \lambda_0^s(y)
\]
where
\[
\lambda_0^s(y) = y^{\frac{\alpha}{\alpha-1}} q_s^{\frac{\alpha}{\alpha-1}} p_s^{\frac{1}{\alpha}}.
\]
Therefore,
\[
\frac{\partial \lambda_0^s(y)}{\partial y} = \frac{\alpha}{\alpha - 1} \cdot y^{\frac{\alpha}{\alpha-1}} q_s^{\frac{\alpha}{\alpha-1}} p_s^{\frac{1}{\alpha}} = \frac{\alpha}{\alpha - 1} \cdot \frac{1}{y} \cdot \lambda_0^s(y)
\]
\[ \tilde{V}'_0(y) = \frac{1 - \alpha}{\alpha} \sum_{s=1}^{k} \frac{\partial \lambda_s^0(y)}{\partial y} \]
\[ = \frac{1 - \alpha}{\alpha} \cdot \frac{\alpha}{\alpha - 1} \cdot \frac{1}{y} \sum_{s=1}^{k} \lambda_s^0(y) \]
\[ = - \frac{1}{y} \sum_{s=1}^{k} \lambda_s^0(y) = - \sum_{s=1}^{k} q_s \theta_s^0(y). \]

**Case U3:** For the logarithmic utility

\[ \tilde{V}_G(y) = \sum_{\ell} p_\ell \ln \left[ \frac{\lambda_s - y q_s g_{\ell}}{y q_s} \right] - \sum_{s=1}^{k} \lambda_s \]

so

\[ \tilde{V}'_G(y) = \sum_{\ell} p_\ell \frac{1}{\lambda_s - y q_s g_{\ell}} \left[ \frac{\partial \lambda_s}{\partial y} - q_s g_{\ell} \right] \]
\[ - \sum_{\ell} p_\ell \frac{1}{y} \sum_{s=1}^{k} \frac{\partial \lambda_s}{\partial y} \]
\[ = - \frac{1}{y} \sum_{\ell} \frac{p_\ell q_s g_{\ell}}{\lambda_s - y q_s g_{\ell}} \]
\[ = - \frac{1}{y} \left[ 1 + \sum_{\ell} \frac{\lambda_s p_\ell}{\lambda_s - y q_s g_{\ell}} - \sum_{\ell} p_\ell \right] \]
\[ = - \frac{1}{y} \sum_{s=1}^{k} \lambda_s^G(y) = - \sum_{s=1}^{k} q_s \theta_s^G(y), \]

as \( \lambda_s = \lambda_s^G(y) \) is the unique solution of

\[ \sum_{\ell \in A_s} \frac{p_\ell}{(\lambda_s - y q_s g_{\ell})} = 1. \]

When \( G = 0 \),

\[ \tilde{V}_0(y) = -\ell n y - 1 - \sum_{s=1}^{k} \tilde{p}_s \ell n \left( \frac{q_s}{p_s} \right) \]

and

\[ \lambda_s^0(y) = \tilde{p}_s. \]
Therefore,
\[ \sum_{s=1}^{k} \lambda_s^0(y) = \sum_{s=1}^{k} \bar{p}_s = 1 \]

so
\[ \bar{V}'_0(y) = -\frac{1}{y} = -\frac{1}{y} \sum_{s=1}^{k} \lambda_s^0(y) = -\sum_{s=1}^{k} q_s \theta_s^0(y). \]

**Case 4:** The general utility.

We have seen that
\[ \bar{V}_G(y) = \sum_\ell p_\ell \bar{U}' \left( \frac{\lambda_s - y q_s g_\ell}{y q_s} \right) - \sum_\ell U' \left( \frac{\lambda_s - y q_s g_\ell}{y q_s} \right) p_\ell g_\ell \]

so
\[ \bar{V}'_G(y) = \sum_\ell p_\ell \bar{U}' \left( \frac{\lambda_s - y q_s g_\ell}{y q_s} \right) \cdot U'' \left( \frac{\lambda_s y q_s g_\ell}{y q_s} \right) \cdot \frac{\partial}{\partial y} \left[ \frac{\lambda_s - y q_s g_\ell}{y q_s} \right] \]
\[ - \sum_\ell p_\ell g_\ell U'' \left( \frac{\lambda_s - y q_s g_\ell}{y q_s} \right) \frac{\partial}{\partial y} \left[ \frac{\lambda_s - y q_s g_\ell}{y q_s} \right]. \]

Recall \( \bar{U}' = -I \). Then
\[ \bar{V}_G(y) = - \sum_\ell p_\ell \lambda_s g_\ell \left( \frac{\lambda_s y q_s g_\ell}{y q_s} \right) \frac{\partial}{\partial y} \left[ \frac{\lambda_s - y q_s g_\ell}{y q_s} \right] \]
\[ - \sum_\ell p_\ell g_\ell U'' \left( \frac{\lambda_s - y q_s g_\ell}{y q_s} \right) \frac{\partial}{\partial y} \left[ \frac{\lambda_s - y q_s g_\ell}{y q_s} \right] \]
\[ = -\frac{1}{y} \sum_\ell \frac{p_\ell \lambda_s}{q_s} U'' \left( \frac{\lambda_s - y q_s g_\ell}{y q_s} \right) \cdot \frac{\partial}{\partial y} \left[ \frac{\lambda_s - y q_s g_\ell}{y q_s} \right]. \]

Now recall \( \lambda_s = \lambda_s^G(y) \) was the unique solution of
\[ \sum_{\ell \in A_s} \frac{p_\ell}{y q_s} U' \left( \frac{\lambda_s - y q_s g_\ell}{y q_s} \right) = 1 \]
for \( s = 1, \ldots, k \).

Differentiating we see
\[ \sum_{\ell \in A_s} p_\ell U'' \left( \frac{\lambda_s - y q_s g_\ell}{y q_s} \right) \cdot \frac{\partial}{\partial y} \left[ \frac{\lambda_s - y q_s g_\ell}{y q_s} \right] = q_s. \]
Therefore, 
\[
\tilde{V}'_G(y) = -\frac{1}{y} \sum_{s=1}^{k} \lambda^G_s(y) = -\sum_{s=1}^{k} q_s \theta^G_s(y).
\]

When \( G = 0 \) in the general case

\[
\tilde{V}'_0(y) = \sum_{s=1}^{k} \tilde{p}_s \tilde{U}\left(y \frac{q_s}{\tilde{p}_s}\right)
\]

so

\[
\tilde{V}'_0(y) = \sum_{s=1}^{k} \tilde{p}_s \tilde{U}'\left(y \frac{q_s}{\tilde{p}_s}\right) \frac{q_s}{\tilde{p}_s}
\]

\[
= -\sum_{s=1}^{k} \tilde{p}_s I\left(y \frac{q_s}{\tilde{p}_s}\right) \frac{q_s}{\tilde{p}_s}.
\]

Now from (24) \( \tilde{y} q_{\tilde{y}} U'(\frac{\lambda_{\tilde{y}}}{y q_{\tilde{y}}}) = 1 \) so

\[
\frac{\lambda_{\tilde{y}}}{y q_{\tilde{y}}} = I\left(\frac{y q_{\tilde{y}}}{\tilde{p}_s}\right)
\]

and

\[
\tilde{V}'_0(y) = -\frac{1}{y} \sum_{s=1}^{k} \lambda^0_s(y) = -\sum_{s=1}^{k} q_s \theta^0_s(y).
\]

7 The Calculation of \( V_0(x) \)

Recall from Theorem 4.7:

\[
V_G(x) = \inf_{y > 0} [\tilde{V}_G(y) + xy]
\]

and \( V_0(x) = \inf_{y > 0} [\tilde{V}_0(y) + xy] \).

The infimum occurs when \( \tilde{V}_G'(y) + x = 0 \) (resp. \( \tilde{V}_0'(y) + x = 0 \)), that is, when

\[
x = \frac{1}{y} \sum_{s=1}^{k} \lambda^G_s(y) = \sum_{s=1}^{k} q_s \theta^G_s(y)
\]

\[
\text{(resp. } x = \frac{1}{y} \sum_{s=1}^{k} \lambda^0_s(y) = \sum_{s=1}^{k} q_s \theta^0_s(y)\text{)}.
\]

If \( \tilde{y} \) is a solution of (25) then \( V_G(x) = \tilde{V}_G(\tilde{y}) + x\tilde{y} \), and similarly for \( V_0(x) \).
We first calculate $V_0(x)$.

**Case U1:** Here $x = \frac{1}{y} \sum_{s=1}^{k} \lambda^0_s(\tilde{y})$ so

$$-x = \frac{1}{\gamma} \ell \ln \frac{\tilde{y}}{\gamma} + \frac{1}{\gamma} \sum_{s=1}^{k} q_s \ln \left[ \frac{q_s}{\bar{p}_s} \right].$$

Therefore,

$$\hat{y} = \gamma \exp \left[ -\gamma x - \sum_{s=1}^{k} q_s \ln \left( \frac{q_s}{\bar{p}_s} \right) \right]$$

$$= \gamma e^{-\gamma x} \prod_{s=1}^{k} \left( \frac{\bar{p}_s}{q_s} \right)^{q_s}$$

$$= \gamma e^{-\gamma x} \mu$$

where $\mu = \prod_{s=1}^{k} \left( \frac{\bar{p}_s}{q_s} \right)^{q_s}$.

So

$$\tilde{V}_0(\tilde{y}) = -\frac{\hat{y}}{\gamma} + \frac{\hat{y}}{\gamma} \ell \ln \frac{\hat{y}}{\gamma} - \frac{\hat{y}}{\gamma} \ell \ln \mu$$

$$= -\frac{\hat{y}}{\gamma} - \hat{y} x$$

and, therefore

$$V_0(x) = \tilde{V}_0(\tilde{y}) + x \hat{y}$$

$$= -\frac{1}{\gamma} \hat{y}$$

$$= -\mu e^{-\gamma x}$$

$$= -e^{-\gamma x} \prod_{s=1}^{k} \left( \frac{\bar{p}_s}{q_s} \right)^{q_s}$$

and this equals $-e^{-\gamma x - h(\hat{Q} \parallel P)}$ as

$$\exp \left( -h(\hat{Q} \parallel P) \right) = \exp \left( -\sum_{\ell} q_s \ell \ln \left( \frac{q_s}{\bar{p}_s} \right) \right)$$

$$= \prod_{s=1}^{k} \left( \frac{\bar{p}_s}{q_s} \right)^{q_s}.$$
Case U2: For the power utility

$$\lambda^0_s(y) = (y)^{\alpha-1} q_s^{\alpha} \tilde{p}_s^{1-\alpha}.$$ 

We are looking for $\hat{y}$ such that

$$x = \frac{1}{y} \sum_{s=1}^{k} \lambda^0_s(\tilde{y})$$

$$= \frac{1}{y^{\alpha-1}} \sum_{s=1}^{k} q_s^{\alpha} \tilde{p}_s^{1-\alpha}$$

$$= \left( \frac{1}{\nu} \right) \cdot \hat{y}^{\alpha-1}$$

where

$$\nu = \left[ \sum_{s=1}^{k} q_s^{\alpha} \tilde{p}_s^{1-\alpha} \right]^{-1}.$$ 

Then $\tilde{y} = (\nu x)^{\alpha-1}$

$$\tilde{V}_0(y) = \left( \frac{1}{\nu} \right) y^{\alpha-1} \frac{1}{\nu}$$

so

$$V_0(x) = \tilde{V}_0(\tilde{y}) + x\tilde{y}$$

$$= \left( \frac{1}{\nu} \right) (\nu x)^{\alpha-1} \cdot \frac{1}{\nu} + x(\nu x)^{\alpha-1}$$

$$= \frac{1}{\nu^{\alpha-1} \cdot x^\alpha}.$$ 

This can be written

$$V_0(x) = \frac{1}{\alpha} x^\alpha \left[ \sum_{s=1}^{k} q_s^{\alpha} \tilde{p}_s^{1-\alpha} \right]^{1-\alpha}$$

$$= \frac{1}{\alpha} x^\alpha \left[ \sum_{s=1}^{k} q_s \left( \frac{p_s}{q_s} \right)^{1-\alpha} \right]^{1-\alpha}$$

$$= \frac{1}{\alpha} x^\alpha \left( \frac{\tilde{Q}}{P} \right)^{1-\alpha}.$$
**Case U3:** For the logarithmic utility

\[
\tilde{V}_0(y) = -\ln y - 1 - \sum_{s=1}^{k} \tilde{p}_s \ln \left( \frac{q_s}{p_s} \right)
\]

so

\[
\tilde{V}_0'(y) = -\frac{1}{y}
\]

and

\[
\tilde{V}_0'(y) + x = 0
\]

when \( \hat{y} = \frac{1}{x} \).

Consequently,

\[
V_0(x) = \tilde{V}_0(\hat{y}) + x \hat{y}
\]

\[
= -\ln \hat{y} - 1 - \sum_{s=1}^{k} \tilde{p}_s \ln \left( \frac{q_s}{p_s} \right) + 1
\]

\[
= \ln x - \sum_{s=1}^{k} \tilde{p}_s \ln \left( \frac{q_s}{p_s} \right).
\]

**Case 4:** For the general utility

\[
\tilde{V}_0(y) = \sum_{s=1}^{k} \tilde{p}_s \tilde{U} \left( y \frac{q_s}{p_s} \right)
\]

and

\[
\lambda_0^s(y) = yq_sI \left( \frac{q_s}{p_s} \right).
\]

Suppose

\[
x = \frac{1}{\hat{y}} \sum_{s=1}^{k} \lambda_0^s(\hat{y})
\]

\[
= \sum_{s=1}^{k} q_sI \left( \frac{\tilde{y}q_s}{p_s} \right). \tag{28}
\]

and suppose this can be solved for \( \hat{y} \).

Then:

\[
V_0(x) = \tilde{V}_0(\hat{y}) + x \hat{y}
\]

\[
= \sum_{s=1}^{k} \tilde{p}_s \tilde{U} \left( \frac{\tilde{y} q_s}{p_s} \right) + \sum_{s=1}^{k} \tilde{y} q_sI \left( \frac{\tilde{y} q_s}{p_s} \right)
\]

\[
= \sum_{s=1}^{k} \left[ \tilde{p}_s \tilde{U} \left( \frac{\tilde{y} q_s}{p_s} \right) + \tilde{y} q_sI \left( \frac{\tilde{y} q_s}{p_s} \right) \right].
\]
From equation (11) in Section 2
\[ \bar{U}(y) + yI(y) = U(I(y)) \]
so
\[ V_0(x) = \sum_{s=1}^{k} \bar{p}_s U\left(I\left(\frac{\bar{y}q_s}{p_s}\right)\right). \]

From (24)
\[ I\left(\frac{\bar{y}q_s}{p_s}\right) = \frac{\lambda_0(\bar{y})}{\bar{y}q_s} = \theta_s(\bar{y}) \]
so
\[ V_0(x) = \sum_{s=1}^{k} \bar{p}_s U(\theta_s(\bar{y})) \]
where
\[ x = \sum_{s=1}^{k} q_s I\left(\frac{\bar{y}q_s}{p_s}\right) \]
\[ = \frac{1}{\bar{y}} \sum_{s=1}^{k} \lambda_0(\bar{y}) \]
\[ = \sum_{s=1}^{k} q_s \theta_s(\bar{y}). \]

The calculation of \( \hat{y} \) is discussed in Section 12.1.

Once \( \hat{y} \) is determined
\[ V_0(x) = \sum_{s=1}^{k} \bar{p}_s U\left(\frac{1}{q_s} f_s(\hat{y})\right). \]

8 The Asking Price for Claims

In Section 7 we have discussed \( V_0(x) \) and the numerical methods are given in Section 12. The asking price of the claim \( G \) at wealth \( x \) is the number \( \nu \) such that \( V_G(x + \nu) = V_0(x) \).

For utility U1 the price \( \nu \) will be independent of \( x \), but this is not always the case.

We now describe an algorithm to compute \( \nu \). Recall that
\[ V_G(x + \nu) = \tilde{V}_G(\hat{y}) + (x + \nu)\hat{y} \]
\[ = \tilde{V}_G(\hat{y}) + f^G(\hat{y})\hat{y} \]

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where
\[ f^G(\hat{y}) = x + \nu = \frac{1}{y} \sum_{s=1}^{k} \lambda_s^G(y) = -\tilde{V}_G'(y). \]

Given \( x \) first compute \( V_0(x) \) as in Section 7. Then find \( \hat{y} \) so that
\[ \tilde{V}_G(\hat{y}) + f^G(\hat{y})\hat{y} = V_0(x). \]
Then \( \nu = f^G(\hat{y}) - x \). Now in the general case:
\[ \tilde{V}_G(y) = \sum_{\ell} p_{\ell} \tilde{U}\left[ U'(\frac{\lambda_s - yq_s g_{\ell}}{yq_s}) \right] \]
\[ - \sum_{\ell} p_{\ell} U'(\frac{\lambda_s - yq_s g_{\ell}}{yq_s}) g_{\ell} \]
\[ = \sum_{\ell} p_{\ell} \tilde{U}\left( U'(\theta^G_s(y) - g_{\ell}) \right) \]
\[ - \sum_{\ell} p_{\ell} U'(\theta^G_s(y) - g_{\ell}) g_{\ell}. \]

Also,
\[ \sum_{\ell \in A_s} \frac{p_{\ell}}{yq_s} U'(\frac{\lambda_s - yq_s g_{\ell}}{yq_s}) = 1 \]
\[ = \sum_{\ell \in A_s} \frac{p_{\ell}}{yq_s} U'(\theta^G_s(y) - g_{\ell}). \]

From (11) of Section 2 \( \tilde{U}(z) = U(I(z)) - yI(z) \) so that \( \tilde{U}[U'(\theta)] = U(\theta) - \theta U'(\theta) \). Therefore,
\[ \tilde{V}_G(y) = \sum_\ell p_\ell \left[ U \left( \frac{\lambda_s - yq_s g_\ell}{yq_s} \right) \right] \]
\[ - \sum_\ell p_\ell \left[ \frac{\lambda_s - yq_s g_\ell}{yq_s} \right] U' \left( \frac{\lambda_s - yq_s g_\ell}{yq_s} \right) \]
\[ - \sum_\ell p_\ell g_\ell U' \left( \frac{\lambda_s - yq_s g_\ell}{yq_s} \right) \]
\[ = \sum_\ell p_\ell U \left[ \frac{\lambda_s - yq_s g_\ell}{yq_s} \right] - \sum_{s=1}^k \lambda_s \sum_{\ell \in \mathcal{A}_s} p_\ell U' \left( \frac{\lambda_s - yq_s g_\ell}{yq_s} \right) \]
\[ = \sum_\ell p_\ell U \left( \frac{\lambda_s - yq_s g_\ell}{yq_s} \right) - \sum_{s=1}^k \lambda_s \]
\[ = \sum_\ell p_\ell U \left( \theta_s^G(y) - g_\ell \right) - y \sum_{s=1}^k q_s \theta_s^G(y). \]

Therefore, \[ V_G(x + \nu) = \tilde{V}_G(\bar{y}) + (x + \nu)\bar{y} \]
\[ = \sum_\ell p_\ell U \left( \frac{\lambda_s - yq_s g_\ell}{yq_s} \right) \]
\[ = \sum_\ell p_\ell U \left( \theta_s^G(g) - g_\ell \right) \]
as
\[ \sum_{s=1}^k \lambda_s = \sum_{s=1}^k \lambda_s(y) = (x + \nu)\bar{y}. \]

Note for \( G = 0 \):
\[ V_0(x) = \sum_\ell p_\ell U \left( \frac{\lambda_s^0(\bar{y})}{yq_s} \right) \]
\[ = \sum_{s=1}^k \bar{p}_s U \left( \frac{\lambda_s^0(\bar{y})}{yq_s} \right) = \sum_{s=1}^k \bar{p}_s U \left( \theta_s^0(\bar{y}) \right). \]

Now
\[ V_G(x + \nu) = \sum_\ell p_\ell U \left( \theta_s^G(\bar{y}) - g_\ell \right) \] (29)
and

\[ x + \nu = \sum_{s=1}^{k} q_s \theta^G_s(\hat{y}) \]  

(30)

so \( \theta^G_s(\hat{y}) \) is the optimum investment in the \( k \) Arrow-Debreu securities.

The existence and computation of \( \hat{y} \) is provided in Section 12.2.

We now consider cases U1, U2, U3.

**Case U1:** \( V_0(x) = -e^{-\gamma x} \prod_{s=1}^{k} \left( \frac{\hat{y}_s}{q_s} \right)^{q_s} \).

\[
V_G(x) = \sum_{\ell} p_\ell U \left( \frac{\lambda^G_s(y)}{q_s} y - g_\ell \right) = \sum_{\ell} p_\ell U \left( \theta^G_s(y) - g_\ell \right).
\]

\[ x + \nu = \frac{1}{y} \sum_{s=1}^{k} \lambda^G_s(y) \]

\[ \lambda^G_s(y) = -q_s \frac{y}{\gamma} \ln \frac{y}{\gamma} - \frac{y}{\gamma} q_s \ln \left( \frac{q_s}{\gamma_s} \right) \]

\[ \gamma_s = \sum_{\ell \in A_s} p_\ell e^{\gamma_\ell}. \]

Then

\[
U \left( \frac{\lambda^G_s(y)}{y q_s} - g_\ell \right) = -\exp \left[ -\gamma \left( \frac{\lambda^G_s(y)}{y q_s} - g_\ell \right) \right]
\]

\[ = -\exp \left[ \ln \frac{y}{\gamma} + \ln \frac{q_s}{\gamma_s} + \gamma g_\ell \right] \]

\[ = -\frac{y}{\gamma} q_s e^{\gamma g_\ell}. \]

Therefore,

\[
\sum_{\ell} p_\ell U \left( \frac{\lambda^G_s(y)}{y q_s} - g_\ell \right) = \sum_{\ell} p_\ell U \left( \theta^G_s(y) - g_\ell \right)
\]

\[ = V_G(x + \nu) \]

\[ = -\frac{y}{\gamma} \sum_{\ell} p_\ell \frac{q_s}{\gamma_s} e^{\gamma g_\ell} \]

\[ = -\frac{y}{\gamma} \sum_{s=1}^{k} q_s = -\frac{y}{\gamma}, \]
and the condition $V_G(x + \nu) = V_0(x)$ is

$$
- \frac{y}{\gamma} = -e^{-\gamma x} \prod_{s=1}^{k} \left( \frac{\bar{p}_s}{q_s} \right)^{q_s}
$$
giving

$$
\hat{y} = \gamma e^{-\gamma x} \prod_{s=1}^{k} \left( \frac{\bar{p}_s}{q_s} \right)^{q_s}.
$$

Then

$$
\frac{1}{y} \sum_{s=1}^{k} \lambda_s^G(\hat{y}) = x + \nu
$$

$$
= - \ell \ln \frac{\hat{y}}{\gamma} - \frac{1}{\gamma} \sum_{s=1}^{k} q_s \ell \ln \left( \frac{q_s}{\gamma_s} \right)
$$

$$
= - \frac{1}{\gamma} \left[ - \gamma x + \sum_{s=1}^{k} q_s \ell \ln \left( \frac{q_s}{\gamma_s} \right) \right]
$$

$$
= x - \frac{1}{\gamma} \sum_{s=1}^{k} q_s \ell \ln \left( \frac{\bar{p}_s}{\gamma_s} \right).
$$

Therefore, the asking pricing for asset $G$ is

$$
\nu = \frac{1}{\gamma} \sum_{s=1}^{k} q_s \ell \ln \left( \frac{\gamma_s}{\bar{p}_s} \right)
$$

with

$$
\gamma_s = \sum_{\ell \in A_s} p_{\ell e^{\gamma \ell_i}}.
$$

**Case U2:** For the power utility we have seen that $V_0(x) = \frac{1}{\alpha} \mu x^{\alpha}$ with

$$
\mu = \left[ \sum_{s=1}^{k} \frac{q_s^{\alpha-1}}{p_s^{1-\alpha}} \right]^{1-\alpha},
$$

and

$$
V_G(x + \nu) = \sum_{\ell} p_{\ell} U \left( \frac{\lambda_s^G(y)}{y q_s} - g_{\ell} \right)
$$

$$
= \sum_{\ell} p_{\ell} U \left( q_s^G(y) - g_{\ell} \right).
$$

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We first find $\hat{y}$ be setting
\[ V_G(x + \nu) = V_0(x), \]
that is
\[
\sum_\ell p_\ell U \left( \frac{\lambda_s^G(\hat{y})}{\hat{y}s} - g_\ell \right) = \sum_\ell p_\ell U \left( \theta_s^G(g) - g_\ell \right) = \frac{1}{\alpha} \mu x^\alpha.
\]
The left side is:
\[
\frac{1}{\alpha} \sum_\ell p_\ell \left( \frac{\lambda_s^G(\hat{y})}{\hat{y}s} - g_\ell \right)^\alpha.
\]
Now $\lambda_s^G(y) = \lambda_s$ is the unique solution of
\[
\sum_{\ell \in A_s} \frac{p_\ell}{(yqs)^\alpha (\lambda_s - yqs g_\ell)^{1-\alpha}} = 1
\]
so
\[
\sum_{\ell \in A_s} p_\ell \left( \frac{\lambda_s^G(y)}{yqs} - g_\ell \right)^{\alpha-1} = yqs.
\]
Therefore,
\[
\frac{d}{dy} \left( \frac{\lambda_s^G(y)}{y} \right) > 0.
\]
Newton’s method, or interval division, can be applied to find $\hat{y}$.
This enables us to calculate
\[
\frac{1}{\hat{y}} \sum_{s=1}^k \lambda_s^G(\hat{y})
\]
which equals $x + \nu$. Therefore, the asking price for $G$ is determined.

**Case U3:** For the logarithmic utility
\[
V_0(x) = \ell n x - \sum_{s=1}^k \bar{p}_s \ell n \left[ \frac{q_s}{\bar{p}_s} \right].
\]
\[
V_G(x + \nu) = \sum_\ell p_\ell U \left( \frac{\lambda_s^G(\hat{y})}{\hat{y}s} - g_\ell \right)
\]
\[
= \sum_\ell p_\ell \ell n \left[ \frac{\lambda_s^G(\hat{y})}{\hat{y}s} - g_\ell \right]
\]
\[
= \sum_\ell p_\ell \ell n \left( \theta_s^G(\hat{y}) - g_\ell \right).
\]
We determine \( \hat{y} \) from the condition

\[ V_G(x + \nu) = V_0(x). \]

Now \( \lambda_s^G(\hat{y}) \) is the unique solution of

\[ \sum_{\ell \in A_s} \frac{p_\ell}{(\lambda_s^G(\hat{y}) - \hat{y}_q g_\ell)} = 1. \]

Therefore,

\[ \sum_{\ell \in A_s} p_\ell \left( \frac{\lambda_s^G(\hat{y})}{\hat{y}_q} - g_\ell \right)^{-1} = y_q \]

implying

\[ \frac{d}{dy} \left( \frac{\lambda_s^G(y)}{y} \right) < 0. \]

Having found \( \hat{y} \) we calculate

\[ \frac{1}{\hat{y}} \sum_{s=1}^k \lambda_s^G(\hat{y}). \]

This equals \( x + \nu \), and so \( \nu \) is found.

Writing

\[ \theta = \theta_s^G(y) = \frac{\lambda_s^G(y)}{y_q}. \]

As before,

\[ \tilde{V}_G(y) = \inf_{\theta \in \mathcal{M}} \left( E_p \left[ \tilde{U} \left( \frac{Q}{P} \right) \right] - y E_Q [G] \right) = \inf_{\theta \in \mathcal{M}} \left( E_p \left[ \tilde{U} \left( \frac{Q}{P} \right) \right] \right). \]

In the calculation of the minimizing \( Q \) in Section 5

\[ \tilde{Q}_G(\ell) = \frac{p_\ell}{y} U' \left( \theta_s^G(y) - g_\ell \right) \quad \text{for} \quad \ell \in A_s, \]

(30)

Also,

\[ \sum_{\ell \in A_s} \frac{p_\ell}{y_q} U' \left( \theta_s^G(y) - g_\ell \right) = 1. \]
As $\tilde{Q}_G(\ell) = q_s p_\ell e^{m_\ell}$ for some $m_\ell$ from (30)

$$\theta^G_s(y) = g_\ell + I(y q_s e^{m_\ell}).$$

Now

$$\sum_{\ell \in A_s} p_\ell e^{m_\ell} = 1$$

so when $G = 0$

$$\sum_{\ell \in A_s} \tilde{Q}(\ell) = \sum_{\ell \in A_s} q_s p_\ell e^{m_\ell}$$

$$= \sum_{\ell \in A_s} \frac{p_\ell}{y} U'(\theta^0_s(y)).$$

That is,

$$q_s = \frac{\tilde{p}_s}{y} U'(\theta^0_s(y))$$

so $$\theta^0_s(y) = I\left(\frac{y q_s}{\tilde{p}_s}\right).$$

Rewriting the results of the U3 case discussed in Section 6

$$\tilde{V}'_G(y) = -\sum_{s=1}^k q_s \theta^G_s(y)$$

$$\tilde{V}'_0(y) = -\sum_{s=1}^k q_s \theta^0_s(y).$$

From Section 8:

$$V_0(x) = \sum_{s=1}^k \tilde{p}_s U(\theta^0_s(y))$$

with $$x = \sum_{s=1}^k q_s \theta^0_s(y).$$

Also,

$$V_G(x) = \sum_{\ell} p_\ell U(\theta^G_s(y) - g_\ell)$$

with $$x = \sum_{s=1}^k q_s \theta^G_s(y).$$
We now solve

\[ V_G(x + \nu) = V_0(x). \]

First compute \( V_0(x) \).

Find \( \hat{y}_1 \) such that

\[ \sum_{s=1}^{k} q_s I\left( \frac{\hat{y}_1 q_s}{p_s} \right) = x. \]

Then

\[ V_0(x) = \sum_{s=1}^{k} \tilde{p}_s U\left( I\left( \frac{\hat{y}_1 q_s}{p_s} \right) \right). \]

Now compute \( \hat{y} \) such that

\[ \sum_{\ell} p_\ell U(\theta_s^G(\hat{y}) - g_\ell) = V_0(x) \]

and

\[ \sum_{\ell} p_\ell U'(\theta_s^G(\hat{y}) - g_\ell) = q_s \hat{y}. \]

Then

\[ \nu = \sum_{s=1}^{k} q_s \theta_s^G(\hat{y}) - x. \]

9 The Bid Price

We now define

\[ V_B^G(x) = \sup_{\alpha} E_p[U(X^{x,\alpha} + G)] \]

and, as before,

\[ V_0(x) = \sup_{\alpha} E_p[U(X^{x,\alpha})]. \]

The bid price, \( \nu^b \), is the number such that \( V_B^G(x - \nu^b) = V_0(x) \).

**Theorem 9.1.** \( \nu^b(G) = -\nu^a(-G) \).

**Proof.** Write \( H = -G \) so

\[ V_B^G(x) = \sup_{\alpha} E_p[U(X^{x,\alpha} - H)] \]

\[ = V_H(x). \]
Then
\[ V_B^G(x - \nu^b(G)) = V_0(x) \]
\[ = V_H(x - \nu^b(G)), \]
and
\[ V_H(x + \nu^a(H)) = V_0(x). \]

As \( y \to V_H(y) \) is monotone
\[ \nu^b(G) = -\nu^a(-G). \]

All the computations for \( \nu^b \) are as before. In fact we have that
\[ \nu^b(G) = -\nu^a(-G). \]

The procedure, therefore, is:

a) Compute \( V_0(x) \)
b) Compute \( \hat{y} \) so that
\[ \sum_{\ell} p_{\ell} U(\tilde{\theta}^G_{\ell}(\hat{y}) + g_{\ell}) = V_0(x) \]
\[ \hat{y}q_s = \sum_{\ell \in A_s} p_{\ell} U'(\tilde{\theta}^G_{\ell}(\hat{y}) + g_{\ell}). \]

Then set
\[ x - \nu^b = \sum_{s=1}^{k} q_s \tilde{\theta}^G_{s}(\hat{y}) \]

to compute \( \nu^b \).

10 Examples

For quadratic and exponential utilities explicit formulae are provided
in the following examples.
Example 10.1. Consider the quadratic utility

\[ U(x) = -(x - a)^2, \quad x < a. \]

Then

\[ U'(x) = -2(x - a) \]
\[ U''(x) = -2. \]

The risk version is

\[ -\frac{U''(x)}{U'(x)} = \frac{1}{a - x}, \]
\[ I(z) = a - \frac{z}{2}. \]

Let us compute \( \theta^G_s(y) \) and \( \tilde{\theta}^G_s(y) \). Now for \( s = 1, \ldots, k \)

\[ -2 \sum_{\ell \in A_s} p_{\ell} [\theta^G_s(y) - g_{\ell} - a] = yq_s \]

so

\[ -2 \tilde{p}_s \theta^G_s(y) + 2 \sum_{\ell \in A_s} p_{\ell} g_{\ell} + 2a \sum_{\ell} p_{\ell} = yq_s \]

and

\[ 2 \tilde{p}_s \theta^G_s(y) = 2 \sum_{\ell \in A_s} p_{\ell} g_{\ell} + 2a \tilde{p}_s - yq_s \]

and

\[ \theta^G_s(y) = \tilde{G}_s + a - \frac{yq_s}{2\tilde{p}_s} \]

where

\[ \tilde{G}_s = \frac{1}{p_s} \sum_{\ell \in A_s} p_{\ell} g_{\ell}. \]

The formula for \( \theta^G_s(y) \) is well defined only if \( \tilde{G}_s - \frac{yq_s}{2\tilde{p}_s} < 0 \) for all \( s \). Then

\[ V^G_G(x) = \sum_{\ell} p_{\ell} U(\theta^G_s(y) - g_{\ell}) \]
\[ = \sum_{\ell} p_{\ell} U(\tilde{G}_s + a - \frac{yq_s}{2\tilde{p}_s} - g_{\ell}) \]
\[ = - \sum_{\ell} p_{\ell} \left( \tilde{G}_s - \frac{yq_s}{2\tilde{p}_s} - g_{\ell} \right)^2 \]
\[ = - \sum_{\ell} p_{\ell} (\tilde{G}_s - g_{\ell})^2 - \frac{1}{4} y^2 \sum_{\ell} q_{\ell}^2 \sum_{s=1}^{k} \frac{q_{s}^2}{p_s} \cdot p_{\ell} \]
\[ = - \sum_{\ell} p_{\ell} (\tilde{G}_s - g_{\ell})^2 - \frac{1}{4} y^2 \sum_{s=1}^{k} \frac{q_{s}^2}{p_s}. \]
and 
\[ x = \sum_{s=1}^{k} q_s^G y q_s \]
\[ = \sum_{s=1}^{k} q_s g_s + a - \frac{y}{2} \sum_{s=1}^{k} \frac{q_s^2}{p_s}. \]

Therefore,
\[ V_G(x) = -\sum_{\ell} p_{\ell} (\tilde{G}_s - g_{\ell})^2 - \left[ \sum_{s=1}^{k} \frac{q_s^2}{p_s} \right]^{-1} \left[ x - a - \sum_{s=1}^{k} q_s \tilde{G}_s \right]^2. \]

For \( G = 0 \)
\[ V_0(x) = -\left[ \sum_{s=1}^{k} \frac{q_s^2}{p_s} \right]^{-1} [x - a]^2. \]

The asking price \( \nu = \nu^a \) is defined by
\[ V_G(x + \nu^a) = V_0(x). \]

That is:
\[ [x + \nu^a - a - \sum_{s=1}^{k} q_s \tilde{G}_s]^2 = [x - a]^2 - \Gamma \Delta \]

where
\[ \Gamma = \sum_{s=1}^{k} \frac{q_s^2}{p_s} \quad \text{and} \quad \Delta = \sum_{\ell} p_{\ell} (\tilde{G}_s - g_{\ell})^2. \]

Therefore,
\[ \nu^a = -(x - a) + \sum_{s=1}^{k} q_s \tilde{G}_s - [(x - a)^2 - \Gamma \Delta]^{1/2}. \]

Note when \( G \) is \( F_s \) measurable
\[ g_{\ell} = \tilde{G}_s \quad \text{for} \quad \ell \in A_s \]
so then \( \Delta = 0 \) and
\[ \nu = \nu^a = \sum_{s=1}^{k} q_s \tilde{G}_s. \]

Therefore, the negative sign must be taken in the square root term.
The bid price $\nu^b$ is defined by

$$V^B_G(x - \nu^b) = V_0(x)$$

where

$$V^B_G(x) = \sum_{\ell} p_{\ell} U(\tilde{\theta}^G_s(y) + g_{\ell})$$

where

$$\tilde{\theta}^G_s(y) = -\tilde{G}_s + a - \frac{yq_s}{2p_s}.$$  

Therefore,

$$V^B_G(x - \nu^b) = -\sum_{\ell} p_{\ell}(\tilde{G}_s - g_{\ell})^2 - \Gamma^{-1}\left[ x - \nu^b - a + \sum_{s=1}^k q_s \tilde{G}_s \right].$$

With

$$V_0(x) = -\Gamma^{-1}(x - a)^2$$

the price $\nu^b$ is determined by

$$x - \nu^b - a + \sum_{s=1}^k q_s \tilde{G}_s = [(x - a)^2 - \Gamma\Delta]^{1/2}$$

so that

$$\nu^b = (x - a) + \sum_{s=1}^k q_s \tilde{G}_s + [((x - a)^2 - \Gamma\Delta]^{1/2}.$$  

Here the positive sign is taken in the square root so that when $\Delta = 0$ $\nu^b$ reduces to

$$\sum_{s=1}^k q_s \tilde{G}_s.$$  

Note that

$$\nu^a - \nu^b = 2[(a - x) - [(a - x)^2 - \Gamma\Delta]^{1/2}]$$

and

$$\frac{\partial \nu^a}{\partial x} = \frac{a - x}{[(a - x)^2 - \Gamma\Delta]^{1/2}}.$$  

Note this is 0 when $\Delta = 0$. Now

$$\frac{\partial \nu^a}{\partial x} = -1 + \frac{a - x}{[(a - x)^2 - \Gamma\Delta]^{1/2}}$$

and

$$\frac{\partial \nu^a}{\partial x} = \frac{-\Gamma\Delta}{[(a - x)^2 - \Gamma\Delta]^{1/2}[(a - x) + [(a - x)^2 - \Gamma\Delta]^{1/2}]} < 0.$$  

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\[ \frac{\partial \nu^b}{\partial x} = 1 - \frac{a - x}{[(a - x)^2 - \Gamma\Delta]^{1/2}} \]

\[ = \frac{\Gamma\Delta}{[(a - x)^2 - \Gamma\Delta]^{1/2}[(a - x) + [(a - x)^2 - \Gamma\Delta]^{1/2}]} > 0. \]

Also, \( \frac{\partial}{\partial x} (\nu^a - \nu^b) < 0. \)

Now any \( Q \in \mathcal{M} \) is of the form

\[ Q(\ell) = q_s p_\ell e^{m_\ell} \]

with

\[ \sum_{\ell \in A_s} p_\ell e^{m_\ell} = 1 \]

and

\[ \sum_{\ell} Q(\ell) \theta^G_s(y) = \sum_{s=1}^k q_s \theta^G_s(y) = x + \nu^a. \]

Therefore, \( \theta^G_s(y) \) is an attainable claim.

**Example 10.2.** Consider Case U1, the exponential utility. Then

\[ U(x) = -e^{-\gamma x} \]

and \( U'(x) = \gamma e^{-\gamma x}. \)

\[ yq_s = \sum_{\ell \in A_s} p_\ell \gamma \exp \left[ -\gamma (\theta^G_s(y) - g_\ell) \right] \]

\[ = e^{-\gamma \theta^G_s(y)} \gamma \sum_{\ell \in A_s} p_\ell e^{\gamma g_\ell}. \]

Then

\[ e^{\gamma \theta^G_s(y)} = \frac{\gamma}{q_s y} \sum_{\ell \in A_s} p_\ell e^{\gamma g_\ell} \]

and

\[ \theta^G_s(y) = \frac{1}{\gamma} \ln \left[ \frac{\gamma}{q_s y} \sum_{\ell \in A_s} p_\ell e^{\gamma g_\ell} \right]. \]
When $G = 0$

$$\theta^0_s(y) = \frac{1}{\gamma} \ln \left[ \frac{\gamma p_s}{q_s y} \right].$$

Now

$$\sum_{s=1}^{k} q_s \theta^G_s(\bar{y}) = x = \frac{1}{\gamma} \ln \gamma - \frac{1}{\gamma} \ln y + \frac{1}{\gamma} M$$

where

$$M = \sum_{s=1}^{k} q_s \ln \left[ \frac{1}{q_s} \sum_{\ell \in A_s} p_{\ell e^{\gamma g_{\ell}}} \right]$$

so

$$\theta^G_s(y) = \frac{1}{\gamma} \ln \gamma - \frac{1}{\gamma} \ln y + \frac{1}{\gamma} \ln \left[ \frac{1}{q_s} \sum_{\ell \in A_s} p_{\ell e^{\gamma g_{\ell}}} \right]$$

$$= x + \frac{1}{\gamma} \ln \left[ \frac{1}{q_s} \sum_{\ell \in A_s} p_{\ell e^{\gamma g_{\ell}}} \right] - \frac{1}{\gamma} M.$$

As

$$V_G(x) = \sum_{\ell} p_{\ell} U(\theta^G_s(y) - g_{\ell})$$

$$= - \sum_{\ell} p_{\ell} \exp \left[ - \gamma \left( x + \frac{1}{\gamma} \ln \left( \frac{1}{q_s} \sum_{\ell \in A_s} p_{\ell e^{\gamma g_{\ell}}} \right) - \frac{1}{\gamma} M - g_{\ell} \right) \right]$$

$$= -e^{-\gamma x} \sum_{s=1}^{k} \left( \sum_{\ell \in A_s} p_{\ell e^{\gamma g_{\ell}}} \right) \cdot q_s \left( \sum_{\ell \in A_s} p_{\ell e^{\gamma g_{\ell}}} \right)^{-1} e^{M}$$

$$= -e^{-\gamma x} e^{M}$$

$$= -e^{-\gamma x} \exp \left[ \sum_{s=1}^{k} q_s \ln \left( \frac{1}{q_s} \sum_{\ell \in A_s} p_{\ell e^{\gamma g_{\ell}}} \right) \right].$$

In particular, when $G = 0$

$$V_0(x) = -e^{-\gamma x} \exp M_0$$

where

$$M_0 = \sum_{s=1}^{k} q_s \ln \left( \frac{\bar{p}_s}{q_s} \right).$$

We are looking for the asking price $\nu = \nu^G$ defined by

$$V_G(x + \nu) = V_0(x).$$
That is, \(-e^{-\gamma(x+\nu)}e^M = -e^{-\gamma x}e^{M_0}\) so \(e^{\gamma \nu} = e^{M-M_0}\) and

\[
\nu = \frac{1}{\gamma} (M - M_0)
\]

\[
= \frac{1}{\gamma} \left[ \sum_{r=1}^{k} q_r \ln \left( \frac{1}{q_r} \sum_{\ell \in A_r} p_{\ell r} e^{\gamma q_{r \ell}} \right) \right] - \frac{1}{\gamma} \left[ \sum_{r=1}^{k} q_r \ln \left( \frac{\bar{p}_r}{q_r} \right) \right]
\]

\[
= \frac{1}{\gamma} \left[ \sum_{r=1}^{k} q_r \ln \left( \frac{1}{p_r} \sum_{\ell \in A_r} p_{\ell r} e^{\gamma q_{r \ell}} \right) \right].
\]

11 Properties of \(\nu\)

We have noted that if \(G\) is \(F_s\) measurable \(\nu^a(G) = \nu^b(G) = \sum_{s=1}^{k} q_s \tilde{G}_s\)
where \(g_\ell = \tilde{G}_s\) for \(\ell \in A_s\).

**Theorem 11.1.** \(G \rightarrow \nu^a(G)\) is convex.

**Proof.**

\(\nu^a(G) = \sum_{s=1}^{k} q_s \theta^G_s(y) - 1\) where

\[
q_s y = \sum_{\ell \in A_s} p_{\ell r} U'(\theta^G_s(y) - g_\ell)
\]

and

\[
\sum_{\ell=1}^{N} p_{\ell r} U(\theta^G_s(y) - g_\ell) = V_0(x).
\]

Write

\[
\phi(t) = \nu^a(tG_1 + (1-t)G_2), \quad 0 \leq t \leq 1.
\]

Write

\[
G(t) = G_2 + t(G_1 - G_2),
\]

with components

\[
g_\ell(t) = g^2_\ell + t(g^1_\ell - g^2_\ell)
\]

and define \(\tilde{g}(t)\) by

\[
\sum_{\ell} p_{\ell r} U\left(\theta^G_s(\tilde{g}(t)) - g_\ell(t)\right) = V_0(x).
\]

Write

\[
\psi_s(t) = \theta^G_s(\tilde{g}(t)).
\]
Then
\[ \nu^a(tG_1 + (1 - t)G_2) = \phi(t) \]
\[ = \sum_{s=1}^{k} q_s \psi_s(t) - x \]
and
\[ \phi''(t) = \sum_{s=1}^{k} q_s \psi_s''(t). \]

Also, as
\[ \sum_{\ell} p_{\ell} U(\psi_s(t) - g_\ell(t)) = V_0(x) \]
differentiating in \( t \) gives
\[ \sum_{\ell} p_{\ell} U'(\psi_s(t) - g_\ell(t))(\psi'_s(t) - g'_\ell(t)) = 0 \quad (31) \]
and
\[ \sum_{\ell} p_{\ell} U''(\psi_s(t) - g_\ell(t))(\psi'_s(t) - g'_\ell(t))^2 \]
\[ + \sum_{\ell} p_{\ell} U'(\psi_s(t) - g_\ell(t))\psi''_s(t) = 0 \]
as \( g''_\ell(t) = 0 \).

Now
\[ q_s \tilde{y}(t) = \sum_{\ell \in A_s} p_{\ell} U'(\psi_s(t) - g_\ell(t)) \]
so we have
\[ \sum_{\ell} p_{\ell} U''(\psi_s(t) - g_\ell(t))(\psi'_s(t) - g'_\ell(t))^2 \]
\[ + \tilde{y}(t) \sum_{s=1}^{k} q_s \psi_s''(t) = 0 \]
giving
\[ \tilde{y}(t) \sum_{s=1}^{k} q_s \psi_s''(t) = \tilde{y}(t)\phi''(t) \]
\[ = - \sum_{\ell} p_{\ell} U''(\psi_s(t) - g_\ell(t))(\psi'_s(t) - g'_\ell(t))^2 \]
\[ \geq 0. \]
Therefore, $\phi(t)$ is convex. That is, the map $G \rightarrow \nu^a(G)$ is convex.

**Example 11.2:** $U(x) = -e^{-\gamma x}$ then

$$\nu^a(G) = \frac{1}{\gamma} \sum_{s=1}^{k} q_s \ln \left[ \frac{1}{p_s} \sum_{\ell \in A_s} p_{\ell} e^{\gamma g_{\ell}} \right]$$

and

$$\nu^a(tG_1 + (1-t)G_2) = \frac{1}{\gamma} \sum_{s=1}^{k} q_s \ln \left[ \frac{1}{p_s} \sum_{\ell \in A_s} p_{\ell} e^{\gamma t g_{\ell}^1 + \gamma (1-t) g_{\ell}^2} \right].$$

By Hölder’s inequality this is

$$\leq \frac{1}{\gamma} \sum_{s=1}^{k} q_s \ln \left[ \left( \sum_{\ell \in A_s} \frac{p_{\ell}}{p_s} e^{\gamma g_{\ell}^1} \right)^t \left( \sum_{\ell \in A_s} \frac{p_{\ell}}{p_s} e^{\gamma g_{\ell}^2} \right)^{1-t} \right]$$

$$= t
\nu^a(G_1) + (1-t)\nu^a(G_2).$$

**Example 11.3.** $U(x) = -(a-x)^2$, with $x \leq a$. Here

$$\nu^a(G) = (a-x) - ((a_x)^2 - \Gamma \Delta)^{1/2} + \sum_{s=1}^{k} q_s \tilde{G}_s$$

where

$$\Delta = \sum_{\ell} p_{\ell} (g_{\ell} - \tilde{G}_s)^2, \quad \Gamma = \sum_{s=1}^{k} \frac{q_s^2}{p_s}.$$

Write

$$\phi(t) = \nu^a(tG_1 + (1-t)G_2)$$

so

$$\phi''(t) = -\frac{d^2}{dt^2} \left( (a-x)^2 - \Gamma \Delta(t) \right)^{1/2}$$

with

$$\Delta(t) = \sum_{\ell} p_{\ell} (g_{\ell}(t) - \tilde{G}_s(t))^2.$$

Then

$$\Delta' = 2 \sum_{\ell} p_{\ell} (g_{\ell}(t) - \tilde{G}_s(t)) \left( g_{\ell}^1 - g_{\ell}^2 - \tilde{G}_s^1 + \tilde{G}_s^2 \right)$$

and

$$\Delta'' = 2 \sum_{\ell} p_{\ell} (g_{\ell}^1 - g_{\ell}^2 + \tilde{G}_s^1 + \tilde{G}_s^2)^2$$

$$\geq 0.$$
Now
\[ \phi''(t) = \frac{1}{4} ((a - x)^2 - \Gamma \Delta)^{-3/2} (\Gamma \Delta'(t))^2 \]
\[ + \frac{1}{2} (a - x)^2 - \Gamma \Delta)^{-1/2} \Gamma \Delta''(t) \]
\[ \geq 0. \]

Consequently,
\[ \phi(t) = \nu^n(tG_1 + (1 - t)G_2) \]
is convex.

We now show \( \nu^b \) is concave. In fact
\[ \nu^b(G) = x - \sum_{s=1}^k q_s \tilde{\theta}_s^G(\tilde{y}) \]
where
\[ \sum_{\ell} p_\ell U(\tilde{\theta}_s^G(\tilde{y}) + g_\ell) = V_0(x) \]
and
\[ yq_s = \sum_{\ell \in A_s} p_\ell U'(\tilde{\theta}_s^G(y) + g_\ell). \]

As before write \( \phi(t) = \nu^b(tG_1 + (1 - t)G_2) \)
\[ g_\ell(t) = g_\ell^2 + t(g_\ell^1 - g_\ell^2). \]

With \( \psi_s^b(t) = \tilde{\theta}_s^G(t)(\tilde{y}) \) we have
\[ \sum_{\ell} p_\ell U(\psi_s^b(t) + g_\ell(t)) = V_0(x) \]
and
\[ \phi''(t) = -\sum_{s=1}^k q_s \psi_s^{bn}(t). \]

We deduce
\[ \sum_{\ell} p_\ell U'(\psi_s^b(t) + g_\ell(t))(\psi_s^{bn}(t) + g_\ell'(t)) = 0 \]
\[ \sum_{\ell} p_{\ell} U''(\psi_s^b(t) + g_{\ell}(t)) (\psi_s^{br}(t) + g'_{\ell}(t))^2 + \sum_{\ell} p_{\ell} U''(\psi_s^b(t) + g_{\ell}(t)) \psi_s^{br}(t) = 0. \]

Then
\[ \hat{y}(t) \sum_{s=1}^{k} q_s \psi_s^{bn}(t) = -\sum_{\ell} p_{\ell} U''(\psi_s^b(t) + g_{\ell}(t)) (\psi_s^{br}(t) + g'_{\ell}(t))^2 \leq 0. \]

Therefore, \( \phi''(t) \leq 0 \) and so \( G \rightarrow \nu^b(G) \) is concave.

**Lemma 11.4.** \( G \rightarrow \nu^a(G) \) is monotonically increasing.

**Proof:** We say \( G_1 \geq G_2 \) if
\[ g_{\ell}^1 \geq g_{\ell}^2 \quad \text{for} \quad \ell = 1, \ldots, N. \]

Suppose \( G_1 \geq G_2 \).

Write
\[ \phi(t) = \nu^a(G_2 + t(G_1 - G_2)). \]

We must show \( \phi(1) \geq \phi(0) \). This is the case if \( \phi'(t) \geq 0 \). Now
\[ \phi'(t) = \sum_{s=1}^{k} q_s \psi_s'(t). \]

However, we have seen in (30) that
\[ \hat{y}(t) \sum_{s=1}^{k} q_s \psi_s'(t) = \sum_{\ell} p_{\ell} U' \left( \psi_s(t) - g_{\ell}(t) \right) (g_{\ell}^1 - g_{\ell}^2) \leq 0. \]

Therefore, \( \phi'(t) \geq 0 \) and the result follows.

**Corollary 11.5.** \( G \rightarrow \nu^b(G) \) is monotonically decreasing.

Here
\[ \phi'(t) = -\sum_{s=1}^{k} q_s \psi_s'(t) \]
and

\[
\hat{y}(t) \sum_{s=1}^{k} q_s \psi_s^b(t) = - \sum_{\ell} p_\ell U'(\psi_s^b(t) + g_\ell(t))(g_\ell^1 - g_\ell^2)
\]

\[
\leq 0
\]

and the result follows.

**Example 11.6.** With \( U(x) = -e^{-\gamma x} \)

\[
\nu^b(G) = - \frac{1}{\gamma} \sum_{s=1}^{k} q_s \ell \ln \left[ \sum_{\ell \in A_s} \frac{p_\ell}{p_s} e^{-\gamma g_\ell} \right]
\]

and clearly \( \nu^b \) is an increasing function of \( G \).

**Lemma 11.7.** Suppose \( G \) is a constant \( C \). Then

\[
\nu^a(C) = C.
\]

**Proof.** For each \( y \) we must determine \( \theta_s^G(y) \) so that

\[
yq_s = \sum_{\ell \in A_s} p_\ell U'(\theta_s^G(y) - C)
\]

\[
= \overline{p}_s U'(\theta_s^G(y) - C).
\]

Therefore,

\[
\theta_s^G(y) = C + I\left(\frac{yq_s}{p_s}\right).
\]

We then wish to find \( \hat{y} \) such that

\[
\sum_{\ell} p_\ell U'(\theta_s^G(\hat{y}) - C) = V_0(x)
\]

\[
= \sum_{\ell} p_\ell U\left( I\left(\frac{\hat{y}q_s}{p_s}\right) \right) = V_0(x).
\]

This same \( \hat{y} \) will work for \( \theta_s^0 \) and

\[
\theta_s^0(\hat{y}) = I\left(\frac{\hat{y}q_s}{p_s}\right).
\]

Then

\[
\nu^a(G) = \sum_{s=1}^{k} q_s (\theta_s^G(\hat{y}) - \theta_s^0(\hat{y}))
\]

\[
= C.
\]
12 Numerical Methods

The procedure to calculate $\nu^a$ is, for a given $x$, to compute $V_0(x)$. Then $y$ is determined so that

$$\sum_\ell p_\ell U(\theta^G_s(y) - g_\ell) = V_0(x).$$

This involves calculating the $\theta^G_s(y)$ for $s = 1, \ldots, k$. Then

$$\nu^a = \sum_{s=1}^k q_s \theta^G_s(y) - x.$$ 

We shall discuss the computation of $\theta^G_s(y)$. Now $\theta_s$ is determined by the condition

$$\sum_{\ell \in A_s} p_\ell U'(\theta_s - g_\ell) = q_s y. \quad (32)$$

12.1 $V_0(x)$

Recall $\theta^0_s(y) = \lambda^0_s(y)/yq_s$. For now write $\theta_s(y) = \theta^0_s(y)$.

Suppose $\hat{y}$ is a solution of

$$x = \sum_{s=1}^k q_s \theta_s(y).$$

Then

$$V_0(x) = \sum_\ell p_\ell U(\theta_s(\hat{y}))$$

$$= \sum_{s=1}^k \tilde{p}_s U(\theta_s(\hat{y})),$$

where again

$$\tilde{p}_s = \sum_{\ell \in A_s} p_\ell.$$

From (24)

$$\tilde{p} U'(\theta_s(\hat{y})) = q_s \hat{y}$$

and from (28)

$$\sum_{s=1}^k q_s \theta_s(y) = x.$$
Write
\[ \theta(y) := \sum_{s=1}^{k} q_s \theta_s(y). \]

We shall use the Newton-Rhapson method to compute \( \hat{y} \).
From (24)
\[ U'(\theta_s(y)) = \frac{q_s}{p_s} y \]
so
\[ U''(\theta_s(y)) \theta'_s(y) = \frac{q_s}{p_s} > 0, \]
so \( \theta'_s(y) < 0 \) as \( U''(z) < 0 \). Differentiating again
\[ \theta''_s(y) = -\frac{U''(\theta_s(y))(\theta'_s(y))^2}{U''(\theta_s(y))} \geq 0 \quad \text{if} \quad U''' \geq 0. \]

Remark: \( U'''(\cdot) \geq 0 \) is not really necessary. However, it is satisfied by the utility functions \( U_1, U_2, \) and \( U_3 \) and it ensures monotone convergence of the Newton-Rhapson method.

Set
\[ \hat{y}_0 = U'(x) \min_s \left\{ \frac{\tilde{p}_s}{q_s} \right\} \]
and
\[ \hat{y}_{n+1} = \hat{y}_n - \left[ \theta(\hat{y}_n) - x \right] \theta'(\hat{y}_n)^{-1}. \]

With the above choice of \( \hat{y}_0 \)
\[ \theta(\hat{y}_0) = \sum_{s=1}^{k} q_s \theta_s(\hat{y}_0) \]
\[ = \sum_{s=1}^{k} q_s I \left( \frac{\tilde{p}_s q_s}{p_s} \right) \quad \text{by (25)} \]
\[ \geq \sum_{s=1}^{k} q_s I \left( U'(x) \frac{\tilde{p}_s}{q_s} \cdot \frac{q_s}{p_s} \right) \quad \text{as} \quad I' < 0, \]
\[ = \sum_{s=1}^{k} q_s I(U'(x)) \]
\[ = x \sum_{s=1}^{k} q_s \]
\[ = x, \quad \text{as} \quad \sum_{s=1}^{k} q_s = 1. \]
Consequently, $\hat{y}_0 \leq \hat{y}$ and $\hat{y}_n$ converges to $\hat{y}$ as $n \to \infty$.

12.2 $V_G(x)$

From equation (29) we wish to find $\hat{y}$ which solves:

$$\sum_{\ell} p \ell U(\theta^G_s(y) - g \ell) = V_G(x + \nu)$$

$$= V_0(x).$$

Then, as in (31), with this choice of $\hat{y}$

$$x + \nu = \sum_{s=1}^{k} q_s \theta^G_s(\hat{y}).$$

Consequently $\nu = \nu^a$, the asking price, can be found.

a) The existence and uniqueness of the solution $\hat{y}$ of (32).

Write

$$F(y) := \sum_{\ell} p \ell U(\theta^G_s(y) - g \ell) - V_0(x).$$

Then

$$F'(y) = \sum_{\ell} p \ell U'(\theta^G_s(y) - g \ell)(\theta^G_s)'(y).$$

As in Section 5,

$$\sum_{\ell \in A_s} p \ell U'(\theta^G_s(\hat{y}) - g \ell) = y q_s$$

so

$$\sum_{\ell \in A_s} p \ell U''(\theta^G_s(\hat{y}) - g \ell)(\theta^G_s)'(y) = q_s$$

and, therefore, $(\theta^G_s)'(y) < 0$ for $s = 1, 2, \ldots, k$. Further,

$$F'(y) = \sum_{s=1}^{k} y q_s (\theta^G_s)'(y)$$

$$< 0,$$

so the solution of (32) is unique if it exists. If a finite solution of (32) exists then $F$ must have a sign change.

Let us recall some properties of $\theta^G_s(y)$ as $y \to 0+$ and $y \to +\infty$. 

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Now for each $s = 1, 2, \ldots, k$
\[ \sum_{\ell \in A_s} p_{\ell} U'(\theta_s^G(y) - g_\ell) = yq_s. \]

**U1:** If $U(x) = -e^{-\gamma x}$, $\gamma > 0$, this states that
\[ \gamma \sum_{\ell \in A_s} p_{\ell} \exp \left( -\gamma (\theta_s^G(y) - g_\ell) \right) = yq_s \]
so that
\[ \theta_s^G(y) = \frac{1}{\gamma} \left[ \frac{\ell_n}{\ln} \sum_{\ell \in A_s} p_{\ell} e^{\gamma g_\ell} - \ln \frac{yq_s}{\gamma} \right]. \]
Therefore, as $y \to 0^+$ we see $\theta_s^G(y) \to \infty$ and as $y \to +\infty$, $\theta_s^G(y) \to \infty$.

**U2 and U3:** If $U(x) = \frac{1}{\alpha} x^\alpha$ for $\alpha < 1$ or $U(x) = \ell n x$, $x > 0$, (corresponding to $\alpha = 0$), then for $s = 1, \ldots, k$,
\[ \sum_{\ell \in A_s} p_{\ell} (\theta_s^G(y) - g_\ell)^{\alpha - 1} = yq_s. \]
Then
\[ \theta_s^G(y) \to +\infty \quad \text{as} \quad y \to 0^+ \]
and
\[ \theta_s^G(y) \to \max_{\ell \in A_s} \{g_\ell\}, \quad \text{as} \quad y \to +\infty. \]

**Quadratic Utility:** If $U(x) = -(a - x)^2$, for $x \leq a$ then
\[ U'(x) = -2(x - a) \]
and the condition
\[ \sum_{\ell \in A_s} p_{\ell} U'(\theta_s^G(y) - g_\ell) = yq_s \]
becomes
\[ -2 \sum_{\ell \in A_s} p_{\ell} \left[ \theta_s^G(y) - a - g_\ell \right] = yq_s. \]
This gives
\[ \theta_s^G(y) = a + \frac{1}{p_s} \sum_{\ell \in A_s} p_{\ell} g_\ell - \frac{1}{2} \frac{q_s}{p_s} y. \]
However, we must have $\theta_s^G(y) \leq a$ so $\theta_s^G(y)$ is only defined when
\[
y \geq \frac{2}{q_s} \sum_{\ell \in A_s} p_\ell g_\ell \quad \text{for each} \quad s.
\]
That is, when
\[
y \geq \max_s \left( \frac{2}{q_s} \sum_{\ell \in A_s} p_\ell g_\ell \right) = \bar{y}.
\]
We then require $F(\bar{y}) \geq 0$. However, for $y \geq \bar{y}$
\[
\sum_{\ell \in A_s} p_\ell U(\theta_s^G(y) - g_\ell)
\]
is then independent of $a$. If $a$, (or $a - x$), is large enough then $F(\bar{y}) > 0$.

For case U1, \( \lim_{y \to \infty} F(y) = -\infty \).

For case U3, when $U(x) = \ell n x$,
\[
\lim_{y \to \infty} F(y) = -\infty.
\]

For case U2, when $U(x) = \frac{1}{\alpha} x^\alpha$, $\alpha < 1$,
\[
\lim_{y \to \infty} F(y) = \sum_{\ell} p_\ell U \left( \max_{\ell' \in A_s} g_{\ell'} - g_\ell \right) - V_0(x)
\]
\[
= \sum_{s=1}^{k} \left[ \sum_{\ell \in A_s} p_\ell U \left( \max_{\ell' \in A_s} g_{\ell'} - g_\ell \right) \right] - V_0(x).
\]
This may not be negative, in which case $F(y) = 0$ does not have a solution. However, $V_0(x) \geq U(x)$ which goes to $+\infty$ as $x \to +\infty$. Therefore, if $x$ is large enough $F(+\infty)$ is negative. Indeed, in Section 7, Case U2 it is shown that $V_0(x) = Cx^\alpha$ for a constant $C$ independent of $x$, so it is possible to estimate how large $x$ must be to ensure $\lim_{y \to \infty} F(y) < 0$.

**Quadratic Utility:** For the quadratic utility \( \lim_{y \to \infty} F(y) = -\infty \).

b) The computation of $\theta_s^G(y)$. 

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Recall for each \( s = 1, 2, \ldots, k \), \( \theta = \theta^G_s(y) \) is the unique solution of

\[
\sum_{\ell \in A_s} p_{\ell} U'(\theta^G_s(y) - g_{\ell}) = q_s y. \tag{33}
\]

For case U1 we know

\[
\theta^G_s(y) = \frac{1}{\gamma} \left[ \ell_n \sum_{\ell \in A_s} p_{\ell} e^{y g_{\ell}} - \ell_n \frac{y q_s}{\gamma} \right].
\]

For the quadratic utility, if \( y \geq \overline{y} \),

\[
\theta^G_s(y) = a + \frac{1}{p_s} \sum_{\ell \in A_s} p_{\ell} g_{\ell} - \frac{1}{2} \overline{q_s} \frac{y}{p_s}.
\]

Consider then cases U2 and U3, the power and logarithm utilities. Then (33) becomes

\[
\sum_{\ell \in A_s} p_{\ell} (\theta^G_s(y) - g_{\ell})^{\alpha - 1} = q_s y,
\]

with \( \alpha = 0 \) in case U3. We know \( \theta^G_s(y) \) is unique and

\[
\max_{\ell \in A_s} g_{\ell} < \theta^G_s(y) < \infty \quad \text{for each} \quad y > 0.
\]

Also

\[
\lim_{y \to 0^+} \theta^G_s(y) = \max_{\ell \in A_s} q_{\ell}.
\]

As

\[
\sum_{\ell \in A_s} p_{\ell} U'''(\theta^G_s(y) - g_{\ell}) (\theta^G_s)'(y) = q_s \tag{34}
\]

for each \( s = 1, 2, \ldots, k \), we must have

\[
(\theta^G_s)'(y) < 0.
\]

Differentiating (34) in \( y \) gives

\[
\left( \sum_{\ell \in A_s} p_{\ell} U'''(\theta^G_s(y) - g_{\ell}) \right) (\theta^G_s)'(y)^2 + \left( \sum_{\ell \in A_s} p_{\ell} U'''(\theta^G_s(y) - g_{\ell}) \right) (\theta^G_s)'(y) = 0
\]

so if \( U''' \geq 0 \) then \( (\theta^G_s)'(y) \geq 0 \).
Note that \( U''' \geq 0 \) holds for all the utilities we have considered. Recall the Arrow-Pratt risk aversion \( A(x) \) is given in terms of the utility as
\[
A(x) = -\frac{U''(x)}{U'(x)}.
\]
Then \( A'(x) \leq 0 \) implies \( U'''(x) \geq 0 \). To find \( \hat{\theta} = \theta^G(y) \), fix \( y \) and solve \( G(\theta) = 0 \) where
\[
G(\theta) = \sum_{\ell \in A_s} p_\ell U'(\theta - g_\ell) - q_s y.
\]
Then
\[
G'(\theta) = \sum_{\ell \in A_s} p_\ell U''(\theta - g_\ell) < 0
\]
and
\[
G''(\theta) = \sum_{\ell \in A_s} p_\ell U'''(\theta - g_\ell) \geq 0.
\]
The equation \( G(\theta) \) will be solved by the Newton-Rhapson method: Choose \( \theta_0 \) with \( F(\theta_0) \geq 0 \) so that \( \theta_{n+1} \geq \theta_n \) for all \( n \) and so \( \theta_n \) is an increasing sequence with limit \( \hat{\theta} = \theta^G(y) \) where \( \theta_{n+1} = \theta_n - F(\theta_n)/F'(\theta_n) \).

To select \( \theta_0 \) write \( \hat{g}_s = \max_{\ell \in A_s} g_\ell = g_{s'} \) and choose
\[
\theta_0 = \sigma_s + I \left( \frac{yq_s}{p_{s'}} \right).
\]
Then
\[ F(\theta_0) = p_\ell U'(\theta - g_\ell) - yq_s \]
\[ = p_\ell U'(I(\frac{yq_s}{p_\ell})) - yq_s \]
\[ = 0, \]
and
\[ \theta_{n+1} = \theta_n - \left[ \sum_{\ell \in A_s} p_\ell U'(\theta_n - g_\ell) - yq_s \right] \]
\[ \left[ \sum_{\ell \in A_s} p_\ell U''(\theta_n - g_\ell) \right]. \]

**Lemma.** $G(\theta_n) > 0$ implies $G(\theta_{n+1}) \geq 0$.

**Proof:** With $\hat{\theta} = \theta_s^G(y)$

\[ \frac{G(\hat{\theta}) - G(\theta_n)}{\hat{\theta} - \theta_n} = G'(\theta_n) \geq G''(\theta_n) \]

for some $\theta_n < \theta_n < \hat{\theta}$, and as $G''(\geq 0)$. Then

\[ \theta_{n+1} = \theta_n - \frac{G(\theta_n)}{G'(\theta_n)} \]
\[ \geq \frac{\theta_n G'(\theta_n) - G(\theta_n)}{G'(\theta_n)} \]
\[ \leq \frac{\hat{\theta} G'(\theta_n)}{G'(\theta_n)} = \hat{\theta}. \]

Therefore, $G(\theta_{n+1}) \geq 0$ as $G' < 0$.

The Lemma implies that $\theta_n$ is an increasing sequence with limit $\hat{\theta}$.

c) We now discuss the computation of $y$, the solution of

\[ F(y) = 0. \]

We have shown

\[ F'(y) = \sum_{\ell} p_\ell U'(\theta_s^G(y) - g_\ell)(\theta_s^G)'(y) \]
\[ = \sum_{s=1}^k yq_s(\theta_s^G)'(y). \]

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Therefore,
\[ F''(y) = \sum_{s=1}^{k} q_s \frac{d}{dy} [y(\theta_s^G)'(y)]. \]

Write
\[ N_k(y) := \frac{1}{p_s} \sum_{\ell \in A_s} p_{\ell} U^{(k)}(\theta(y) - g_{\ell}) \]
for \( k = 1, 2, \ldots \), where \( \theta(y) = \theta_s^G(y) \). Now
\[ \sum_{\ell \in A_s} p_{\ell} U'(\theta_s^G(\hat{y}) - g_{\ell}) = y q_s \]
so
\[ N_1(y) = \frac{y q_s}{p_s}. \]

Also,
\[ \sum_{\ell \in A_s} p_{\ell} U''(\theta_s^G(\hat{y}) - g_{\ell})(\theta_s^G)'(y) = q_s \]
so
\[ y(\theta_s^G)'(y) = \frac{q_s y}{\sum_{\ell \in A_s} p_{\ell} U''(\theta(y) - g_{\ell})} = \frac{N_1(y)}{N_2(y)}. \]

Then
\[ (y \theta'(y))' = \left[ \frac{N_2(y)^2 - N_1(y)N_3(y)}{N_2(y)^2} \right] \theta'(y) \]
as
\[ N'_1(y) = N_2(y) \theta'(y) \]
and
\[ N'_2(y) = N_3(y) \theta'(y). \]

We wish to show \( G''(y) > 0 \) which is the case if \( (y \theta'(y))' > 0 \). That is, if \( N_2(y)^2 \leq N_1(y)N_3(y) \). Now this is not the case when \( U(x) = -(a - x)^2 \) as then \( N_3(y) = 0 \). However, it is true for the cases U1, U2 and U3, because for these utilities
\[ (U''(x))^2 \leq U'(x)U'''(x) \]
with \( U' \geq 0 \) and \( U''' \geq 0 \).
**U1:** With \( U(x) = -e^{-\gamma x}, \gamma > 0, \)

\[
U'(x) = \gamma e^{-\gamma x}, \quad U''(x) = -\gamma^2 e^{-\gamma x}
\]

and

\[
U'''(x) = \gamma^3 e^{-\gamma x}
\]

so

\[
(U''(x))^2 = U'(x)U'''(x).
\]

Then

\[
\pm N_2(y) = \pm \sum_{\ell \in A_s} \frac{p_{\ell}}{p_s} U''(\theta(y) - g_{\ell})
\]

\[
\leq \sum_{\ell \in A_s} \frac{p_{\ell}}{p_s} |U''(\theta(y) - g_{\ell})|
\]

\[
\leq \sum_{\ell \in A_s} \frac{p_{\ell}}{p_s} \sqrt{U'(\theta(y) - g_{\ell})} \sqrt{U'''(\theta(y) - g_{\ell})}
\]

\[
\leq \left[ \sum_{\ell \in A_s} \frac{p_{\ell}}{p_s} U'(\theta(y) - g_{\ell}) \right]^{1/2} \left[ \sum_{\ell \in A_s} \frac{p_{\ell}}{p_s} U'''(\theta(y) - g_{\ell}) \right]^{1/2}
\]

\[
= N_1(y)^{1/2} N_3(y)^{1/2}.
\]

Therefore

\[
|N_2(y)| \leq N_1(y)^{1/2} N_3(y)^{1/2}
\]

and so

\[
N_2(y)^2 \leq N_1(y) N_3(y)
\]

as required.

**U2:** With \( U(x) = \frac{1}{\alpha} x^\alpha, \alpha < 1, \)

\[
U'(x) = x^{\alpha - 1}, \quad U''(x) = (\alpha - 1) x^{\alpha - 2}
\]

and

\[
U'''(x) = (\alpha - 1)(\alpha - 2) x^{\alpha - 3}.
\]

Then

\[
(U''(x))^2 = (\alpha - 1)^2 x^{2\alpha - 4}, \quad x > 0
\]

\[
\leq x^{\alpha - 1} (\alpha - 1)(\alpha - 2) x^{\alpha - 3}
\]

\[
= U'(x) U'''(x).
\]

The proof that \( N_2(y)^2 \leq N_1(y) N_3(y) \) then follows as in case U1.
U3: With $U(x) = \ell \ln x$, $x > 0$,

$$U'(x) = \frac{1}{x}, \quad U''(x) = -\frac{1}{x^2},$$

and

$$U'''(x) = \frac{2}{x^3}.$$ 

Then

$$U''(x)^2 = \frac{1}{x^4} \leq \frac{1}{x} \cdot \frac{2}{x^3}, \quad x > 0.$$ 

Again, the proof follows as in case U1.

The Newton-Rhapson Method for $y$: Choose $y_0$ so that

$$\theta^G_s(y_0) \geq g_e + U^{-1}\left(\frac{V_0(x)}{p_e}\right),$$ 

(33)

where $g_e = \max_{\ell'} g_e$, and suppose $\ell' \in A_s'$.

This is possible for U1, U2, and U3 but necessarily for $U(x) = -(x - a)^2$. However, there we can take $y_0 = 7$ provided $(a - x)$ is large enough. Then

$$F(y_0) \geq p_e U\left(g_e + U^{-1}\left(\frac{V_0(x)}{p_e}\right) - g_e\right) - V_0(x)$$

$$= p_e \frac{V_0(x)}{p_e} - V_0(x) = 0.$$ 

Now

$$\sum_{\ell' \in A_s'} p_{\ell'} U'(\theta^G_{s'}(y_0) - g_e) = y_0 q_{s'}$$

so

$$p_e U'(\theta^G_{s'}(y_0) - g_e) \leq y_0 q_{s'}.$$ 

As $I$ is a decreasing function

$$\theta^G_{s'}(y_0) \geq g_e + I\left(\frac{q_{s'} y_0}{p_e}\right).$$

If

$$y_0 = \frac{p_e}{q_{s'}} U'(U^{-1}\left(\frac{V_0(x)}{p_e}\right)),$$

(which can be computed), we have

$$g_e + I\left(\frac{q_{s'} y_0}{p_e}\right) = g_e + U^{-1}\left(\frac{V_0(x)}{p_e}\right).$$
Lemma. Choose

\[ y_0 = \frac{p_{\ell'}}{q_{s'}} U' \left( U^{-1} \left( \frac{V_0(x)}{p_{\ell'}} \right) \right) \]

where \( g_{\ell'} = \max_{\ell} g_\ell \) and \( \ell' \in A_{s'} \). Then

\[ F(y_0) > 0. \]

Set

\[ y_{n+1} := y_n - \frac{F(y_n)}{F'(y_n)}. \]

Then

\[ F(y_n) = \sum_{\ell} p_\ell U(\theta_s^G(y) - g_\ell) - V_0(x) \]
\[ F'(y_n) = \sum_{\ell} p_\ell U'(\theta_s^G(y_n) - g_\ell)(\theta_s^G)'(y_n) \]
\[ = \sum_{s=1}^k q_s y_n (\theta_s^G)'(y_n) \]

and

\[ (\theta_s^G)'(y_n) = \frac{q_s}{\sum_{\ell \in A_s} p_\ell U''(\theta_s^G(y_n) - g_\ell)}. \]

Summary: The algorithm is as follows: Start with the \( y_0 \) determined above. If \( y_n \) has been obtained compute \( \theta_s^G(y_n) \) for each \( s = 1, \ldots, k \) by the algorithm in part b). Then \( F(y_n) \) and \( F'(y_n) \) can be computed.

\( y_{n+1} \) is then given as \( y_n - \frac{F(y_n)}{F'(y_n)} \). We have \( y_n \) is monotonic increasing to \( y \) as \( n \to \infty \). As \( \theta_s^G(\cdot) \) is a decreasing function \( \sum_{s=1}^k q_s \theta_s^G(y_n) - x \) is decreasing to the asking price \( \nu^a \).

Note that for \( U(x) = -e^{-\gamma x} \), \( (\gamma > 0) \), and \( U(x) = -(a - x)^2 \) \( (x \leq a) \), explicit formulae are available for \( \theta_s^G(y) \).

13 Approximate Formulae

We shall obtain approximate values for \( \nu^a(G) \), \( \nu^b(G) \). Here we focus on \( \nu^b(G) \); \( \nu^a(G) \) is treated similarly.
Lemma 13.1.

$$\nu^b \cong \sum_s q_s E[G|A_s]$$

$$+ \frac{1}{2} \sum_s q_s \frac{U''(\theta^0_s)}{U'(\theta^0_s)} \text{var}[G|A_s].$$

(34)

Proof. We first recall:

(i) $$E[G|A_s] = \frac{1}{\tilde{p}_s} \sum_{\ell \in A_s} p_{\ell} g_{\ell}.$$ 

$$\text{var}[G|A_s] = \frac{1}{\tilde{p}_s} \sum_{\ell \in A_s} p_{\ell} [g_{\ell} - E[G|A_s]]^2.$$ 

(ii) The Arrow-Pratt risk aversion is given by

$$A(x) = -\frac{U''(x)}{U'(x)}$$

so the second term in (34) is

$$-\frac{1}{2} \sum_s q_s A(\theta^0_s) \text{var}[G|A_s].$$

Now $$\theta^0_s$$ is the solution of

$$U'(\theta^0_s) = \lambda_1 \frac{q_s}{\tilde{p}_s}$$

$$\sum_{s=1}^k q_s \theta^0_s = x$$

(35)

(36)

where

$$\tilde{p}_s = \sum_{\ell \in A_s} p_{\ell}.$$ 

Also $$\theta^G_s(y)$$ is the solution of

$$\sum_{\ell \in A_s} p_{\ell} U'(\theta^G_s(y) + G_{\ell}) = \lambda_2 q_s$$

$$\sum_{s=1}^k q_s \theta^G_s(y) = y.$$ 

(37)

(38)

We then select $$y$$ so that

$$E[U(\theta^G(y) + G)] = E[U(\theta^0)] = V_0(x)$$

(39)
and then
\[ y = x - \nu^b(G). \]

Suppose we approximate (37) by
\[ \sum_{\ell \in A_s} p_{\ell} \{ U' (\theta^0_s) + U'' (\theta^0_s) [\theta^G_s (y) + g_{\ell} - \theta^0_s] \} \equiv \lambda_2 q_s \] (40)
or
\[ \tilde{p}_s U' (\theta^0_s) + \tilde{p}_s U'' (\theta^0_s) [\theta^G_s (y) + E[G|A_s] - \theta^0_s] \equiv \lambda_2 q_s. \] (41)

Then
\[ (\theta^G_s (y) - \theta^0_s + E[G|A_s]) U'' (\theta^0_s) = \lambda q_s \tilde{p}_s \]
where \( \lambda = \lambda_2 - \lambda_1 \) by (35).

Therefore,
\[ \theta^G_s (y) - \theta^0_s + E[G|A_s] = \lambda q_s \frac{1}{p_s} \frac{q_s}{p_s} \frac{1}{U'' (\theta^0_s)}. \] (42)

This implies, using (36) and (38), that
\[ y - x + \sum_{s=1}^{k} q_s E[G|A_s] = \lambda \sum_{s=1}^{k} q_s \frac{2}{p_s} \frac{1}{U'' (\theta^0_s)} = \lambda M, \] (43)
where
\[ M = \sum_{s=1}^{k} \frac{q_s^2}{p_s} \frac{1}{U'' (\theta^0_s)} < 0. \]

Consequently, \( \lambda = M^{-1} (y - x + \sum_{s=1}^{k} q_s E[G|A_s]) \) so from (42):
\[ \theta^G_s (y) - \theta^0_s + E[G|A_s] = \frac{y - x + \sum_{s=1}^{k} q_s E[G|A_s]}{M} \frac{q_s}{p_s} \frac{1}{U'' (\theta^0_s)}. \] (44)
We now approximate:

\[ V^G_B(x - \nu^b) = \sum_{\ell} p_\ell U(\theta^G_s(y) + g_\ell) \]

\[ \approx = \sum_{\ell} p_\ell [U(\theta^0_s) + U'(\theta^0_s)[\theta^G_s(y) + g_\ell - \theta^0_s]] \]

\[ + \frac{1}{2} U''(\theta^0_s)[\theta^G_s(y) + g_\ell - \theta^0_s]^2 \]

\[ = V_0(x) + \sum_{s=1}^k \tilde{p}_s U'(\theta^0_s) [\theta^G_s(y) + E[G|A_s] - \theta^0_s] \]

\[ + \frac{1}{2} \sum_{s=1}^k \tilde{p}_s U''(\theta^0_s) \{ \text{var}\ (G|A_s) + (\theta^G_s(y) + E[G|A_s] - \theta^0_s)^2 \}. \]

Write

\[ Z \triangleq y - x + \sum_{s} q_s E[G|A_s] \]

so from (44)

\[ \theta^G_s(y) + E[G|A_s] - \theta^0_s = \frac{Z}{M} \frac{1}{U''(\theta^0_s)} \frac{q_s}{\tilde{p}_s}. \] (45)

Then

\[ \sum_{s=1}^k \tilde{p}_s U'(\theta^0_s) [\theta^G_s(y) + E[G|A_s] - \theta^0_s] \]

\[ = \sum_{s=1}^k \tilde{p}_s \frac{U'(\theta^0_s)}{U''(\theta^0_s)} \frac{q_s}{\tilde{p}_s} \frac{Z}{M} \text{ from (45)} \] (46)

\[ = \left( \sum_{s=1}^k q_s \frac{U'(\theta^0_s)}{U''(\theta^0_s)} \right) \frac{Z}{M}. \]

Also, using (45) again:

\[ \frac{1}{2} \sum_{s=1}^k \tilde{p}_s U''(\theta^0_s) [\theta^G_s(y) + E[G|A_s] - \theta^0_s]^2 \]

\[ = \frac{1}{2} \sum_{s=1}^k \tilde{p}_s U''(\theta^0_s) \frac{Z^2}{M^2} \frac{1}{(U''(\theta^0_s))^2} \frac{q_s^2}{\tilde{p}_s^2} \] (47)

\[ = \frac{1}{2} \frac{Z^2}{M^2} \sum_{s=1}^k q_s^2 \frac{1}{\tilde{p}_s U''(\theta^0_s)} \]

\[ = \frac{1}{2} \frac{Z^2}{M} \]
from the definition of $M$.

Now we are looking for the $\theta_s^G(y), s = 1, \ldots, k$, so that $V_B^G(x - \nu) = V_0(x)$ so that implies that approximately

$$\sum \ell p_\ell \left[ U'(\theta_\ell^0)(\theta_s^G(y) + g_\ell - \theta_\ell^0) + \frac{1}{2} U''(\theta_\ell^0)(\theta_s^G(y) + g_\ell - \theta_\ell^0)^2 \right] = 0$$

and we have

$$\frac{Z^2}{M} \sum_{s=1}^k q_s \frac{U'(\theta_s^0)}{U''(\theta_s^0)} + \frac{1}{2} \sum_{s=1}^k \bar{p}_s U''(\theta_{0s}) \text{var} [G|A_s] + \frac{1}{2} \frac{Z^2}{M} = 0. \quad (48)$$

That is

$$Z^2 + 2Z \sum_{s=1}^k q_s \frac{U'(\theta_s^0)}{U''(\theta_s^0)} + M \sum_{s=1}^k \bar{p}_s U''(\theta_{0s}) \text{var} [G|A_s] = 0.$$ 

Using (35) we have

$$\sum_{s=1}^k q_s \frac{U'(\theta_s^0)}{U''(\theta_s^0)} = \lambda_1 M.$$ 

Also from (35)

$$\bar{p}_s U''(\theta_s^0) = \lambda_1 q_s \frac{U''(\theta_s^0)}{U'(\theta_s^0)}$$

so

$$Z^2 + 2\lambda_1 M Z + \lambda_1 M \sum_{s=1}^k q_s \frac{U''(\theta_s^0)}{U'(\theta_s^0)} \text{var} [G|A_s] = 0. \quad (49)$$

Therefore,

$$(Z + \lambda_1 M)^2 = \lambda_1^2 M^2 - \lambda_1 M \sum_{s=1}^k q_s \frac{U''(\theta_s^0)}{U'(\theta_s^0)} \text{var} [G|A_s]$$

$$= \lambda_1^2 M^2 \left[ 1 - \frac{1}{\lambda_1 M} \sum_{s=1}^k q_s \frac{U''(\theta_s^0)}{U'(\theta_s^0)} \text{var} [G|A_s] \right].$$

Consequently,

$$Z + \lambda_1 M = \lambda_1 M \left[ 1 - \frac{1}{\lambda_1 M} \sum_{s=1}^k q_s \frac{U''(\theta_s^0)}{U'(\theta_s^0)} \text{var} [G|A_s] \right]^{1/2}$$

$$\approx \lambda_1 M \left[ 1 - \frac{1}{2} \lambda_1 M \sum_{s=1}^k q_s \frac{U''(\theta_s^0)}{U'(\theta_s^0)} \text{var} [G|A_s] \right].$$
Then
\[ Z \sim - \frac{1}{2} \sum_{s=1}^{k} q_s \frac{U''(\theta_s^0)}{U'(\theta_s^0)} \text{var}(G|A_s). \quad (50) \]

So
\[ Z = y - x + \sum_{s=1}^{k} q_s E[G|A_s] \approx - \frac{1}{2} \sum_{s=1}^{k} q_s \frac{U''(\theta_s^0)}{U'(\theta_s^0)} \text{var}(G|A_s). \quad (51) \]

As \( y = x - \nu^b(G) \)
\[ \nu^b(G) \approx \sum_{s=1}^{k} q_s E[G|A_s] + \frac{1}{2} \sum_{s=1}^{k} q_s \frac{U''(\theta_s^0)}{U'(\theta_s^0)} \text{var}(G|A_s). \quad (52) \]

**Corollary 13.2.** Suppose \( U(x) = -e^{-\gamma x}, \gamma > 0. \) Then
\[ \nu^b(G) \approx \sum_{s=1}^{k} q_s E[G|A_s] - \frac{\gamma}{2} \sum_{s} q_s \text{var}(G|A_s). \quad (53) \]

**Corollary 13.3.** Suppose \( U(x) = \frac{x^\alpha}{\alpha}, \alpha < 1. \) Then
\[ \frac{U''(\theta_s^0)}{U'(\theta_s^0)} = - \left[ \frac{1-\alpha}{x} \right] \left( \sum_{s=1}^{k} q_s' \left( \frac{q_s'}{\bar{p}_s} \right) \left( \frac{\bar{p}_s}{q_s} \right) \right). \quad (54) \]

**Remark 13.4.** Note that
\[ f(\theta_s^0) \sim f(x) + (\theta_s^0 - x)f'(x) \]
so
\[ \sum_{s} q_s f(\theta_s^0) \sim f(x) \quad (55) \]

since
\[ \sum_{s} q_s = 1 \]

and
\[ \sum_{s} q_s (\theta_s^0 - x) = 0. \]
Corollary 13.5. If
\[ A(\theta_0) = - \frac{U''(\theta_0)}{U'(\theta_0)} \]
is (approximately) independent of \( s \) then from Remark 13.4
\[ \nu^b(G) \cong \sum_{s=1}^{k} q_s E[G|A_s] - \frac{1}{2} A(x) \sum_{s=1}^{k} q_s \text{var} (G|A_s). \quad (56) \]

Corollary 13.6. If \( G \) is attainable it is \( F_s \) measurable so \( \text{var} (G|A_s) = 0 \) and
\[ \nu^b(G) = \sum_{s=1}^{k} q_s E[G|A_s]. \quad (57) \]

Corollary 13.7. If there are no hedging instruments then
\[ \nu^b(G) \cong E(G) - \frac{1}{2} A(x) \text{var} (G). \quad (58) \]

Remark. This is a classic actuarial approximation.

Lemma 13.8. Another approximation is
\[ \nu^b(G) \cong \sum_{s} q_s \left\{ U^{-1} \left( E[U(\theta_0^s + G)|A_s]\right) - \theta_0^s \right\}. \quad (59) \]

Proof.
\[ E[U(\theta_0^s + g_\varepsilon)|A_s] \cong U(\theta_0^s) + U'(\theta_0^s) [E[G|A_s]] + \frac{1}{2} U''(\theta_0^s) E[G^2|A_s] \]
\[ = U(\theta_0^s) + \varepsilon_s, \text{ say}. \]

Now for small \( \varepsilon_s \), again using the Taylor expansion:
\[ U^{-1}(U(\theta_0^s) + \varepsilon_s) \cong U^{-1}(U(\theta_0^s)) + \varepsilon_s(U^{-1})'(U(\theta_0^s)) \]
\[ + \frac{1}{2} \varepsilon_s^2(U^{-1})''(U(\theta_0^s)) + \ldots \]
\[ = \theta_0^s + \frac{\varepsilon_s}{U'(\theta_0^s)} + \frac{1}{2} \varepsilon_s^2 \left[ \frac{U''(\theta_0^s)}{\left(U'(\theta_0^s)^3\right)} \right] + \ldots \quad (60) \]

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since

\[(U^{-1})'(U(\theta_s^0)) = \frac{1}{U'(\theta_s^0)}\]

and \[(U^{-1})''(U(\theta_s^0)) = -\frac{U''(x)}{U'(x)^3}.\]

In fact, using the chain rule,

\[U^{-1}(U(x)) = x\]

so

\[(U^{-1})'(U(x))U'(x) = 1.\]

Therefore,

\[(U^{-1})'(U(x)) = \frac{1}{U'(x)}.\]

Differentiating again

\[(U^{-1})''(U(x))U'(x) = -\frac{1}{U'(x)^2} U''(x)\]

giving

\[(U^{-1})''(U(x)) = -\frac{1}{U'(x)^3} U''(x).\]

Therefore, from (60):
\[ \sum_{s=1}^{k} q_s \left\{ U^{-1} \left( E[U(\theta_s^0 + G)|A_s] \right) - \theta_s^0 \right\} \]

\[ \cong \sum_{s=1}^{k} q_s \left\{ \frac{\varepsilon_s}{U''(\theta_s^0)} - \frac{1}{2} \varepsilon_s^2 \frac{U'''(\theta_s^0)}{U'(\theta_s^0)^3} \right\} \]

\[ = \sum_{s=1}^{k} q_s \left\{ E[G|A_s] + \frac{1}{2} \frac{U''(\theta_s^0)}{U'(\theta_s^0)} E[G^2|A_s] \right\} \]

\[ - \frac{1}{2} \left( U'(\theta_s^0) E[G|A_s] + \frac{1}{2} U''(\theta_s^0) E[G^2|A_s] \right)^2 \frac{U'''(\theta_s^0)}{U'(\theta_s^0)^3} \}

\[ = \sum_{s=1}^{k} q_s E[G|A_s] + \frac{1}{2} \sum_{s=1}^{k} q_s \frac{U''(\theta_s^0)}{U'(\theta_s^0)} E[G^2|A_s] \]

\[ - \frac{1}{2} \sum_{s=1}^{k} q_s \frac{U''(\theta_s^0)}{U'(\theta_s^0)} \left( E[G|A_s] \right)^2 \]

\[ - \frac{1}{2} \sum_{s=1}^{k} q_s \left[ \frac{U''(\theta_s^0)}{U'(\theta_s^0)} \right]^2 E[G|A_s] E[G^2|A_s] \]

\[ - \frac{1}{8} \sum_{s=1}^{k} q_s \left[ \frac{U''(\theta_s^0)}{U'(\theta_s^0)} \right]^3 E[G^2|A_s]^2 \]

\[ \cong \sum_{s=1}^{k} q_s E[G|A_s] + \frac{1}{2} \sum_{s=1}^{k} q_s \frac{U''(\theta_s^0)}{U'(\theta_s^0)} \text{var} (G|A_s). \]

(61)

This follows because, if we assume that the Arrow-Pratt risk aversion \( A(\theta_s^0) = - \frac{U''(\theta_s^0)}{U'(\theta_s^0)} \) is small, we can ignore the higher powers of \( A(\theta_s^0) \). This last expression (61) is the same as the approximation established in Lemma 1 and hence Lemma 13.8 follows.

**Remark 13.9.** The interpretation of equation (59) is this: Compute the certainty equivalent of \( \theta_s^0 + G \) given \( A_s \) and then subtract \( \theta_s^0 \). This is an attainable claim whose present value is the right hand side of (59).

**Corollary 13.10.** If \( G \) is attainable, then

\[ U^{-1} \left( E[U(\theta_s^0 + G)|A_s] \right) = \theta_s^0 + G \]
and so
\[ \nu^h(G) = \sum_{s=1}^{k} q_s E[G | A_s] = \sum_{s=1}^{k} q_s g_s \]
where \( g_\ell = g_s \) for \( \ell \in A_s \).

**Corollary 13.11.** If there is no tradable assets then \( \nu^h(G) \cong U^{-1} E[U(G)] \)
which agrees with actuarial pricing.

14 An Alternative Representation for \( V_G(x) \)

We shall now derive an alternative expression for \( V_G(x) \) using our duality formulae. Some generalizations of pricing formulae given by Musiela and Zariphopoulou [4] will then be given.

Recall from Theorem 4.7
\[ V_G(x) = \inf_{y > 0} \left[ \tilde{V}_G(y) + xy \right]. \]  
(62)

From Definition 4.5
\[ \tilde{V}_G(y) = \inf_{Q \in \mathcal{M}} E_P \left[ \tilde{U} \left( \frac{Q}{P} y \right) - yE_Q(G) \right]. \]  
(63)

From Equation (11), for \( 0 < z < \infty \),
\[ \tilde{U}(z) = U(I(z)) - zI(z). \]  
(64)

Substituting (64) in (63)
\[ \tilde{V}_G(y) = \inf_{Q \in \mathcal{M}} \left[ E_P \left[ U \left( I \left( \frac{Q}{P} y \right) \right) \right] - yE_Q \left( I \left( \frac{Q}{P} y \right) \right) \right] - yE_Q(G) \]  
(65)

and substituting in (62):
\[ V_G(x) = \inf_{y > 0} \inf_{Q \in \mathcal{M}} \left[ E_P \left[ U \left( I \left( \frac{Q}{P} y \right) \right) \right] - yE_Q \left( I \left( \frac{Q}{P} y \right) \right) - yE_Q(G) + xy \right]. \]  
(66)

Reversing the order of the infima:
\[ V_G(x) = \inf_{Q \in \mathcal{M}} \inf_{y > 0} \left[ E_P \left[ U \left( I \left( \frac{Q}{P} y \right) \right) \right] - yE_Q \left( I \left( \frac{Q}{P} y \right) \right) - yE_Q(G) + xy \right]. \]  
(67)
Given $Q$, the first order condition for the minimizing $\hat{y} = \hat{y}(Q)$ is:

$$\mathbb{E}_P\left[U'(I(Q_P \hat{y})) I'(\frac{Q}{P} \hat{y}) \frac{Q}{P}\right] - \mathbb{E}_Q\left[I(\frac{Q}{P} \hat{y})\right] - \hat{y} \mathbb{E}_Q[I'(\frac{Q}{P} \hat{y}) \frac{Q}{P}] - \mathbb{E}_Q(G) + x = 0.$$  

That is,

$$\mathbb{E}_Q\left[I(\frac{Q}{P} \hat{y})\right] = x - \mathbb{E}_Q(G).$$  

(68)

Here we used $U'(I(Q_P \hat{y})) = \frac{Q}{P} \hat{y}$ and

$$\mathbb{E}_P\left[\frac{Q}{P} I'(\frac{Q}{P} \hat{y}) \frac{Q}{P}\right] = \mathbb{E}_Q\left[I'(\frac{Q}{P} \hat{y}) \frac{Q}{P}\right].$$

Therefore, with this choice of $\hat{y} = \hat{y}(Q)$ the final three terms in (67) cancel so

$$V_G(x) = \inf_{Q \in \mathcal{M}} \mathbb{E}_P\left[U(I(Q_P \hat{y}(Q)))\right].$$  

(69)

Write

$$\phi(y) := \mathbb{E}_P\left[U\left(I\left(\frac{Q}{P} y\right)\right)\right] - y \mathbb{E}_Q\left[I\left(\frac{Q}{P} y\right)\right] - y \mathbb{E}_Q(G) + xy.$$

Then

$$\phi'(y) = -\mathbb{E}_Q\left[I\left(\frac{Q}{P} y\right)\right] - \mathbb{E}_Q(G) + x$$

and

$$\phi''(y) = -\mathbb{E}_Q\left[I'(\frac{Q}{P} y) \frac{Q}{P}\right] > 0$$

as $U''(I(y)) I'(y) = 1$ implies $I'(y) < 0$.

Consequently $y \rightarrow \phi(y)$ is convex. Thus (68) characterizes the minimization with respect to $y$. We summarize our results as

**Theorem 14.1.**

$$V_G(x) = \inf_{Q \in \mathcal{M}} \mathbb{E}_P\left[U\left(I\left(\frac{Q}{P} \hat{y}(Q)\right)\right)\right]$$  

(70)

where $\hat{y}(Q)$ is the solution of

$$\mathbb{E}_Q\left[I\left(\frac{Q}{P} \hat{y}(Q)\right)\right] = x - \mathbb{E}_Q(G).$$

This result provides a new recipe for the asking price for $G$

$$\nu = \nu^\alpha(G),$$

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(and similarly for \( \nu^b(G) = -\nu^a(-G) \)). This is:

\[
\inf_{Q \in M} E_P \left[ U \left( I \left( \frac{Q}{P} \hat{y}_1(Q) \right) \right) \right] = \inf_{Q \in M} E_P \left[ U \left( I \left( \frac{Q}{P} \hat{y}_2(Q) \right) \right) \right]
\]

(71)

where

\[
E_Q \left[ I \left( \frac{Q}{P} \hat{y}_1(Q) \right) \right] = x + \nu - E_Q(G)
\]

(72)

and

\[
E_Q \left[ I \left( \frac{Q}{P} \hat{y}_2(Q) \right) \right] = x.
\]

(73)

**Example 14.1.** With \( U(x) = -e^{-\gamma x}, \ (\gamma > 0, \ x \in \mathbb{R}) \)

\[
U'(x) = \gamma e^{-\gamma x}, \quad I(y) = -\frac{1}{\gamma} \ln \frac{y}{\gamma}
\]

and \( U(I(y)) = -\frac{y}{\gamma} \), as before.

Writing

\[
h(Q \parallel P) = E_Q \left[ \ln \frac{Q}{P} \right]
\]

we can show

\[
\hat{y}_1(Q) = \gamma \exp \left[ -\gamma (x + \nu - E_Q(G)) - h(Q \parallel P) \right]
\]

and

\[
\hat{y}_2(Q) = \gamma \exp \left[ -\gamma x - h(Q \parallel P) \right].
\]

Then

\[
E_P \left[ U \left( I \left( \frac{Q}{P} \hat{y}_1(Q) \right) \right) \right] = -\exp \left[ -\gamma (x + \nu - E_Q(G)) - h(Q \parallel P) \right]
\]

and

\[
E_P \left[ U \left( I \left( \frac{Q}{P} \hat{y}_2(Q) \right) \right) \right] = -\exp \left[ -\gamma x - h(Q \parallel P) \right]
\]

so from (71):

\[
-\gamma \nu + \sup_{Q \in \mathcal{M}} \left[ \gamma E_Q(G) - h(Q \parallel P) \right] = -\inf_{Q \in \mathcal{M}} h(Q \parallel P).
\]

Therefore

\[
\nu = \sup_{Q \in \mathcal{M}} \left[ E_Q(G) - \frac{1}{\gamma} \left[ h(Q \parallel P) - \inf_{Q \in \mathcal{M}} h(Q \parallel P) \right] \right]
\]

\[
= \sup_{Q \in \mathcal{M}} \left[ E_Q(G) - \theta(Q) \right]
\]

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where
\[ \theta(Q) = \frac{1}{\gamma} \left[ h(Q\|P) - \inf_{Q \in \mathcal{M}} h(Q\|P) \right]. \]

This expression was given by Musiela in his presentation at the 2002 Bachelier conference.

**Example 14.2.** With \( U(x) = \frac{1}{\alpha} x^\alpha \), \((0 < \alpha < 1, x > 0)\)

\[ U'(x) = x^{\alpha-1}, \quad I(y) = y^{\frac{1}{\alpha-1}}, \quad y > 0 \]

and

\[ U(I(y)) = \frac{1}{\alpha} y^{\frac{\alpha}{\alpha-1}}. \]

In this case we can easily find

\[ \hat{y}_1(Q) = \left( \frac{E_Q \left( \frac{P}{Q} \right)^{\frac{1}{1-\alpha}} \right) \frac{1}{x + \nu - E_Q(G)} \right)^{1-\alpha} \]

(74)

\[ \hat{y}_2(Q) = \frac{1}{x^{1-\alpha}} \left[ E_Q \left( \frac{P}{Q} \right)^{\frac{1}{1-\alpha}} \right]^{1-\alpha} \]

(75)

where, as noted in \( \S \)12.2, \( \hat{y}_1(Q) \) is defined provided \( x \) is sufficiently large.

Then

\[ E_P \left[ U \left( I \left( \frac{Q}{P} \hat{y}_1(Q) \right) \right) \right] = \frac{1}{\alpha} \hat{y}_1(Q)^{\frac{\alpha}{\alpha-1}} E_P \left[ \left( \frac{P}{Q} \right)^{\frac{1}{1-\alpha}} \right] \]

\[ = \frac{1}{\alpha} \hat{y}_1(Q)^{\frac{\alpha}{\alpha-1}} E_Q \left[ \frac{P}{Q} \left( \frac{P}{Q} \right)^{\frac{1}{1-\alpha}} \right] \]

\[ = \frac{1}{\alpha} \hat{y}_1(Q)^{\frac{\alpha}{\alpha-1}} E_Q \left[ \left( \frac{P}{Q} \right)^{\frac{1}{1-\alpha}} \right] \]

\[ = \frac{1}{\alpha} \left( x + \nu - E_Q(G) \right)^\alpha \left[ E_Q \left[ \left( \frac{P}{Q} \right)^{\frac{1}{1-\alpha}} \right] \right]^{1-\alpha} \]

and

\[ E_P \left[ U \left( I \left( \frac{Q}{P} \hat{y}_2(Q) \right) \right) \right] = \frac{1}{\alpha} x^{\alpha} \left[ E_Q \left[ \left( \frac{P}{Q} \right)^{\frac{1}{1-\alpha}} \right] \right]^{1-\alpha}. \]

Then from (71)

\[ \inf_{Q \in \mathcal{M}} \left( x + \nu - E_Q(G) \right)^\alpha \left[ E_Q \left[ \left( \frac{P}{Q} \right)^{\frac{1}{1-\alpha}} \right] \right]^{1-\alpha} = x^{\alpha} \inf_{Q \in \mathcal{M}} \left[ E_Q \left[ \left( \frac{P}{Q} \right)^{\frac{1}{1-\alpha}} \right] \right]^{1-\alpha} \]

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or
\[
\inf_{Q \in \mathcal{M}} \left[ (x + \nu - E_Q(G))^\alpha \left[ E_P \left[ \left( \frac{P}{Q} \right)^{\frac{\alpha}{1 - \alpha}} \right] \right]^{1 - \alpha} \right] = x^\alpha \inf_{Q \in \mathcal{M}} \left[ E_P \left[ \left( \frac{P}{Q} \right)^{\frac{\alpha}{1 - \alpha}} \right] \right]^{1 - \alpha}.
\]

The characterization of \( Q \in \mathcal{M} \) given in Theorem 2.3 can then be used to calculate these infima.

**Example 14.3.** With \( U(x) = \ln x, \ x > 0 \)
\[
U'(x) = \frac{1}{x}, \quad I(y) = \frac{1}{y}
\]
\[
U(I(y)) = -\ln y.
\]
We can then check that
\[
\hat{y}_1(Q) = \frac{1}{(x + \nu - E_Q(G))} \quad \text{and} \quad \hat{y}_2(Q) = \frac{1}{x}.
\]
Then
\[
E_P \left[ U \left( I \left( \frac{Q}{P} \hat{y}_1(Q) \right) \right) \right] = -E_P \left[ \ln \left( \frac{Q}{P} \hat{y}_1(Q) \right) \right]
\]
\[
= -\ln \hat{y}_1(Q) - E_P [\ln \left( \frac{Q}{P} \right)]
\]
\[
= -\ln \hat{y}_1(Q) + E_P [\ln \left( \frac{Q}{P} \right)]
\]
\[
= -\ln \hat{y}_1(Q) + h(P\|Q)
\]
\[
= -\ln \hat{y}_1(Q) + h(P\|Q) = +\ln (x + \nu - E_Q(G)) + h(P\|Q).
\]
Also,
\[
E_P \left[ U \left( I \left( \frac{Q}{P} \hat{y}_2(Q) \right) \right) \right] = \ln x + h(P\|Q).
\]
Therefore, (71) becomes
\[
\inf_{Q \in \mathcal{M}} \left[ \ln (x + \nu - E_Q(G)) + h(P\|Q) \right] = \inf_{Q \in \mathcal{M}} \left[ \ln x + h(P\|Q) \right]
\]
\[
= \ln x + \inf_{Q \in \mathcal{M}} \ln h(P\|Q).
\]

**Example 14.4.** With \( U(x) = -(a - x)^2, \ x \leq a, \)
\[
U'(x) = -2(a - x), \quad I(y) = a - \frac{y}{2},
\]
\[
U(I(y)) = -\frac{y^2}{4}.
\]
We can then check that
\[
\hat{y}_1(Q) = \frac{2[a - \nu - x + EQ(G)]}{EP[(\frac{Q}{P})^2]}
\]
and
\[
\hat{y}_2(Q) = \frac{2(a - x)}{EP[(\frac{Q}{P})^2]}.
\]
Then
\[
EP\left[U\left(I(Q \hat{y}_1(Q))\right)\right] = -\frac{1}{4} EP\left[(\frac{Q}{P})^2 \cdot \left(\frac{4(a - x - \nu + EQ(G))^2}{EP(\frac{Q}{P})^2}\right)\right]
\]
\[
= - \frac{(a - x - \nu + EQ(G))^2}{EP(\frac{Q}{P})^2},
\]
and
\[
EP\left[U\left(I(Q \hat{y}_2(Q))\right)\right] = \frac{-(a - x)^2}{EP(\frac{Q}{P})^2}.
\]
Therefore, (71) in this case becomes
\[
\max_{Q \in M} \left[\frac{(a - x - \nu + EQ(G))^2}{EP(\frac{Q}{P})^2}\right] = \max_{Q \in M} \left[\frac{(a - x)^2}{EP(\frac{Q}{P})^2}\right].
\]
Again, the characterization of \( Q \in M \) given in Theorem 2.3 can be used to calculate these maxima.

References


